

XI Hotine-Marussi Symposium, Genova, 7-11 September 2026

SPHERICAL SHAPE FUNCTIONS FOR A SIX-NODE TRI-RECTANGULAR PRISM AND AN EIGHT-NODE QUADRANGULAR RIGHT PRISM

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DEGLI STUDI DI MILANO



**POLITECNICO
MILANO 1863**

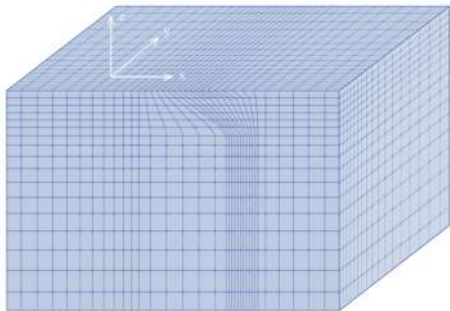
Math. Comput. Appl. (2025) - doi.org:10.3390/mca30040088

FEM - WHY DO WE NEED SPHERICAL SHAPE FUNCTIONS?

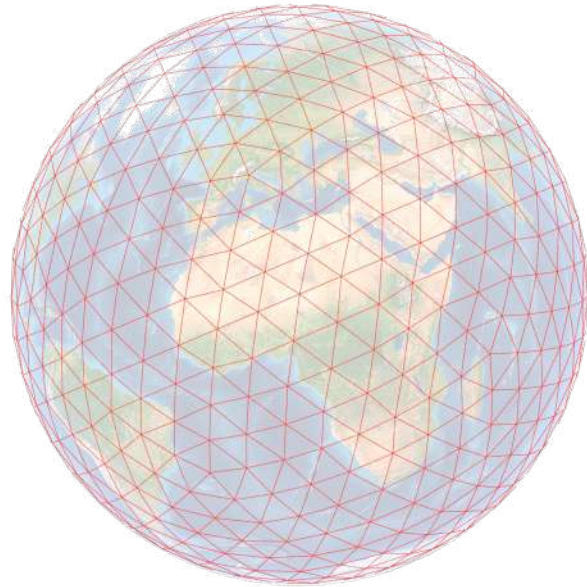
NUMERICAL MODELLING OF GEODYNAMIC PROCESSES

Lithosphere deformation
Mantle convection
Seismic wave propagation

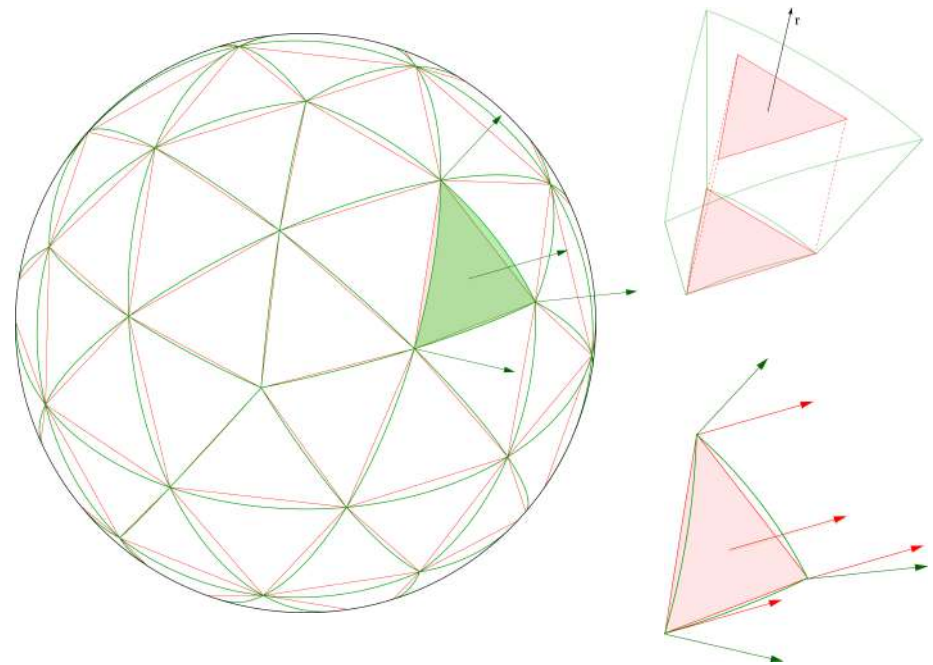
FLAT EARTH



SPHERICAL EARTH

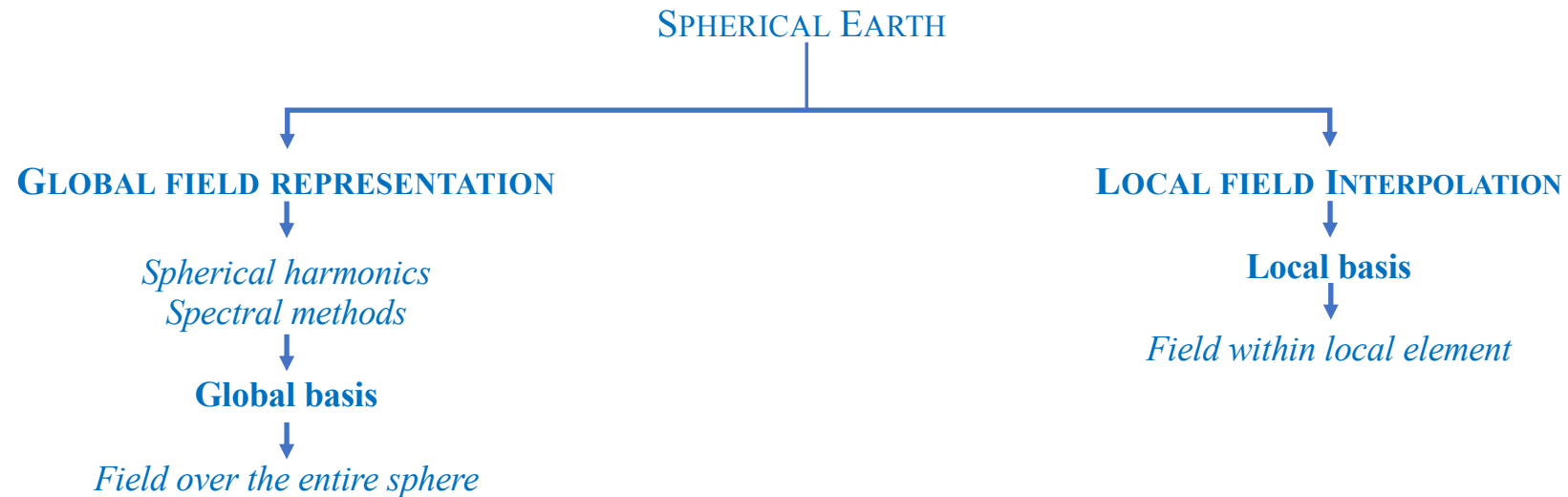


FLAT VS SPHERICAL EARTH REPRESENTATION



Classical Cartesian finite elements cannot
exactly represent spherical geometry.

FIELDS ON A SPHERICAL EARTH - from Global Representation to Local Interpolation



Different mathematical representations address different spatial scales.

BUT

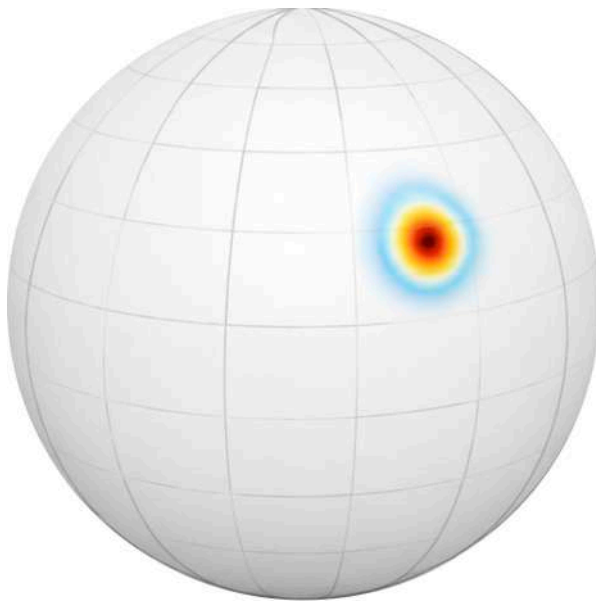
HOW DOES THE CHOICE OF REPRESENTATION AFFECT GEODETIC APPLICATIONS?

