

Extension of the Spectral Finite Element Method to Earth Deformation Caused by Tensile Cracks

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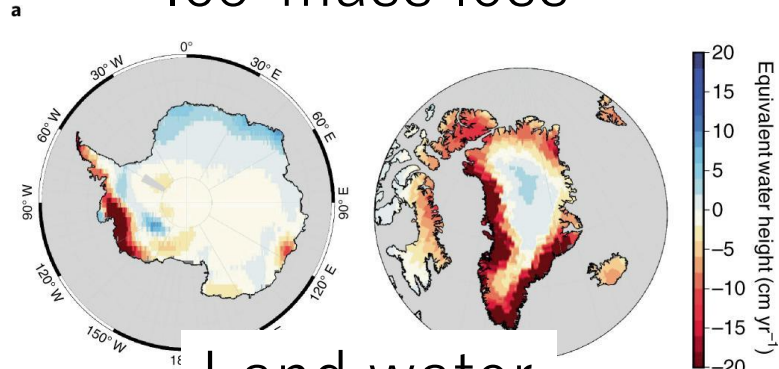
JSG T.46 Deformation, rotation and gravity field modeling for Earth and space

XI Hotine Marussi Symposium, 7-11 Sep, 2026, Genoa

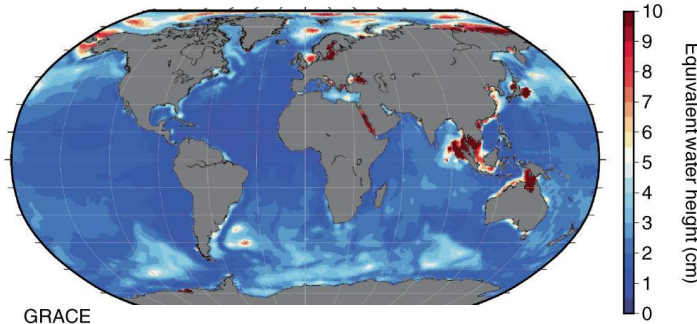
Temporal variations in gravity observed by satellites

Rate for gravity change from GRACE

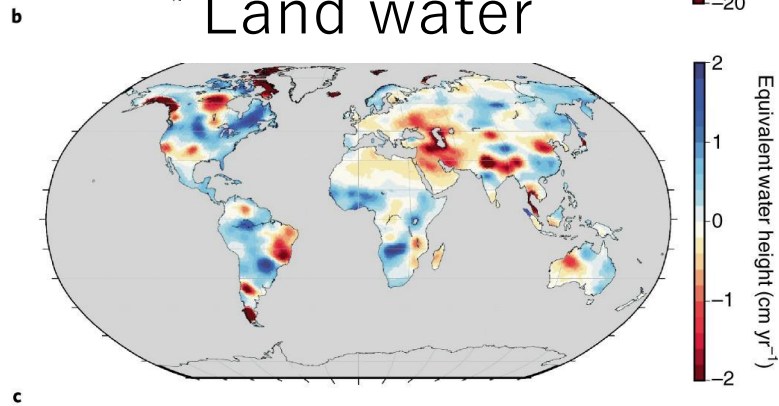
Ice-mass loss



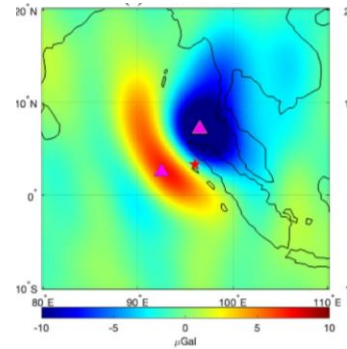
Ocean mass



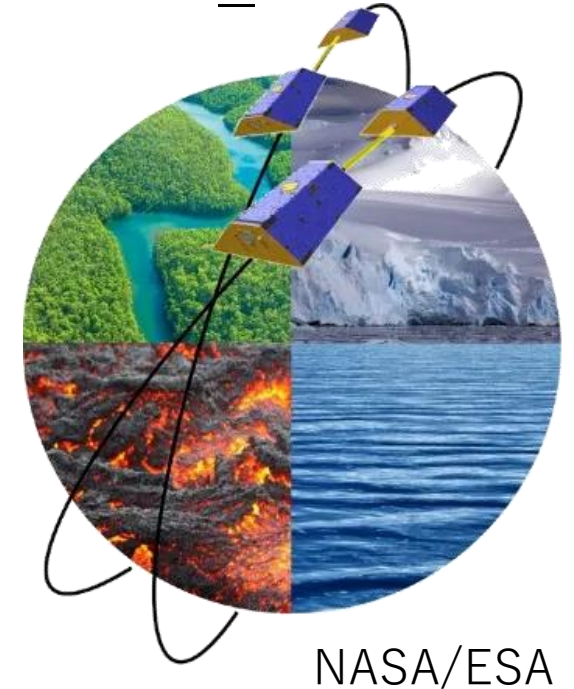
Land water



Earthquakes



Mass-Change and Geosciences
International Constellation

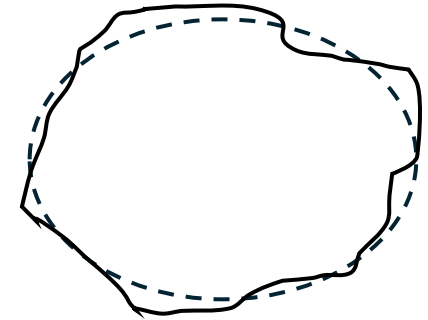


- Various factors cause gravity changes on the order of μGal .
- GRACE: 1 month, **MAGIC (around 2032-) 3-5 days**

Theoretical approach in this study

Geodetic approach

- Surface data $\rightarrow$ shape (GBVP)

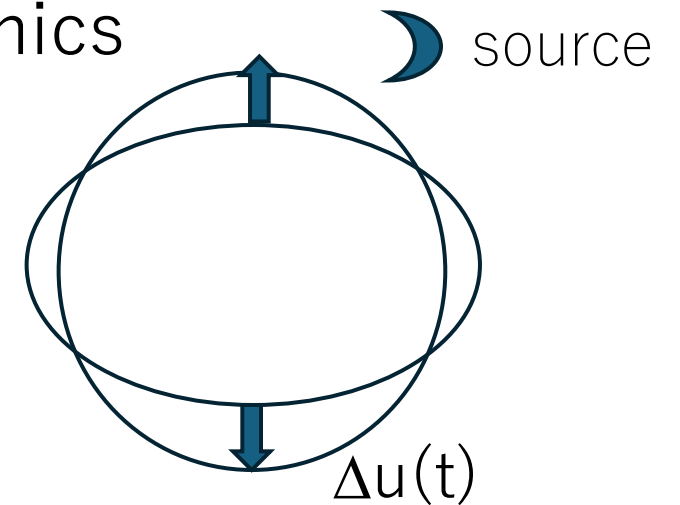


Geophysical approach

- Differential equation from continuum mechanics
e.g., Navier–Stokes equations for fluids