GLOBAL TO LOCAL SPHERICAL REPRESENTATIONS

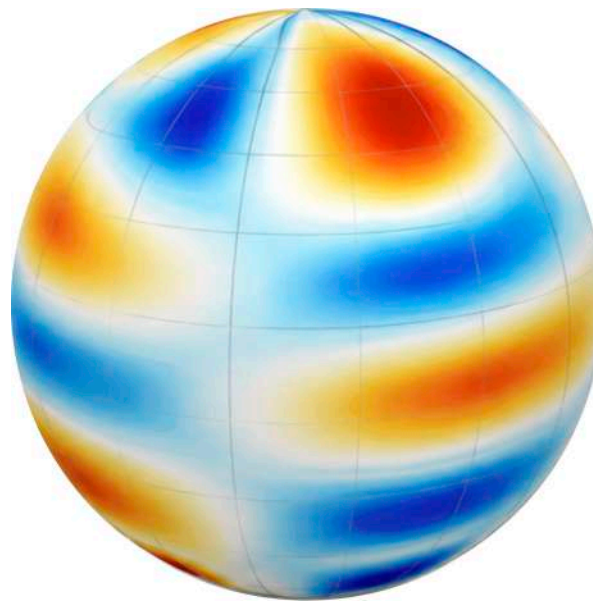


LEAKAGE

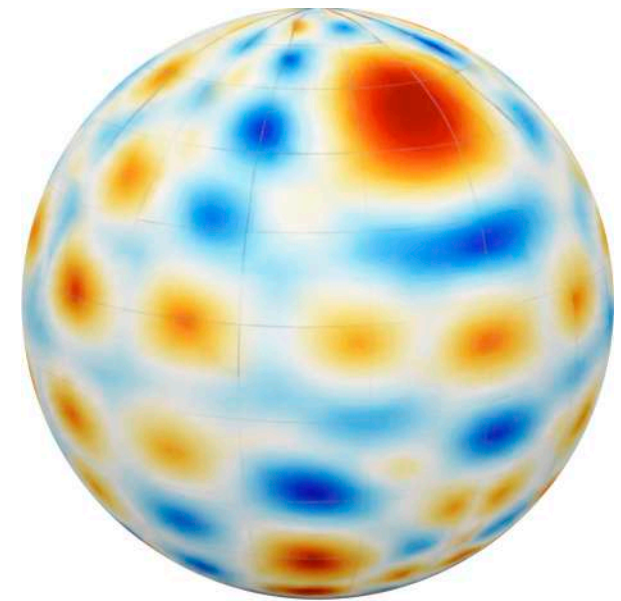
LOCALIZED
SURFACE SIGNAL



GLOBAL SPHERICAL-HARMONIC
REPRESENTATION



SPATIAL LOCALIZATION / TRUNCATION



Spatially local functions on the sphere



ANALYTICAL SPHERICAL SHAPE FUNCTIONS

FROM SATELLITE TO EARTH



GRAVITATIONAL POTENTIAL DOWNWARD CONTINUATION

STANDARD APPROACHES

SPHERICAL HARMONICS → *Global spectral representation*
→ **Limited spatial localization**

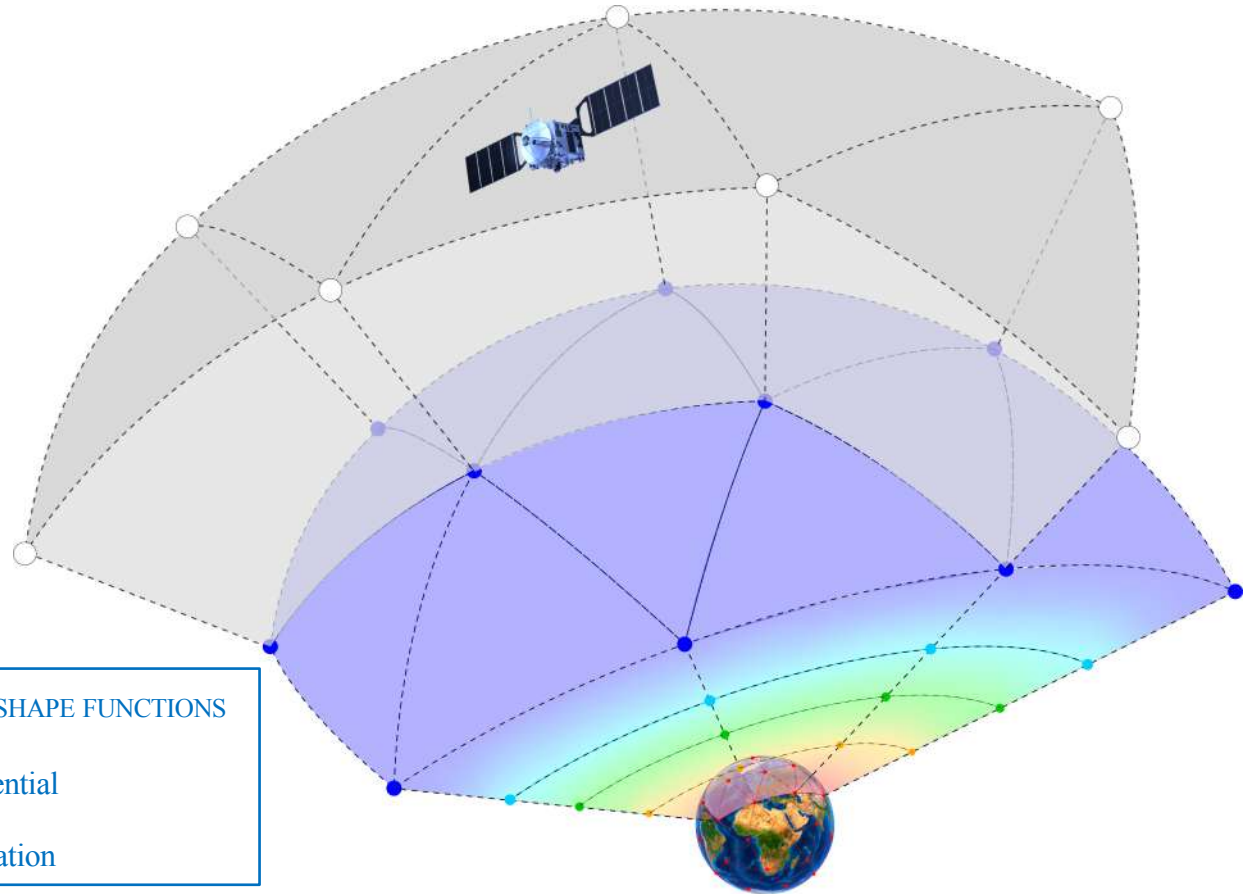
POISSON INTEGRAL / EQUIVALENT SOURCES → *Inversion of upward continuation* → **Non-uniqueness and depth dependence**

REGULARIZATION → *Filtering and stabilization of the ill-posed problem* → **Resolution loss / signal smoothing**

PLANAR-ELEMENT FEM → *Curved spherical domain approximated by flat elements* → **Geometric errors and increased mesh requirements**

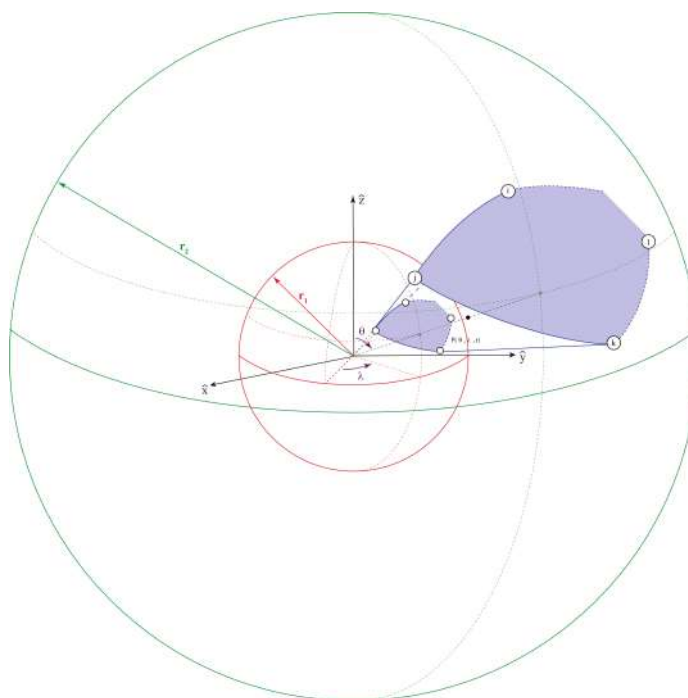
SPHERICAL LOCAL BASIS → ANALYTICAL SPHERICAL SHAPE FUNCTIONS

- Local 3-D spherical FEM representation of the gravitational potential
- Geometrically exact representation of spherical boundaries
- Potential for stable and physically consistent downward continuation



SPHERICAL SHAPE FUNCTIONS - MATHEMATICAL FORMULATION

$$N(\theta, \lambda, r)$$



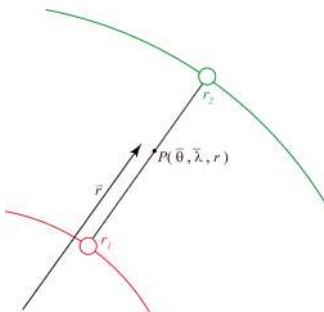
Generic spherical prismatic element

SPHERICAL SHAPE FUNCTIONS - MATHEMATICAL FORMULATION

$$N(\theta, \lambda, r) = R(r) \cdot A(\theta, \lambda) \cdot F(\theta, \lambda)$$

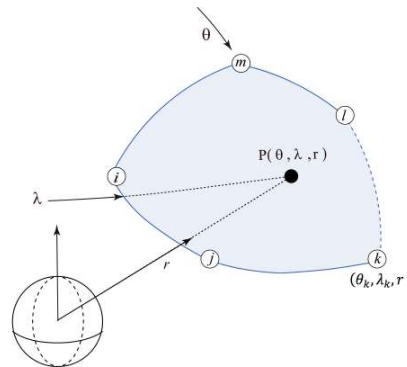
$R(r)$

RADIAL INTERPOLATION
controls SCALAR variation
through depth



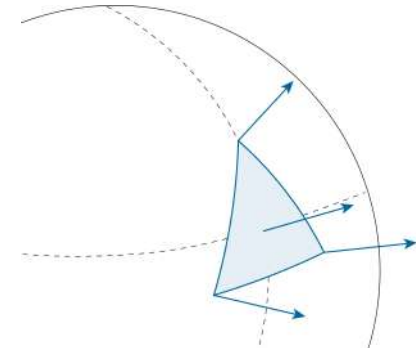
$A(\theta, \lambda)$

SURFACE INTERPOLATION
preserves SCALAR interpolation
over the spherical surface



$F(\theta, \lambda)$

LOCAL FRAME TRANSFORMATION
ensures consistency of DIRECTIONS
over the spherical surface

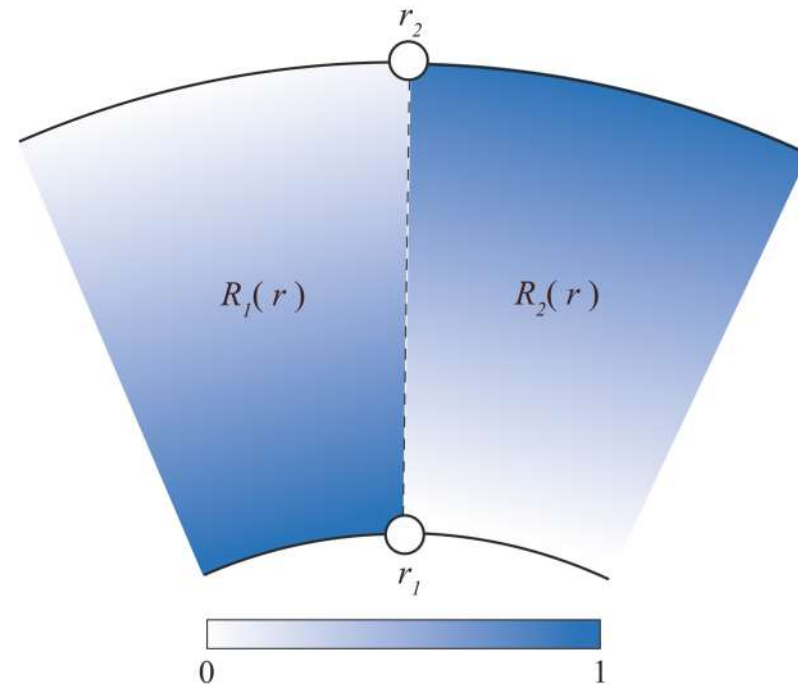
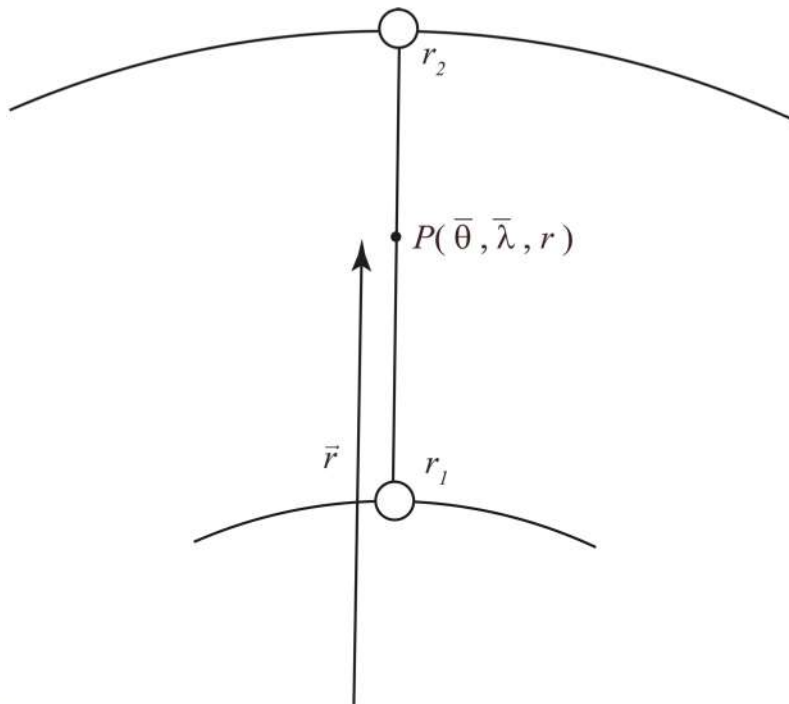


$R(r)$ RADIAL INTERPOLATION - controls SCALAR variation through depth

$$\mathbf{R}(r) = \begin{bmatrix} R_1(r) \\ R_2(r) \end{bmatrix}$$

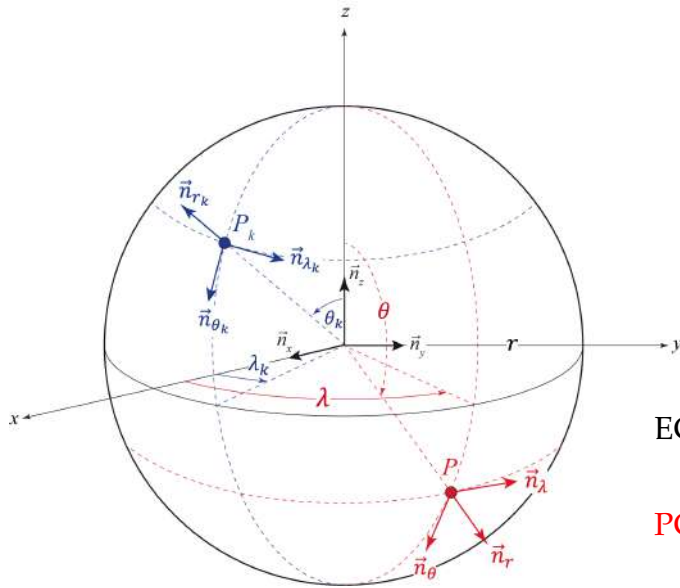
$$R_1(r) = \frac{r_2 - r}{r_2 - r_1} = \begin{cases} 1 & \text{for } r = r_1 \\ 0 & \text{for } r = r_2 \end{cases}$$

$$R_2(r) = \frac{r - r_1}{r_2 - r_1} = \begin{cases} 0 & \text{for } r = r_1 \\ 1 & \text{for } r = r_2 \end{cases}$$



$$F(\theta, \lambda)$$

LOCAL FRAME TRANSFORMATION
ensures consistency of DIRECTIONS
over the spherical surface



Scheme illustrating the variation of the
orientation of unit vector normal to the sphere
surface moving from point P to point Pk.

ECR Earth-Centered Rotational
Reference Frame

PCR Centered P-Rotational
Reference Frame

P_kCR Centered P_k-Rotational
Reference Frame

$$R_{RP}(\theta, \lambda) = \begin{bmatrix} \cos \theta \cos \lambda & \cos \theta \sin \lambda & -\sin \theta \\ -\sin \lambda & \cos \lambda & 0 \\ \sin \theta \cos \lambda & \sin \theta \sin \lambda & \cos \theta \end{bmatrix}$$

$$\begin{bmatrix} n_\theta \\ n_\lambda \\ n_r \end{bmatrix} = R_{RP}(\theta, \lambda) \cdot \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

$$R_{PR}(\theta_k, \lambda_k) = \begin{bmatrix} \cos \theta_k \cos \lambda_k & -\sin \lambda_k & \sin \theta_k \cos \lambda_k \\ \cos \theta_k \sin \lambda_k & \cos \lambda_k & \sin \theta_k \sin \lambda_k \\ -\sin \theta_k & 0 & \theta_k \end{bmatrix}$$

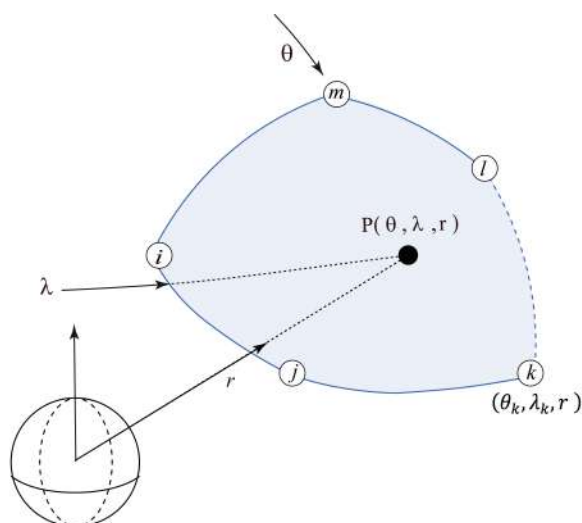
$$\begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = R_{PR}(\theta_k, \lambda_k) \cdot \begin{bmatrix} n_{\theta_k} \\ n_{\lambda_k} \\ n_{r_k} \end{bmatrix}$$

$$R_{RP}(\theta, \lambda) \cdot R_{PR}(\theta_k, \lambda_k) = \mathbf{F}_k(\theta, \lambda) = \begin{bmatrix} F_k^{\theta\theta} & F_k^{\theta\lambda} & F_k^{\theta r} \\ F_k^{\lambda\theta} & F_k^{\lambda\lambda} & F_k^{\lambda r} \\ F_k^{r\theta} & F_k^{r\lambda} & F_k^{rr} \end{bmatrix}$$

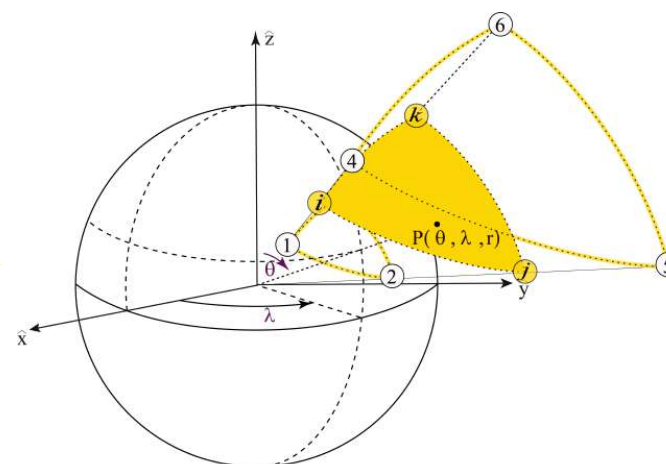
$$\mathbf{F}_k(\theta, \lambda) = \begin{cases} F_k^{\theta\theta}(\theta, \lambda) = \cos \theta \cos \theta_k \cos(\lambda - \lambda_k) + \sin \theta \sin \theta_k \\ F_k^{\theta\lambda}(\theta, \lambda) = \cos \theta \sin(\lambda - \lambda_k) \\ F_k^{\theta r}(\theta, \lambda) = \cos \theta \sin \theta_k \cos(\lambda - \lambda_k) - \sin \theta \cos \theta_k \\ F_k^{\lambda\theta}(\theta, \lambda) = -\cos \theta_k \sin(\lambda - \lambda_k) \\ F_k^{\lambda\lambda}(\theta, \lambda) = \cos(\lambda - \lambda_k) \\ F_k^{\lambda r}(\theta, \lambda) = -\sin \theta_k \sin(\lambda - \lambda_k) \\ F_k^{r\theta}(\theta, \lambda) = \sin \theta \cos \theta_k \cos(\lambda - \lambda_k) - \cos \theta \sin \theta_k \\ F_k^{r\lambda}(\theta, \lambda) = \sin \theta \sin(\lambda - \lambda_k) \\ F_k^{rr}(\theta, \lambda) = \sin \theta \sin \theta_k \cos(\lambda - \lambda_k) + \cos \theta \cos \theta_k \end{cases}$$