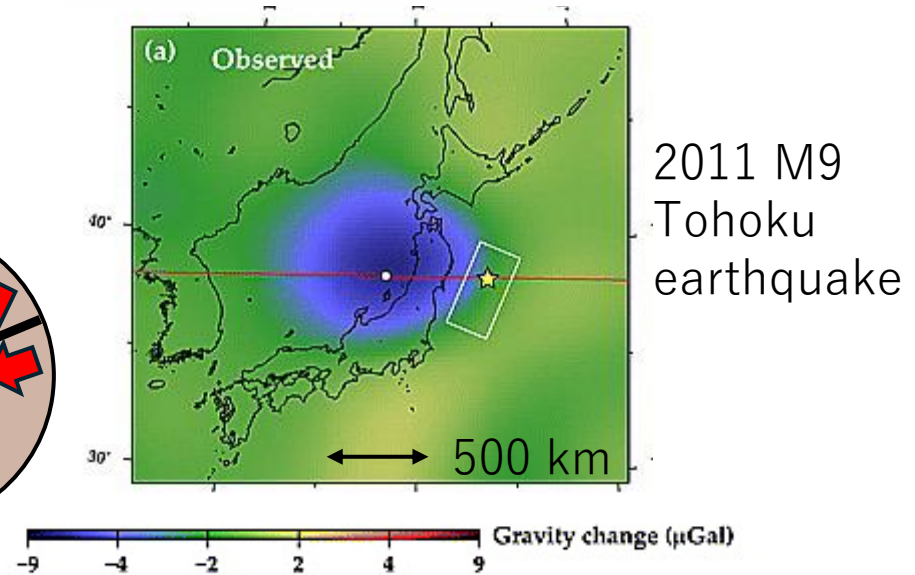
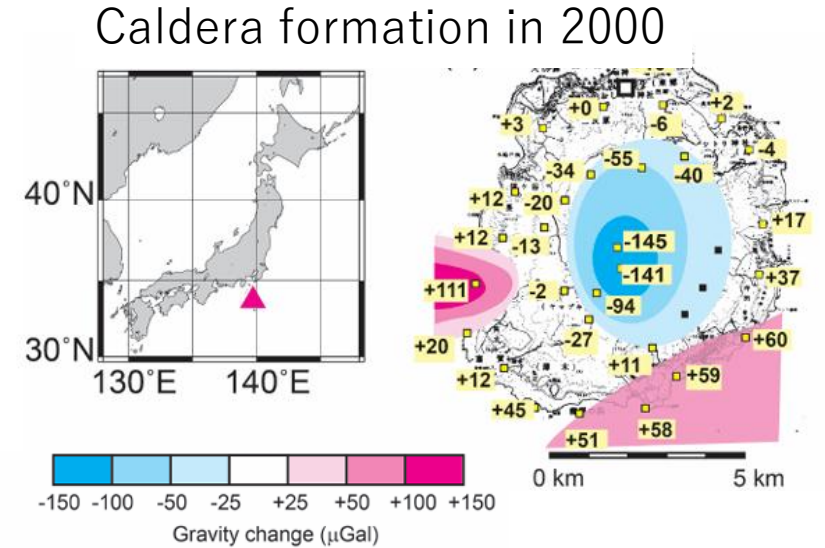
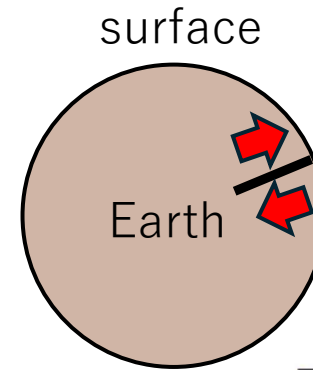
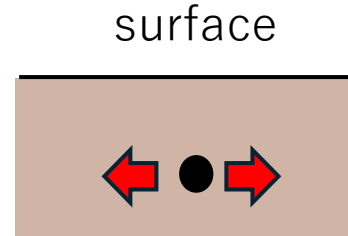
- Source $\rightarrow$ deformation (Initial-BVP)

e.g., Tides, surface loads, internal forces



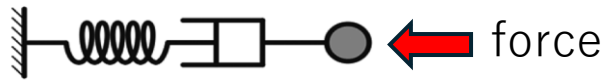
Gravity changes due to earthquakes & volcanoes

- Modeled as deformation caused by internal forces within a medium, often assumed to be elastic
- Half-space models for local deformation (Okada, 1985; Okubo, 1992)
- **Spherical models** for broad-scale deformation (e.g., Sun and Okubo, 1993)

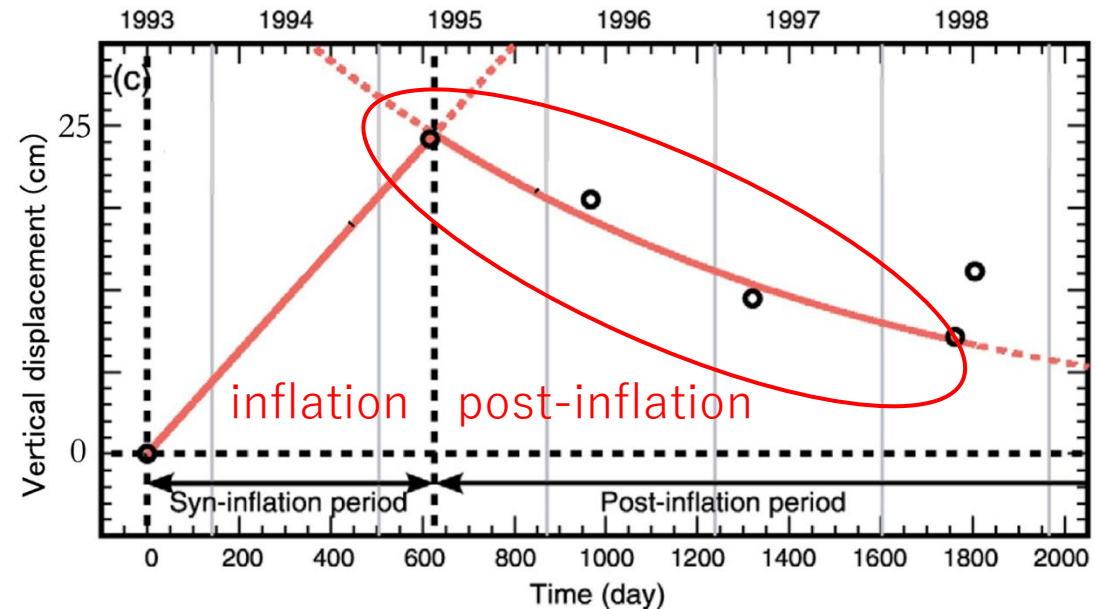
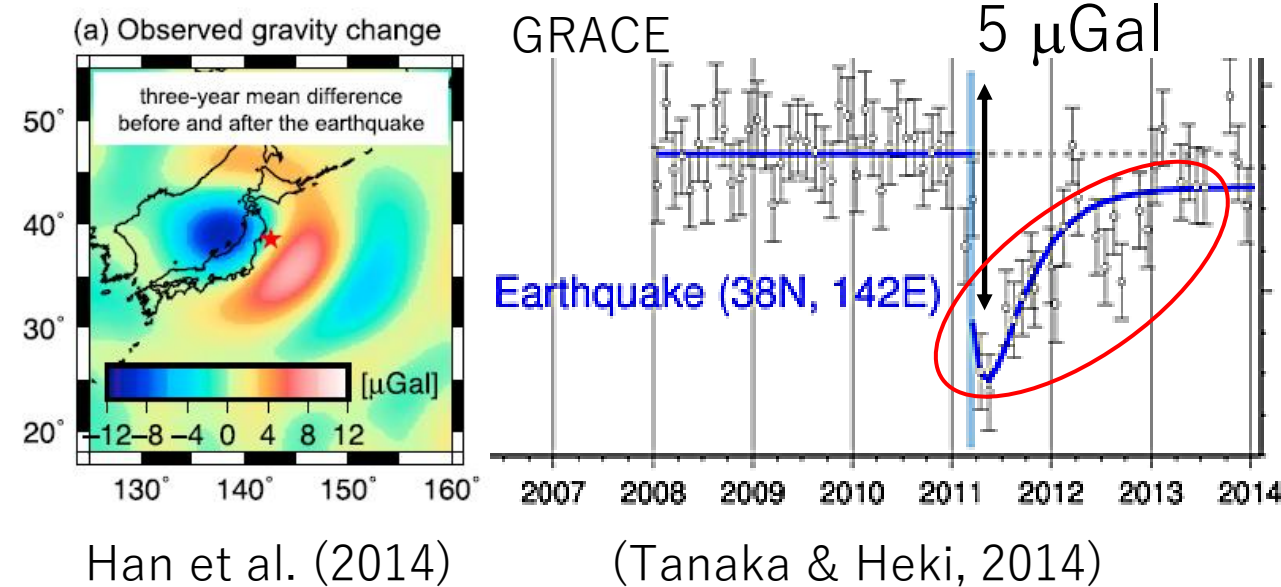


Okubo (2005), Heki & Matsuo (2011)

Deformation of viscoelastic medium

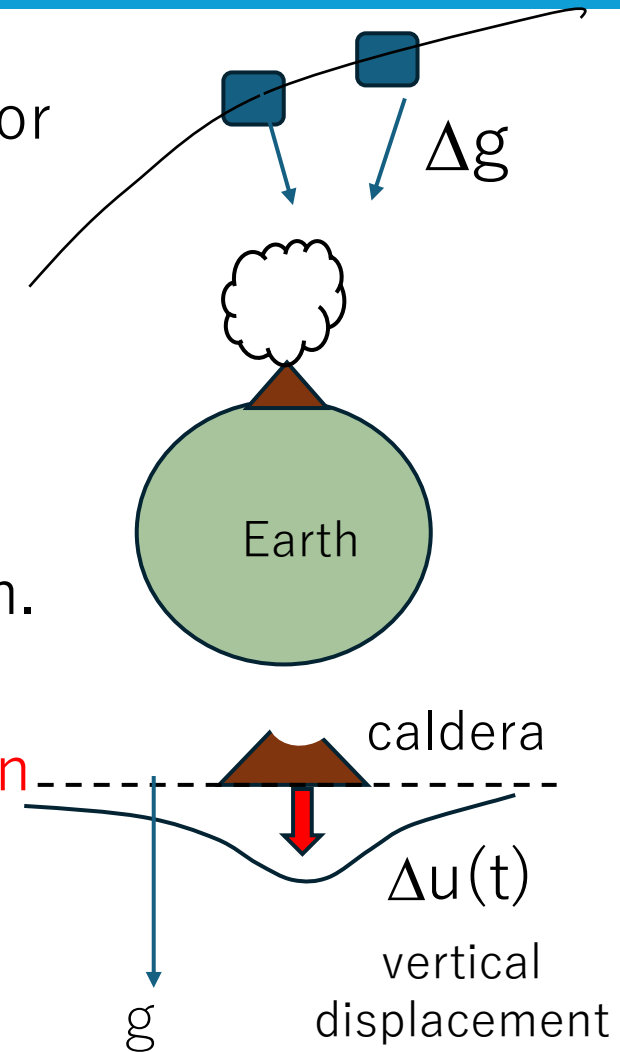
- Mantle **viscoelasticity** causes transient **post-seismic deformation**
 - Elastic spring + Newtonian fluid
- 
- **Spherical models** for large earthquakes (e.g., Han et al., 2014)

- **Post-inflation/deflation of a volcano**
- A viscoelastic lower crust (e.g., Yamazaki et al. 2019)
- **Few spherical models** because small spatial-scale deformation is dominant



Purpose of this study

- Develop a spherical model to calculate volcanic deformation for two reasons:
- Estimates **long-wavelength components precisely**, which may be detected by **future satellite gravimetry**
- Regional modeling by FEM often approximates self-gravitation.
- Spherical models can treat **self-gravitation naturally**
- Examine self-gravitational (or **isostatic**) **effects on deformation** caused by volcanic cycles **over a longer time scale**
- Applications associated with these two



Proposed method: Spectral FEM

Stress equilibrium equation

$$\nabla \cdot \tau - \rho_0 \nabla \phi_1 + \nabla \cdot (\rho_0 \vec{u}) \nabla \phi_0 - \nabla (\rho_0 \vec{u} \cdot \nabla \phi_0) = 0$$

$$\nabla^2 \phi_1 + 4\pi G \nabla \cdot (\rho_0 \vec{u}) = 0$$

Poisson's eq

$$\longrightarrow \nabla \cdot \tau^{E,i+1} - \rho_0 \nabla \phi_1^{i+1} + \nabla \cdot (\rho_0 \vec{u}^{i+1}) \nabla \phi_0 - \nabla (\rho_0 \vec{u}^{i+1} \cdot \nabla \phi_0) = -\nabla \cdot \tau^{V,i} + \rho_0 \vec{f}^{i+1}$$