$$A(\theta, \lambda)$$

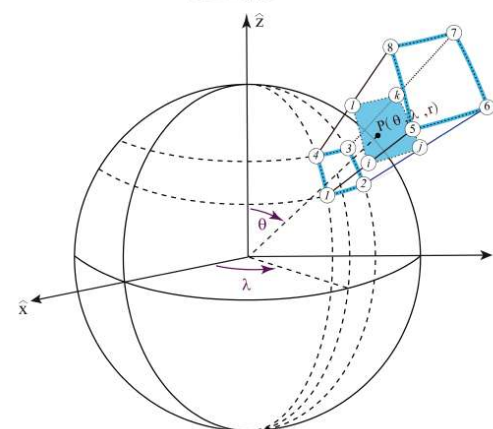
SURFACE INTERPOLATION
preserves SCALAR interpolation
over the spherical surface



**Spherical Six-Node
Triangular Prism**

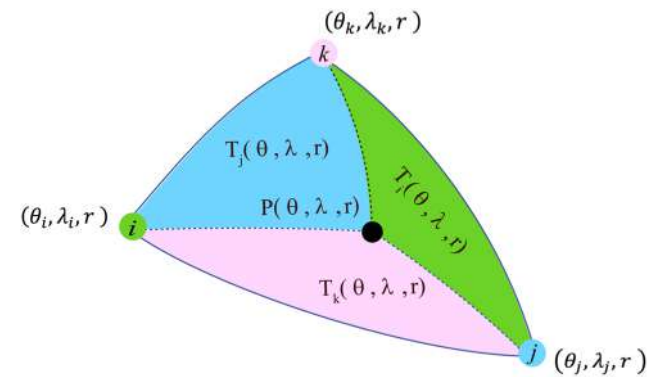
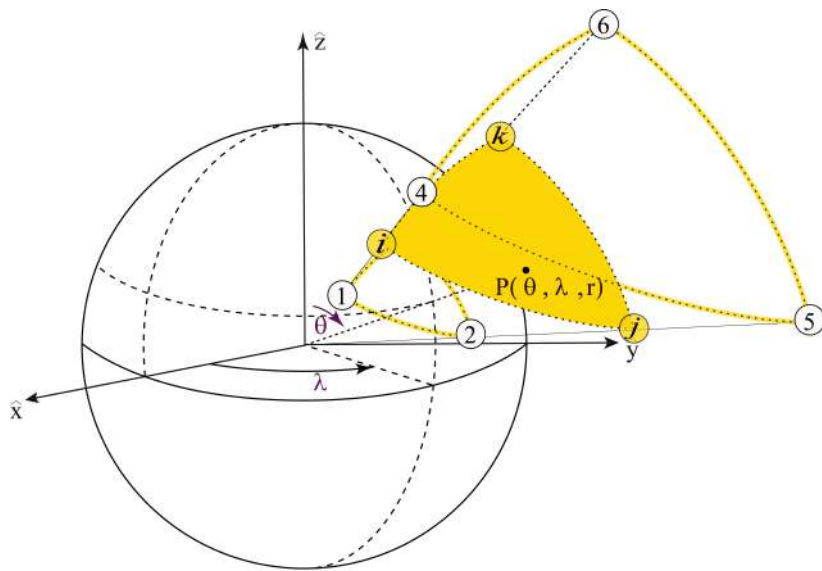


**Spherical Eight-Node
Quadrangular Right Prism**



$A(\theta, \lambda)$

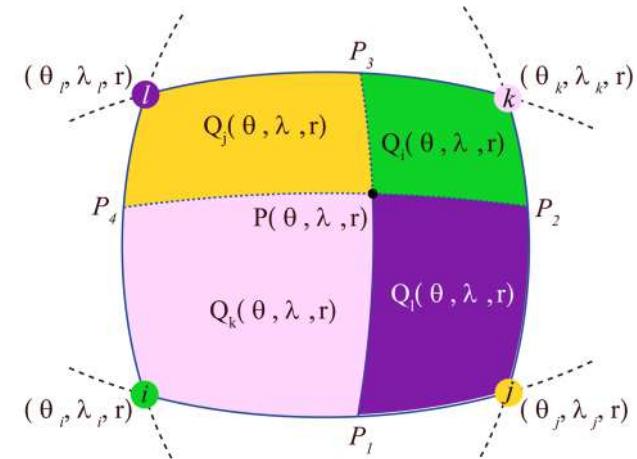
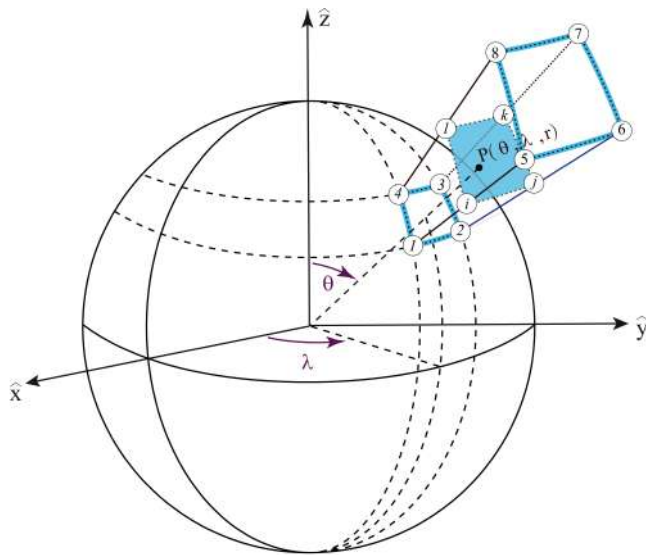
Spherical Six-Node Triangular Prism



$$A(\theta, \lambda) = \begin{cases} A_i(\theta, \lambda) = \frac{\text{area of spherical triangle } (Pjk)}{\text{area of spherical triangle } (ijk)} = \frac{T_i(\theta, \lambda, r)}{T_{ijk}(r)} \\ A_j(\theta, \lambda) = \frac{\text{area of spherical triangle } (Pki)}{\text{area of spherical triangle } (ijk)} = \frac{T_j(\theta, \lambda, r)}{T_{ijk}(r)} \\ A_k(\theta, \lambda) = \frac{\text{area of spherical triangle } (Pij)}{\text{area of spherical triangle } (ijk)} = \frac{T_k(\theta, \lambda, r)}{T_{ijk}(r)} \end{cases}$$

$A(\theta, \lambda)$

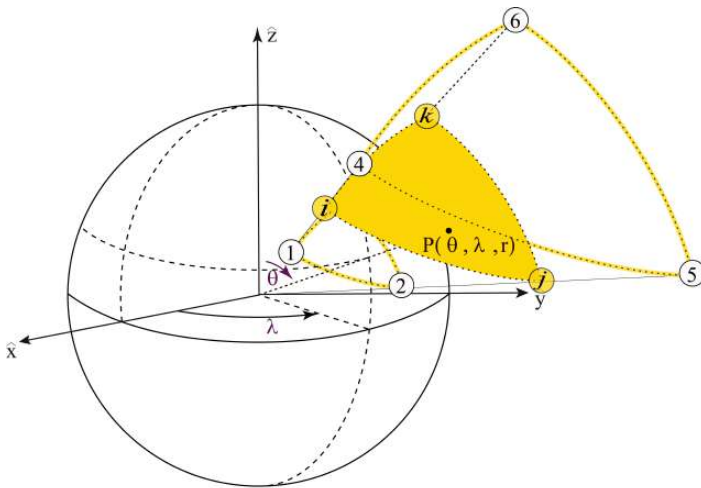
Spherical Eight-Node Quadrangular Right Prism



$$A(\theta, \lambda) = \begin{cases} A_i(\theta, \lambda) = \frac{\text{area of spherical right rectangle } (PP_2kP_3)}{\text{area of spherical right rectangle } (ijkl)} = \frac{Q_i(\theta, \lambda, r)}{Q_{ijkl}(r)} \\ A_j(\theta, \lambda) = \frac{\text{area of spherical right rectangle } (PP_3lP_4)}{\text{area of spherical right rectangle } (ijkl)} = \frac{Q_j(\theta, \lambda, r)}{Q_{ijkl}(r)} \\ A_k(\theta, \lambda) = \frac{\text{area of spherical right rectangle } (PP_4iP_1)}{\text{area of spherical right rectangle } (ijkl)} = \frac{Q_k(\theta, \lambda, r)}{Q_{ijkl}(r)} \\ A_l(\theta, \lambda) = \frac{\text{area of spherical right rectangle } (PP_1jP_2)}{\text{area of spherical right rectangle } (ijkl)} = \frac{Q_l(\theta, \lambda, r)}{Q_{ijkl}(r)} \end{cases}$$