Source term

$$\dot{\tau} = \dot{\tau}^E - \frac{\mu}{\eta} \left(\tau - \frac{1}{3} \text{tr}(\tau) I \right)$$

stress-strain for viscoelastic medium
Peltier (1974)

$$\dot{\tau}^E = \lambda (\nabla \vec{u}) I + 2\mu \varepsilon$$

$$\vec{u}(r, \theta, \varphi) = \sum_{jm} \left[\underline{U_{jm}}(r) S_{jm}^{(-1)}(\theta, \varphi) + \underline{V_{jm}}(r) S_{jm}^{(1)}(\theta, \varphi) + \underline{W_{jm}}(r) S_j^{(0)}(\theta, \varphi) \right]$$

$$\phi_1(r, \theta, \varphi) = \sum_{jm} \underline{F_{jm}}(r) Y_{jm}(\theta, \varphi)$$



Finite elements for the radial coordinate only.

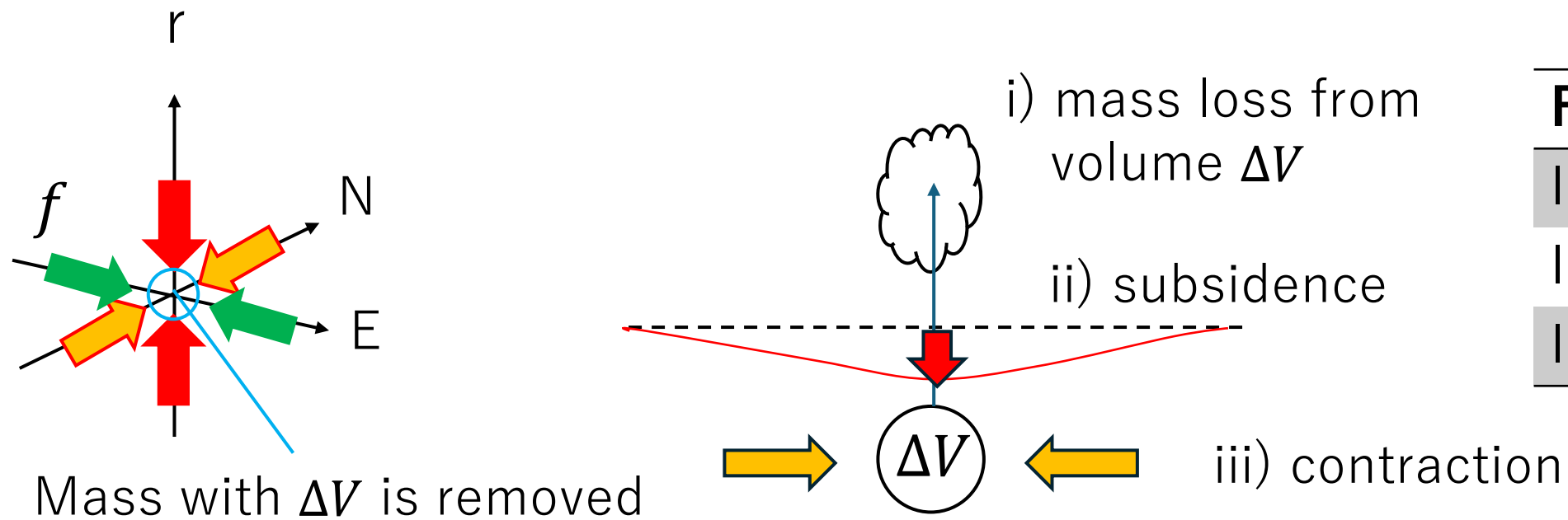
Horizontal resolutions change with j and m .

Martinec (2000)

- 1D FEM and analytical expressions based on spherical harmonics enables low-cost global modeling and natural treatment of self-gravitation.
- Initial & boundary conditions: free-surface and Heaviside-type forces at the volcanic source for $t \geq 0$

Source term and causes of gravity change

- Expression of the internal force f for earthquakes: Tanaka et al. (2014)
- Eruption causes mass loss and contraction of ΔV . The corresponding deformation is represented by the three coupled forces (Mogi source)

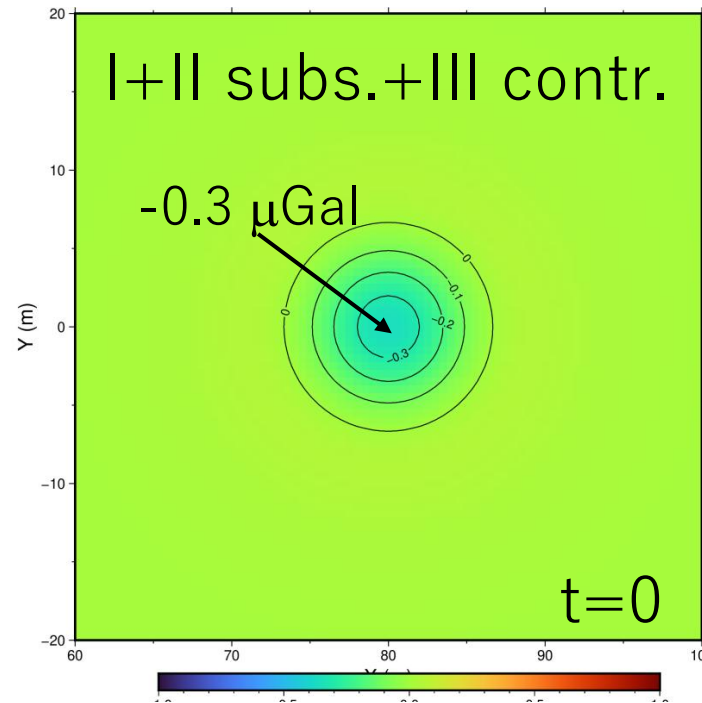
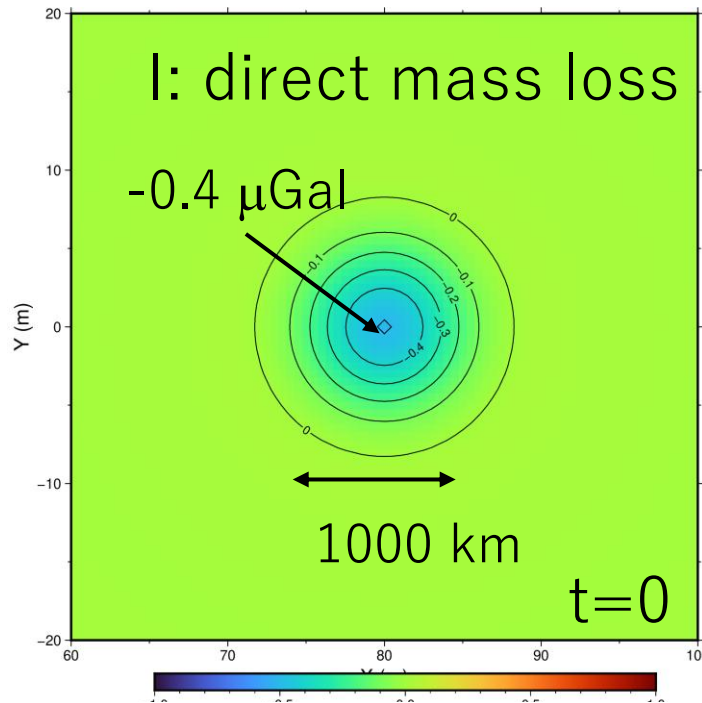


Factor	Δg
I	< 0
II	< 0
III	> 0

The details will be presented in a special issue dedicated to this symposium.

Result I: simulation assuming satellite gravimetry

- $\Delta V = 6 \times 10^9 \text{ m}^3$ (2022 Hunga Tonga-Hunga Ha'apai eruption)
- Lithospheric thickness = 10 km, source is just above the viscoelastic lower crust. Characteristic time $\tau_{1,2} = 0.7$ and 7 years (Burgers rheology has short and long time scales)
- GRACE resolution (cut-off SH degree = 40, Gaussian radius = 500 km)



co-eruptive step-like change

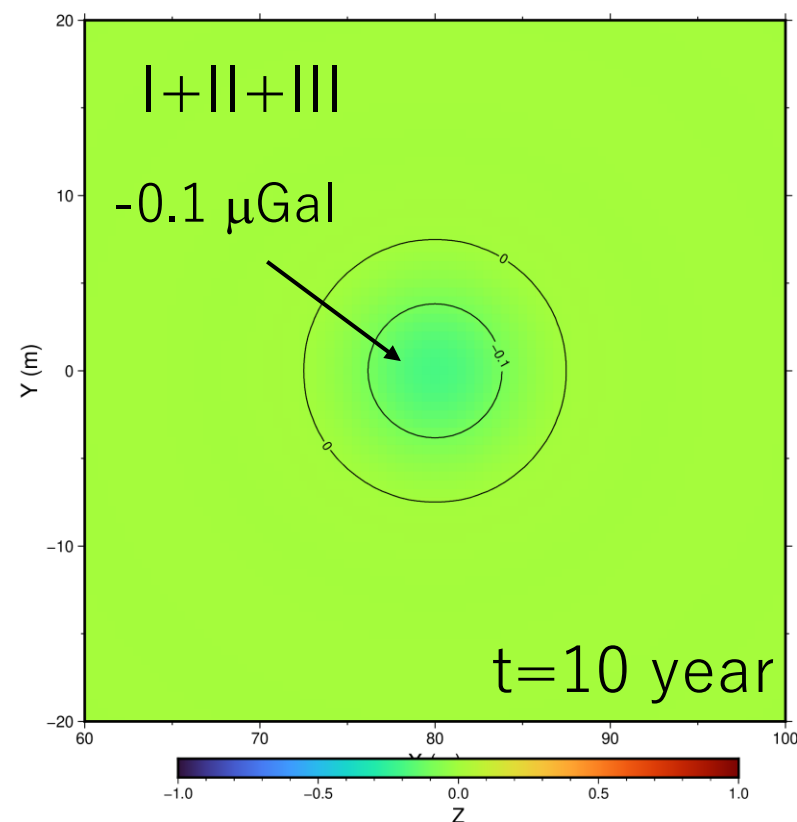
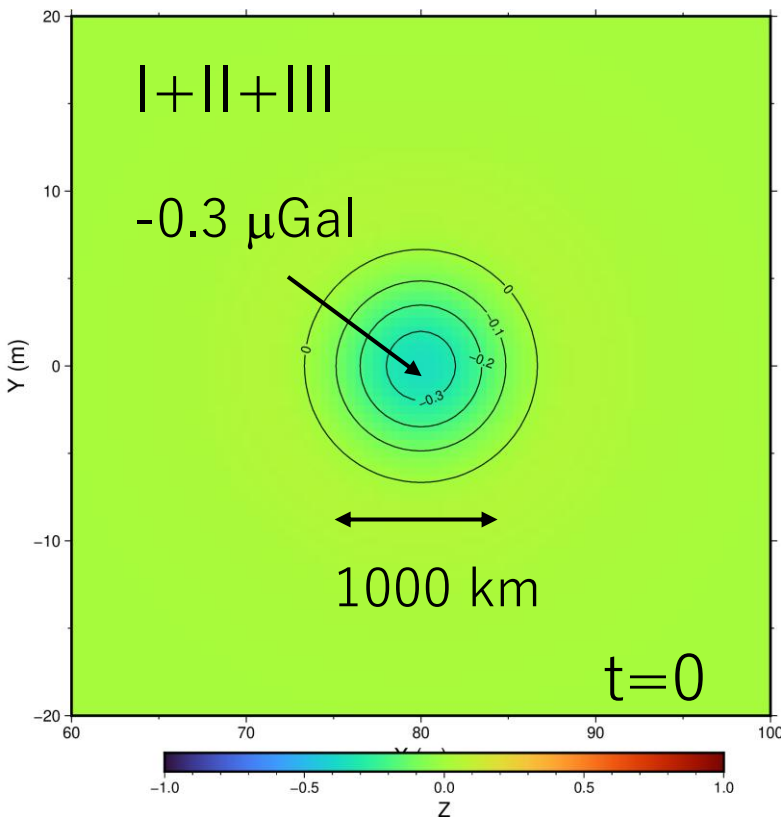
Factor	Δg
I	< 0
II	< 0
III	> 0

Deformation-induced gravity change (II+III) is $+0.1 \mu\text{Gal}$.

The effect of contraction is larger than of subsidence, consistent with the case of earthquakes (Han & Simons, 2008)

Post-eruptive gravity change

- Viscoelastic deformation causes uplift and contraction, which produces $+0.2 \mu\text{Gal}$.
- $0.2 \mu\text{Gal}$ in 10 years is not detectable by GRACE.