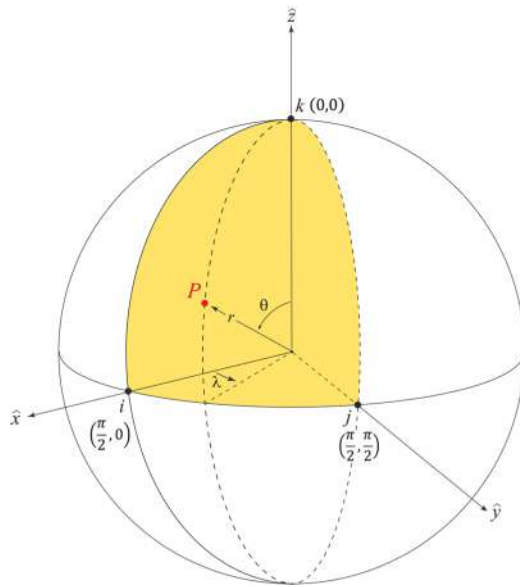
SPHERICAL SHAPE FUNCTIONS - Spherical 6-Node Triangular Prism

$$N(\theta, \lambda, r) = R(r) \cdot A(\theta, \lambda) \cdot F(\theta, \lambda)$$



$$\begin{cases} N_1(\theta, \lambda, r) = R_1(r) \cdot A_1(\theta, \lambda) \cdot F_1(\theta, \lambda) \\ N_2(\theta, \lambda, r) = R_1(r) \cdot A_2(\theta, \lambda) \cdot F_2(\theta, \lambda) \\ N_3(\theta, \lambda, r) = R_1(r) \cdot A_3(\theta, \lambda) \cdot F_3(\theta, \lambda) \\ N_4(\theta, \lambda, r) = R_2(r) \cdot A_1(\theta, \lambda) \cdot F_1(\theta, \lambda) \\ N_5(\theta, \lambda, r) = R_2(r) \cdot A_2(\theta, \lambda) \cdot F_2(\theta, \lambda) \\ N_6(\theta, \lambda, r) = R_2(r) \cdot A_3(\theta, \lambda) \cdot F_3(\theta, \lambda) \end{cases}$$

SPHERICAL SHAPE FUNCTIONS - Spherical 6-Node Tri-Rectangular Prism



$$(\theta_i, \lambda_i, r) = \left(\frac{\pi}{2}, 0, r\right), \text{ nodes 1 and 4}$$

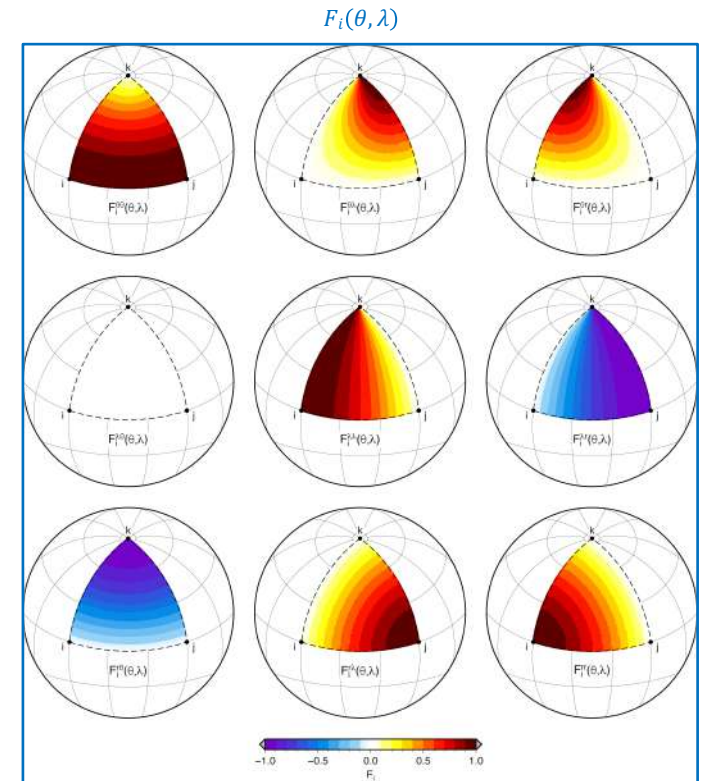
$$(\theta_j, \lambda_j, r) = \left(\frac{\pi}{2}, \frac{\pi}{2}, r\right), \text{ nodes 2 and 5}$$

$$(\theta_k, \lambda_k, r) = (0, 0, r), \text{ nodes 3 and 6}$$

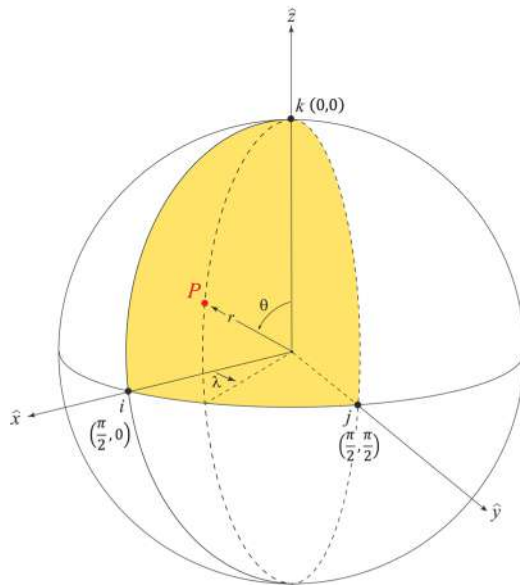
$$\begin{cases} F_i(\theta, \lambda) \\ \downarrow \\ F_1(\theta, \lambda) = F_4(\theta, \lambda) \end{cases} = \begin{bmatrix} \sin \theta & \cos \theta \sin \lambda & \cos \theta \cos \lambda \\ 0 & \cos \lambda & -\sin \lambda \\ -\cos \theta & \sin \theta \sin \lambda & \sin \theta \cos \lambda \end{bmatrix}$$

$$\begin{cases} F_j(\theta, \lambda) \\ \downarrow \\ F_2(\theta, \lambda) = F_5(\theta, \lambda) \end{cases} = \begin{bmatrix} \sin \theta & -\cos \theta \cos \lambda & \cos \theta \sin \lambda \\ 0 & \sin \lambda & \cos \lambda \\ -\cos \theta & -\sin \theta \cos \lambda & \sin \theta \sin \lambda \end{bmatrix}$$

$$\begin{cases} F_k(\theta, \lambda) \\ \downarrow \\ F_3(\theta, \lambda) = F_6(\theta, \lambda) \end{cases} = \begin{bmatrix} \cos \theta \cos \lambda & \cos \theta \sin \lambda & -\sin \theta \\ -\sin \lambda & \cos \lambda & 0 \\ \sin \theta \cos \lambda & \sin \theta \sin \lambda & \cos \theta \end{bmatrix}$$



SPHERICAL SHAPE FUNCTIONS - Spherical 6-Node Tri-Rectangular Prism

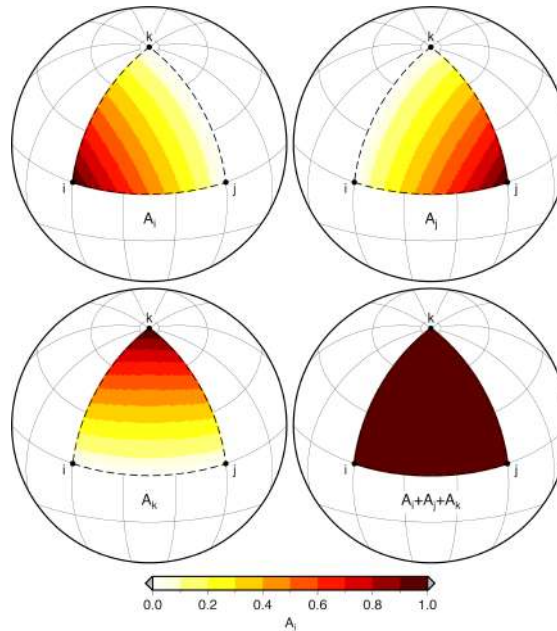


$$(\theta_i, \lambda_i, r) = \left(\frac{\pi}{2}, 0, r\right), \text{ nodes 1 and 4}$$

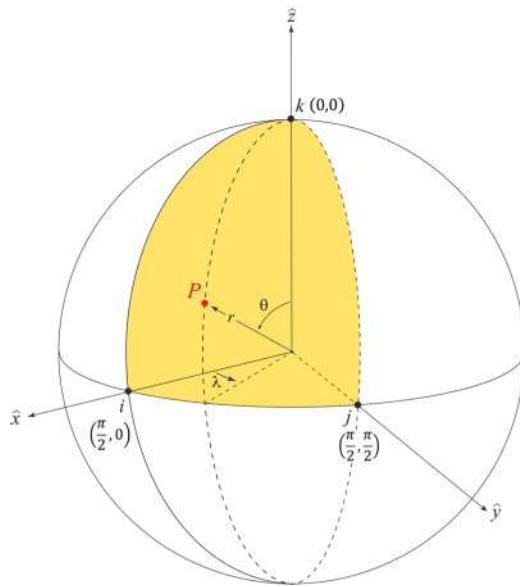
$$(\theta_j, \lambda_j, r) = \left(\frac{\pi}{2}, \frac{\pi}{2}, r\right), \text{ nodes 2 and 5}$$

$$(\theta_k, \lambda_k, r) = (0, 0, r), \text{ nodes 3 and 6}$$

$$\begin{cases} A_i(\theta, \lambda) = \frac{2}{\pi} \left\{ \arccos \left[\frac{-\cos(\theta) \sin(\lambda)}{\sqrt{1 - \sin^2(\theta) \sin^2(\lambda)}} \right] + \arccos \left[\frac{\cos(\theta)}{\sqrt{1 - \sin^2(\theta) \sin^2(\lambda)}} \right] + \arccos[\sin(\lambda)] - \pi \right\} \\ A_j(\theta, \lambda) = \frac{2}{\pi} \left\{ \arccos \left[\frac{\cos(\theta)}{\sqrt{1 - \sin^2(\theta) \cos^2(\lambda)}} \right] + \arccos \left[-\frac{\cos(\theta) \cos(\lambda)}{\sqrt{1 - \sin^2(\theta) \cos^2(\lambda)}} \right] + \arccos[\cos(\lambda)] - \pi \right\} \\ A_k(\theta, \lambda) = 1 - A_i(\theta, \lambda) - A_j(\theta, \lambda) \end{cases}$$