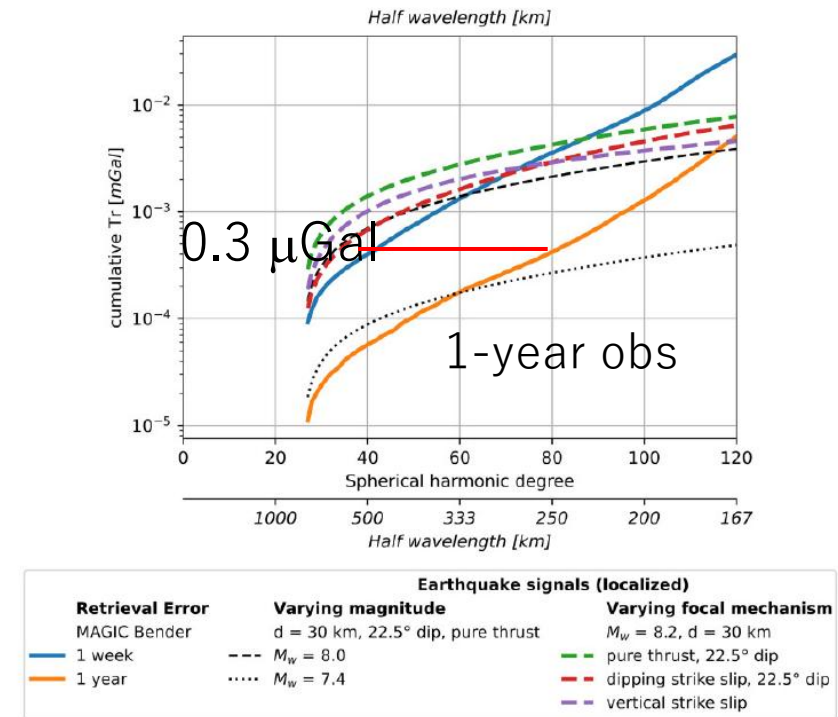
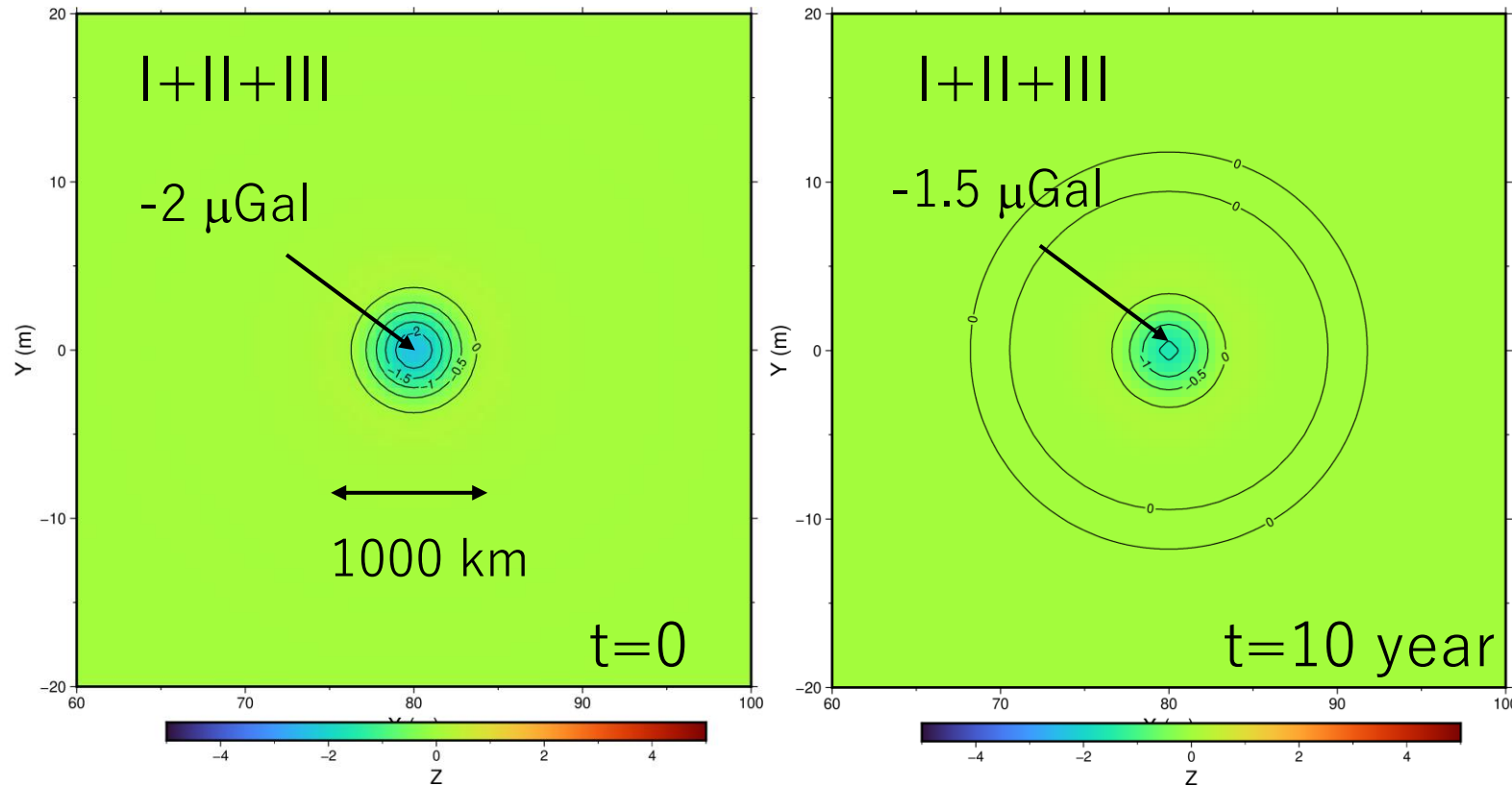


post-eruptive change

Factor	Δg
I	< 0 (constant)
II uplift	> 0
III contr.	> 0

Post-eruptive gravity change (MAGIC)

- Cut-off degree=80, Gaussian radius=250 km

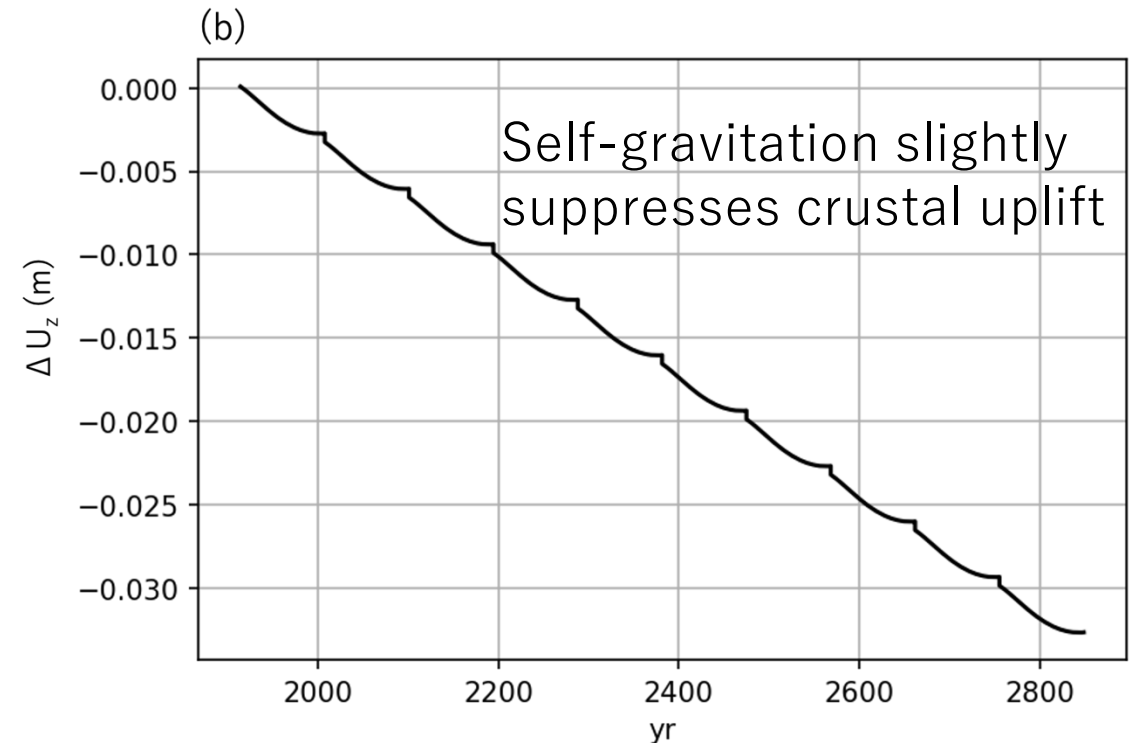
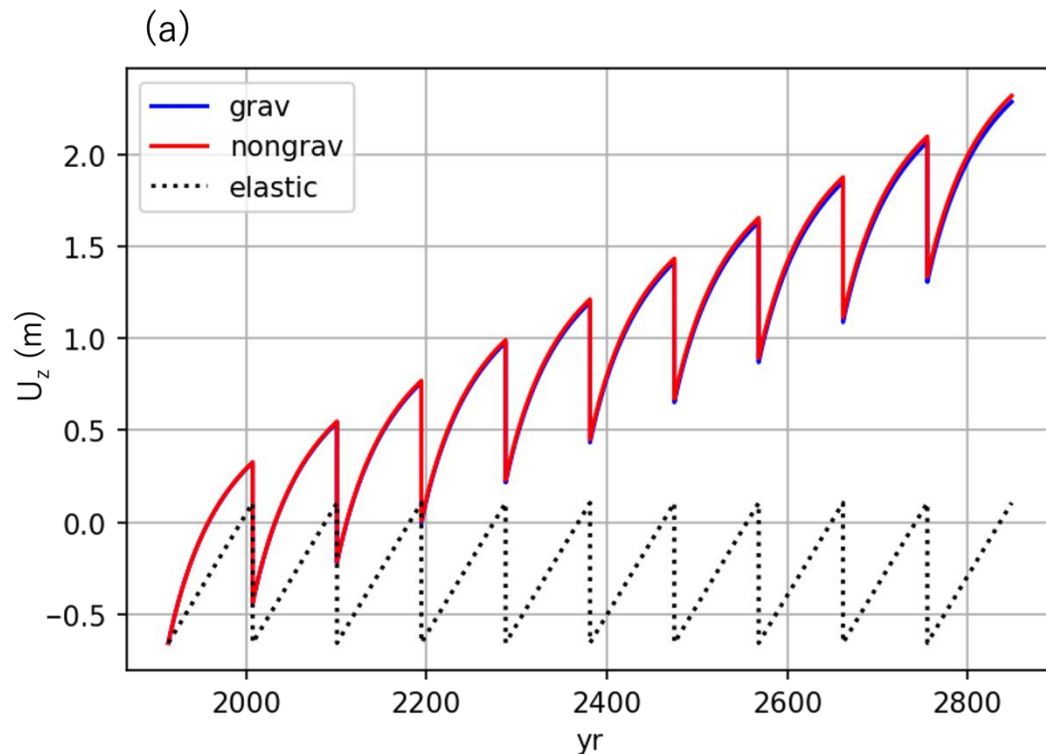


Daras et al. (2024)

- Both co- and post-eruptive changes are detectable by MAGIC ($0.5 \mu\text{Gal}$ over 500 km by 10 year observation)