SPHERICAL SHAPE FUNCTIONS - Spherical 6-Node Tri-Rectangular Prism



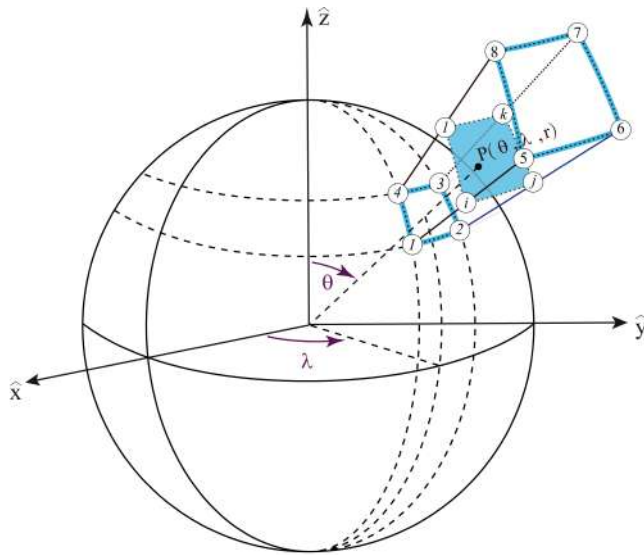
$$N(\theta, \lambda, r) = R(r) \cdot A(\theta, \lambda) \cdot F(\theta, \lambda)$$

$$\begin{cases} N_1(\theta, \lambda, r) = R_1(r) \cdot A_1(\theta, \lambda) \cdot F_1(\theta, \lambda) \\ N_2(\theta, \lambda, r) = R_1(r) \cdot A_2(\theta, \lambda) \cdot F_2(\theta, \lambda) \\ N_3(\theta, \lambda, r) = R_1(r) \cdot A_3(\theta, \lambda) \cdot F_3(\theta, \lambda) \\ N_4(\theta, \lambda, r) = R_2(r) \cdot A_1(\theta, \lambda) \cdot F_1(\theta, \lambda) \\ N_5(\theta, \lambda, r) = R_2(r) \cdot A_2(\theta, \lambda) \cdot F_2(\theta, \lambda) \\ N_6(\theta, \lambda, r) = R_2(r) \cdot A_3(\theta, \lambda) \cdot F_3(\theta, \lambda) \end{cases}$$

$$\begin{cases} N_k(\theta_k, \lambda_k, r_k) = I \\ N_k(\theta_i, \lambda_i, r_i) = 0, \text{ for } k \neq i \text{ and } k, i = 1, 6 \end{cases}$$

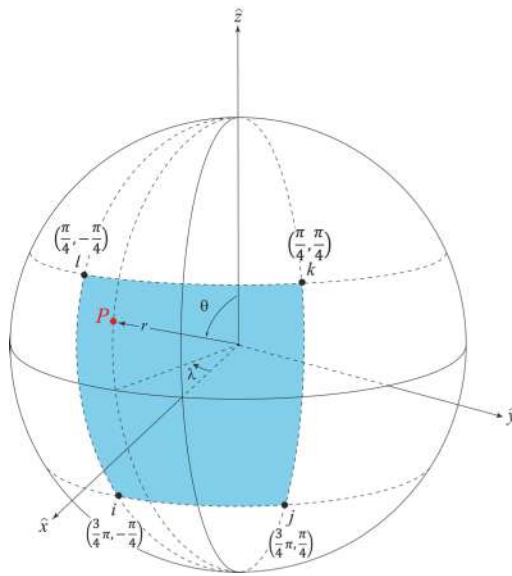
SPHERICAL SHAPE FUNCTIONS - Spherical 8-Node Quadrangular Right Prism

$$N(\theta, \lambda, r) = R(r) \cdot A(\theta, \lambda) \cdot F(\theta, \lambda)$$



$$\begin{cases} N_1(\theta, \lambda, r) = R_1(r) \cdot A_1(\theta, \lambda) \cdot F_1(\theta, \lambda) \\ N_2(\theta, \lambda, r) = R_1(r) \cdot A_2(\theta, \lambda) \cdot F_2(\theta, \lambda) \\ N_3(\theta, \lambda, r) = R_1(r) \cdot A_3(\theta, \lambda) \cdot F_3(\theta, \lambda) \\ N_4(\theta, \lambda, r) = R_1(r) \cdot A_4(\theta, \lambda) \cdot F_4(\theta, \lambda) \\ N_5(\theta, \lambda, r) = R_2(r) \cdot A_1(\theta, \lambda) \cdot F_1(\theta, \lambda) \\ N_6(\theta, \lambda, r) = R_2(r) \cdot A_2(\theta, \lambda) \cdot F_2(\theta, \lambda) \\ N_7(\theta, \lambda, r) = R_2(r) \cdot A_3(\theta, \lambda) \cdot F_3(\theta, \lambda) \\ N_8(\theta, \lambda, r) = R_2(r) \cdot A_4(\theta, \lambda) \cdot F_4(\theta, \lambda) \end{cases}$$

SPHERICAL SHAPE FUNCTIONS - Spherical 8-Node Quadrangular Right Prism



$$(\theta_i, \lambda_i, r) = \left(\frac{3}{4}\pi, -\frac{\pi}{4}, r\right); \text{ nodes 1 and 5}$$

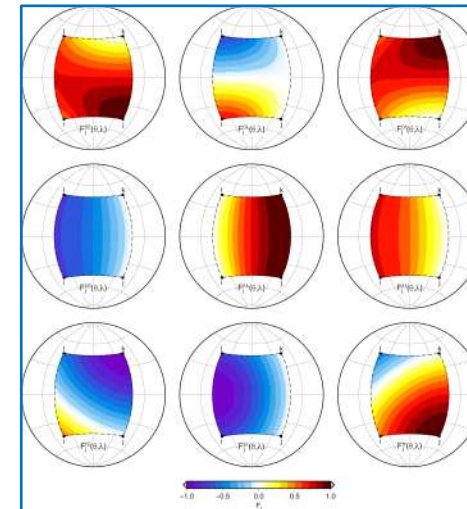
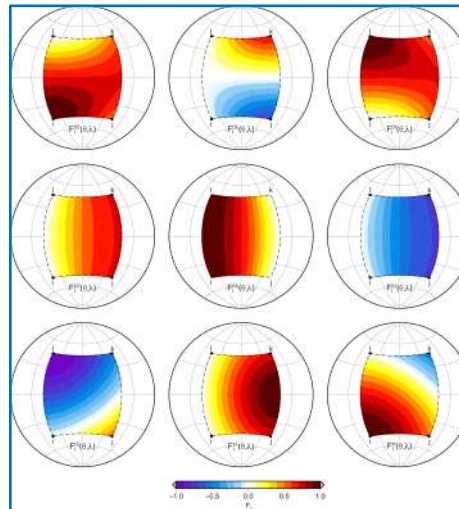
$$(\theta_j, \lambda_j, r) = \left(\frac{3}{4}\pi, \frac{\pi}{4}, r\right); \text{ nodes 2 and 6}$$

$$(\theta_k, \lambda_k, r) = \left(\frac{\pi}{4}, \frac{\pi}{4}, r\right); \text{ nodes 3 and 7}$$

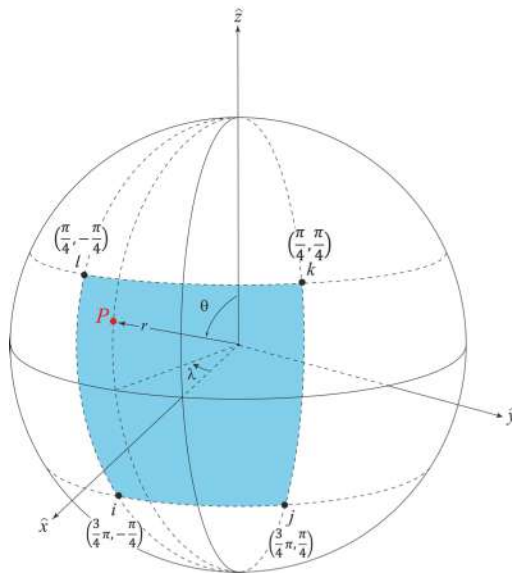
$$(\theta_l, \lambda_l, r) = \left(\frac{\pi}{4}, -\frac{\pi}{4}, r\right); \text{ nodes 4 and 8}$$

$$N(\theta, \lambda, r) = R(r) \cdot A(\theta, \lambda) \cdot F(\theta, \lambda)$$

$$\begin{cases} F_i(\theta, \lambda) \\ \downarrow \uparrow \\ F_1(\theta, \lambda) = F_5(\theta, \lambda) \end{cases} = \begin{bmatrix} \frac{1}{\sqrt{2}} \left[\sin \theta - \cos \theta \cos \left(\frac{\pi}{4} + \lambda \right) \right] & \cos \theta \sin \left(\frac{\pi}{4} + \lambda \right) & \frac{1}{\sqrt{2}} \left[\sin \theta + \cos \theta \cos \left(\frac{\pi}{4} + \lambda \right) \right] \\ \frac{1}{\sqrt{2}} \sin \left(\frac{\pi}{4} + \lambda \right) & \cos \left(\frac{\pi}{4} + \lambda \right) & -\frac{1}{\sqrt{2}} \sin \left(\frac{\pi}{4} + \lambda \right) \\ -\frac{1}{\sqrt{2}} \left[\cos \theta + \sin \theta \cos \left(\frac{\pi}{4} + \lambda \right) \right] & \sin \theta \sin \left(\frac{\pi}{4} + \lambda \right) & \frac{1}{\sqrt{2}} \left[-\cos \theta + \sin \theta \cos \left(\frac{\pi}{4} + \lambda \right) \right] \end{bmatrix}$$



SPHERICAL SHAPE FUNCTIONS - Spherical 8-Node Quadrangular Right Prism



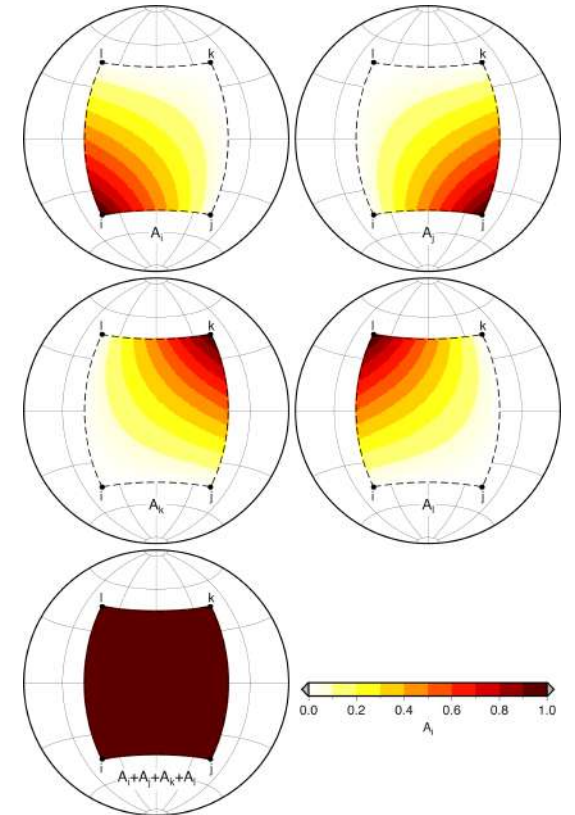
$$(\theta_i, \lambda_i, r) = \left(\frac{3}{4}\pi, -\frac{\pi}{4}, r\right); \text{ nodes 1 and 5}$$

$$(\theta_j, \lambda_j, r) = \left(\frac{3}{4}\pi, \frac{\pi}{4}, r\right); \text{ nodes 2 and 6}$$

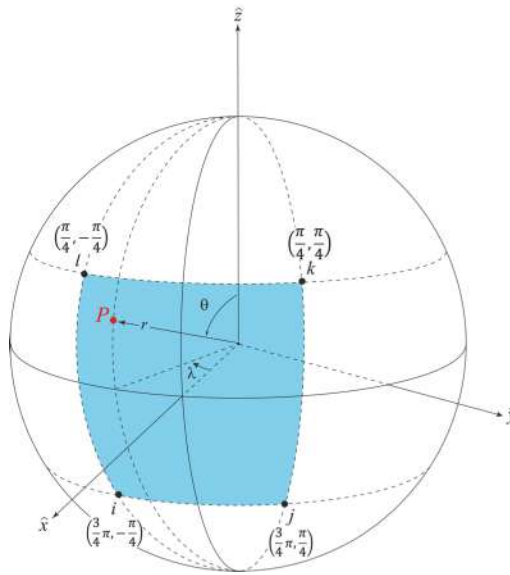
$$(\theta_k, \lambda_k, r) = \left(\frac{\pi}{4}, \frac{\pi}{4}, r\right); \text{ nodes 3 and 7}$$

$$(\theta_l, \lambda_l, r) = \left(\frac{\pi}{4}, -\frac{\pi}{4}, r\right); \text{ nodes 4 and 8}$$

$$\begin{cases} A_i(\theta, \lambda) = A_1(\theta, \lambda) = A_5(\theta, \lambda) = -\frac{1}{\pi} \left[\frac{\pi}{4} - \lambda \right] [\sqrt{2} \cos \theta - 1] \\ A_j(\theta, \lambda) = A_2(\theta, \lambda) = A_6(\theta, \lambda) = -\frac{1}{\pi} \left[\frac{\pi}{4} + \lambda \right] [\sqrt{2} \cos \theta - 1] \\ A_k(\theta, \lambda) = A_3(\theta, \lambda) = A_7(\theta, \lambda) = +\frac{1}{\pi} \left[\frac{\pi}{4} + \lambda \right] [\sqrt{2} \cos \theta + 1] \\ A_l(\theta, \lambda) = A_4(\theta, \lambda) = A_8(\theta, \lambda) = +\frac{1}{\pi} \left[\frac{\pi}{4} - \lambda \right] [\sqrt{2} \cos \theta + 1] \end{cases}$$



SPHERICAL SHAPE FUNCTIONS - Spherical 8-Node Quadrangular Right Prism



$$N(\theta, \lambda, r) = R(r) \cdot A(\theta, \lambda) \cdot F(\theta, \lambda)$$

$$\begin{cases} N_1(\theta, \lambda, r) = R_1(r) \cdot A_1(\theta, \lambda) \cdot F_1(\theta, \lambda) \\ N_2(\theta, \lambda, r) = R_1(r) \cdot A_2(\theta, \lambda) \cdot F_2(\theta, \lambda) \\ N_3(\theta, \lambda, r) = R_1(r) \cdot A_3(\theta, \lambda) \cdot F_3(\theta, \lambda) \\ N_4(\theta, \lambda, r) = R_1(r) \cdot A_4(\theta, \lambda) \cdot F_4(\theta, \lambda) \\ N_5(\theta, \lambda, r) = R_2(r) \cdot A_1(\theta, \lambda) \cdot F_1(\theta, \lambda) \\ N_6(\theta, \lambda, r) = R_2(r) \cdot A_2(\theta, \lambda) \cdot F_2(\theta, \lambda) \\ N_7(\theta, \lambda, r) = R_2(r) \cdot A_3(\theta, \lambda) \cdot F_3(\theta, \lambda) \\ N_8(\theta, \lambda, r) = R_2(r) \cdot A_4(\theta, \lambda) \cdot F_4(\theta, \lambda) \end{cases}$$

$$(\theta_i, \lambda_i, r) = \left(\frac{3}{4}\pi, -\frac{\pi}{4}, r\right); \text{ nodes 1 and 5}$$

$$(\theta_j, \lambda_j, r) = \left(\frac{3}{4}\pi, \frac{\pi}{4}, r\right); \text{ nodes 2 and 6}$$

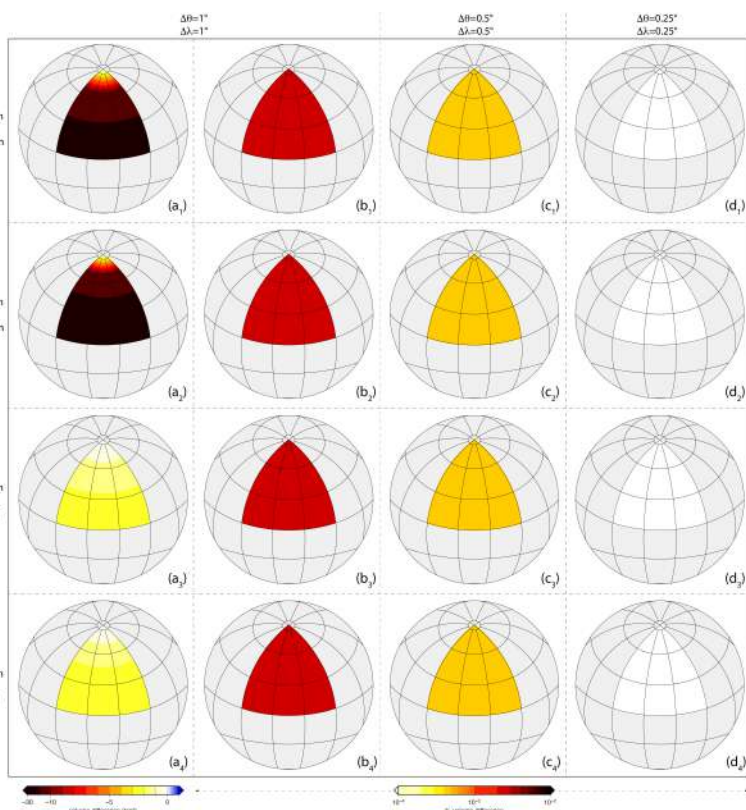
$$(\theta_k, \lambda_k, r) = \left(\frac{\pi}{4}, \frac{\pi}{4}, r\right); \text{ nodes 3 and 7}$$

$$(\theta_l, \lambda_l, r) = \left(\frac{\pi}{4}, -\frac{\pi}{4}, r\right); \text{ nodes 4 and 8}$$

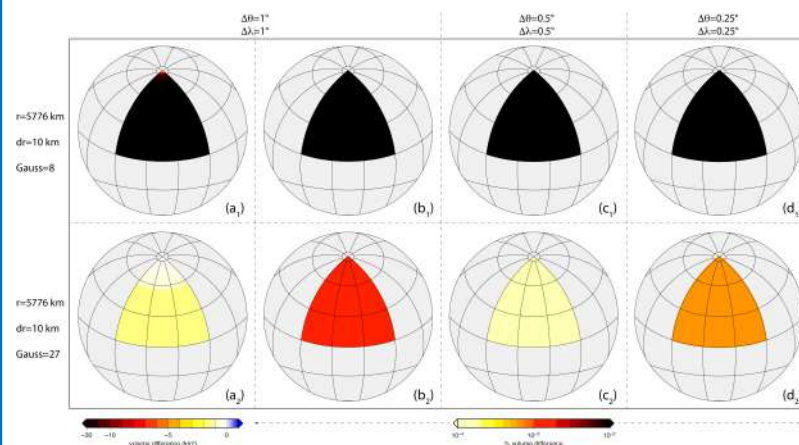
$$\begin{cases} N_k(\theta_k, \lambda_k, r_k) = I \\ N_k(\theta_i, \lambda_i, r_i) = 0, \text{ for } k \neq i \text{ and } k, i = 1, 8 \end{cases}$$

VALIDATION TEST 1 - FINITE VOLUMES OF SPHERICAL 8-NODE QUADRANGULAR RIGHT PRISM

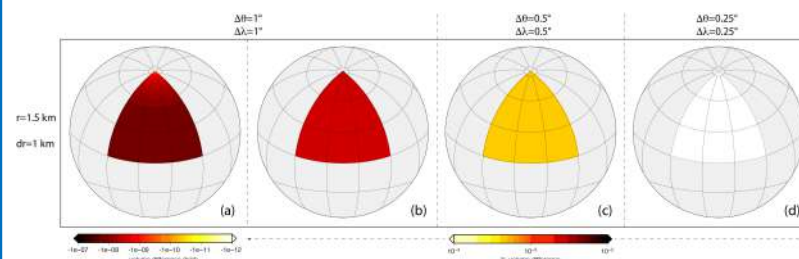
125 Gauss–Legendre quadrature points; $R=6371$ km



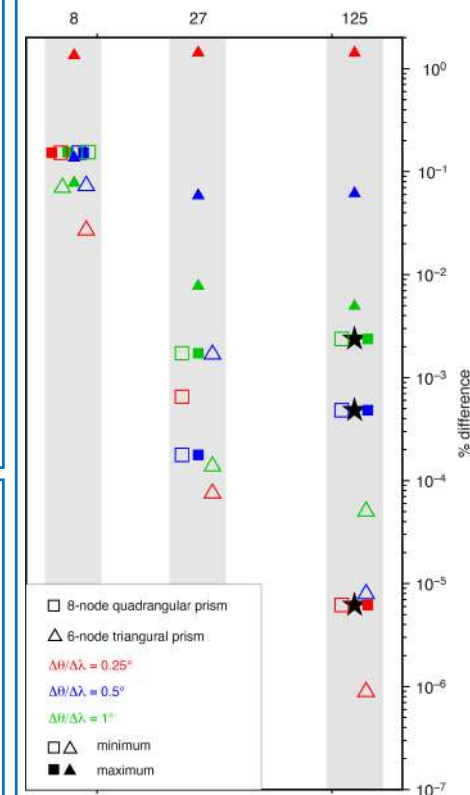
8 and 27 Gauss–Legendre quadrature points; $R=6371$ km



125 Gauss–Legendre quadrature points; $R=2$ km, $dr=1$ km



number of Gauss–Legendre quadrature points



VALIDATION TEST 2 – INTEGRATION OF POISSON EQUATION

STEADY STATE HEAT CONDUCTION IN A SPHERE WITH INTERNAL HEAT PRODUCTION

$$\nabla \cdot (K \nabla T) + \rho H = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 K \frac{\partial T}{\partial r} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta K \frac{1}{r} \frac{\partial T}{\partial \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \lambda} \left(K \frac{1}{r \sin \theta} \frac{\partial T}{\partial \lambda} \right) + Q = 0$$

Assuming that:

- 1) the temperature varies only along the radial distance, $T(r)$
- 2) the internal energy source is uniform

$$3) \begin{cases} T(r = R_s) = T_s \\ T(r = R_b) = T_b \end{cases}$$

$$T(r) = T_s + \left(\frac{1}{r} - \frac{1}{R_s} \right) \left[R_s R_b \left(\frac{T_b - T_s}{R_s - R_b} \right) - \frac{\rho H}{6K} R_s R_b (R_s + R_b) \right] + \frac{\rho H}{6K} (R_s^2 - r^2)$$

VALIDATION TEST 2 **INTEGRATION OF POISSON EQUATION**

$$R_s = 6371 \text{ km}$$

$$R_b = 6341 \text{ km}$$

$$0 \leq \lambda \leq 360$$

$$0 \leq \theta \leq 180$$

$$1^\circ \leq \Delta\lambda \leq 5^\circ$$

$$1^\circ \leq \Delta\theta \leq 5^\circ$$

$$5 \text{ km} \leq \Delta r \leq 10$$

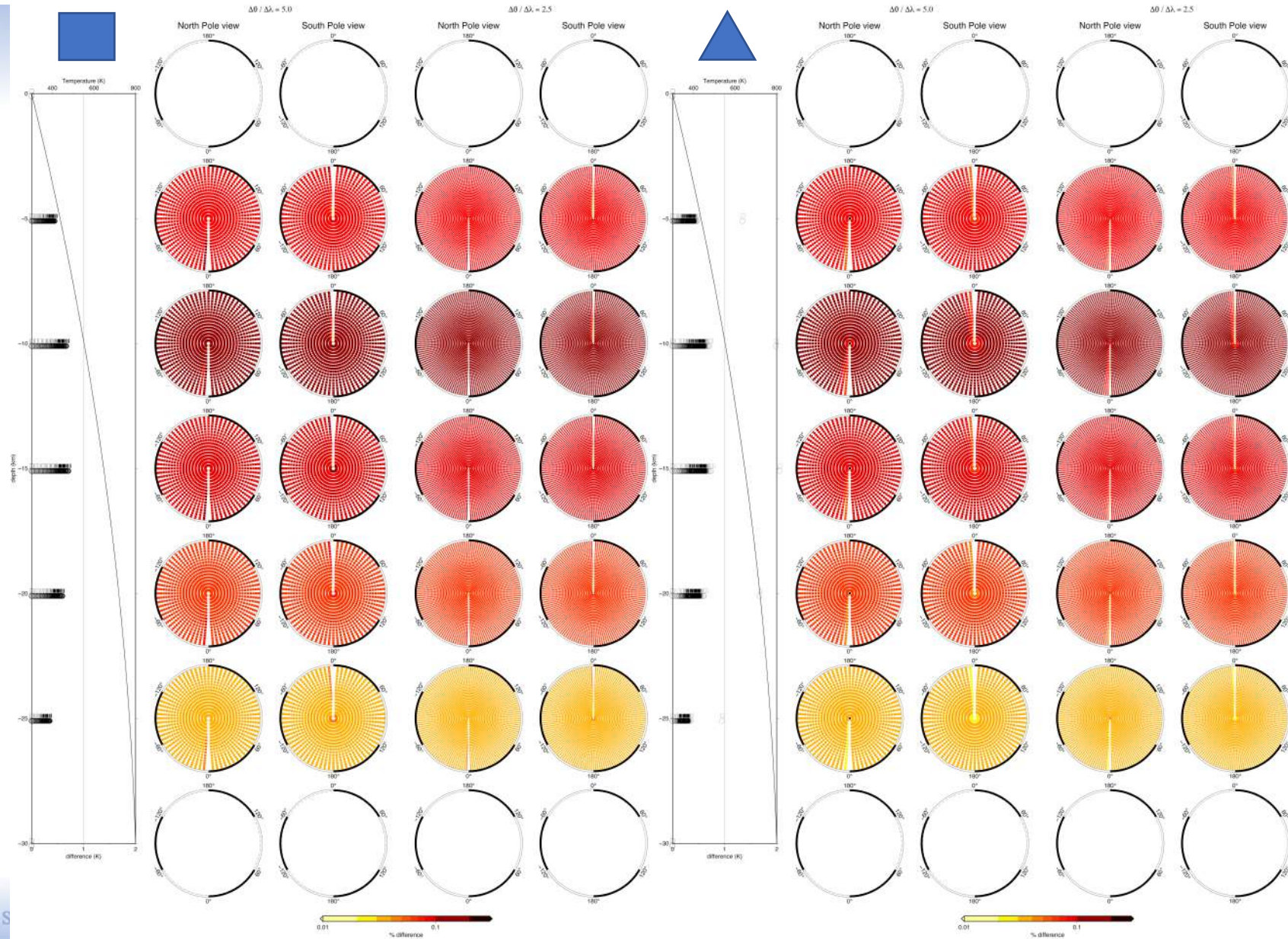
8 Gauss-Legendre quadrature points.

$$Q = 2.59 \mu\text{W}/\text{m}^3$$

$$T_s = 300 \text{ K}$$

$$T_b = 800 \text{ K}$$

$$K = 3 \frac{\text{W}}{\text{Km}}$$





Synthesis

- We have determined rigorous Spherical shape functions for generic spherical prismatic elements and, in particular, for the case of spherical 6-node tri-rectangular and 8-node quadrangular spherical prisms.
- The results of numerical tests demonstrate their full capability and the high degree of accuracy that can be achieved.
- The proposed spherical shape functions can be used to rigorously interpolate scalar and vectoral fields within a spherical domain, which can be taken as reference elements within a finite element numerical code.
- The percentage volume difference between the exact and the numerically evaluated volumes is independent from either the geographical position and the depth and ranges from 10^{-3} to lower than 10^{-4} for angular dimensions ranging from $1^\circ \times 1^\circ$ to $0.25^\circ \times 0.25^\circ$.
- A satisfactory accuracy is attained for 8 Gauss–Legendre quadrature points.
- Numerical integration of Poisson equation shows that, even with a relatively coarse grid, our shape functions are capable to provide a satisfactory fit between numerical and analytical solutions, with a maximum difference of the order of 0.2% of the exact value.

Funding: Research reported in this publication was supported by the ASI (Italian Space Agency) funded project “NGGM/MAGIC - a breakthrough in the understanding of the dynamics of the Earth”, contr. n. CI-UOT-2023-057

Conclusions & perspectives

01

GEOMETRIC CONSISTENCY

Local FEM formulation
intrinsically adapted to
spherical geometry

02

NUMERICAL FRAMEWORK

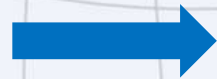
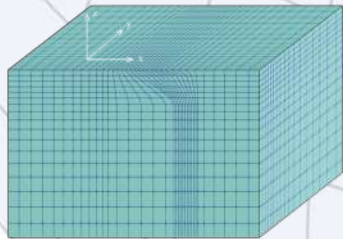
Analytical spherical shape functions
for accurate representation of
distances, directions and vector fields

03

GEODETTIC PERSPECTIVE

Towards FEM modelling
consistent with the
geometry of the Earth

PLANAR FEM



SPHERICAL FEM



GEODETTIC MODELLING



From planar approximations to finite elements that follow the geometry of the Earth.