Result II: Isostatic effect on long-term deformation

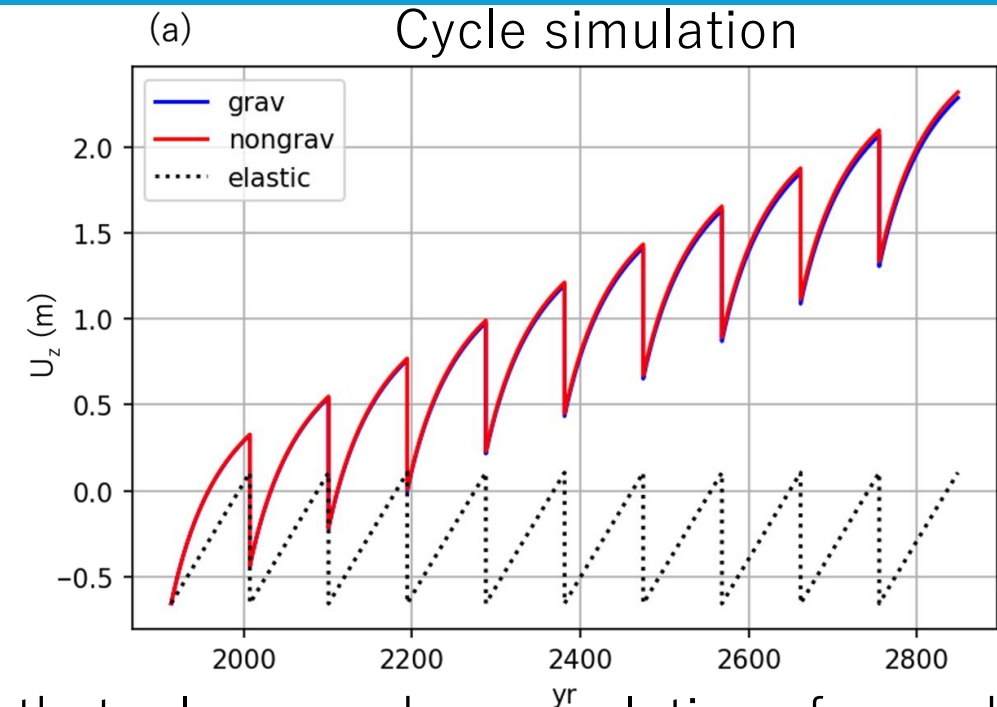
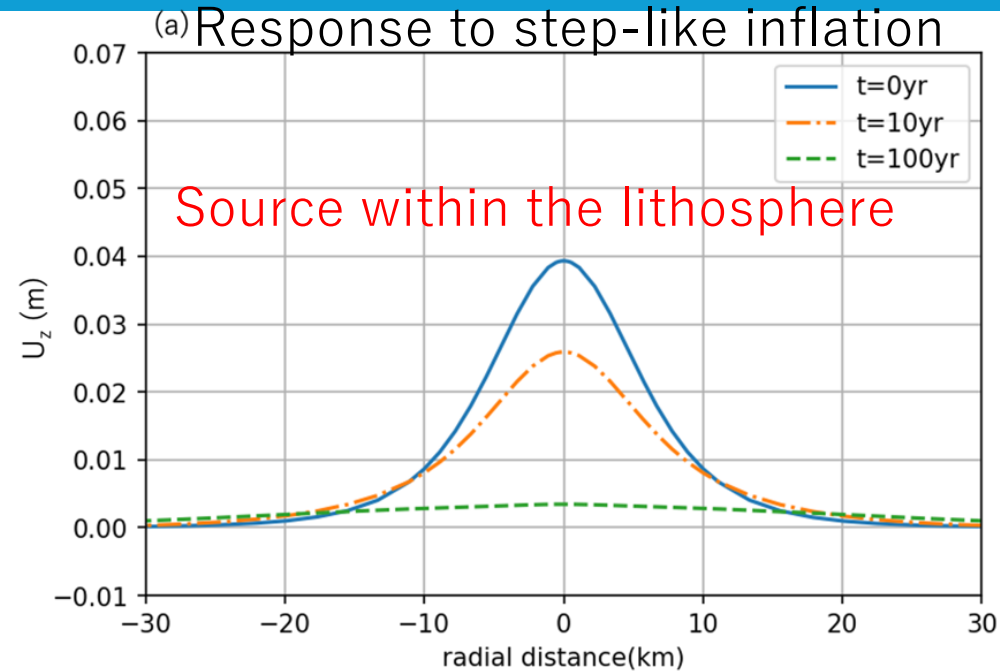
- Assuming a relatively large eruption at Sakurajima in 1914 (southern Japan) every 100 years, we calculated cumulative viscoelastic deformation.
- Cut-off degree $>10,000$ (land observation, spherically symmetric structure)
- The difference between self-gravitating and no self-gravitating model is 3 cm after 800 years, practically difficult to detect.



Summary and future work

- Spectral FEM has been extended to calculate volcanic deformation within a spherical, self-gravitating Earth model.
- For satellite observations, co- and post-eruptive gravity changes associated with a large eruption may be detectable by the MAGIC mission.
- A more sophisticated analysis shows the validity of future satellite gravimetry (Braitenberg & Pastorutti, 2024)
- For ground-based observations, the effect of self-gravitation on viscoelastic deformation due to repeated large eruptions is practically negligible.
- Applications to large-scale topography (rift zones, ridges, Martian volcanoes, etc.)

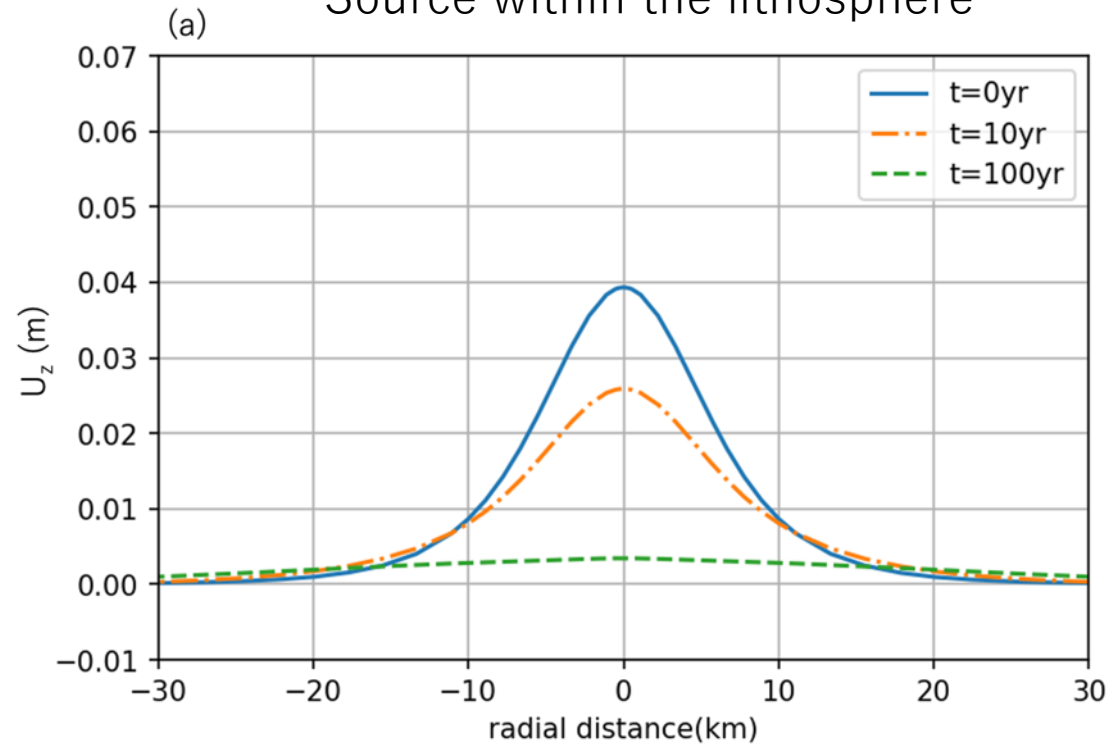
Why does the surface move upward due to viscoelastic deformation?



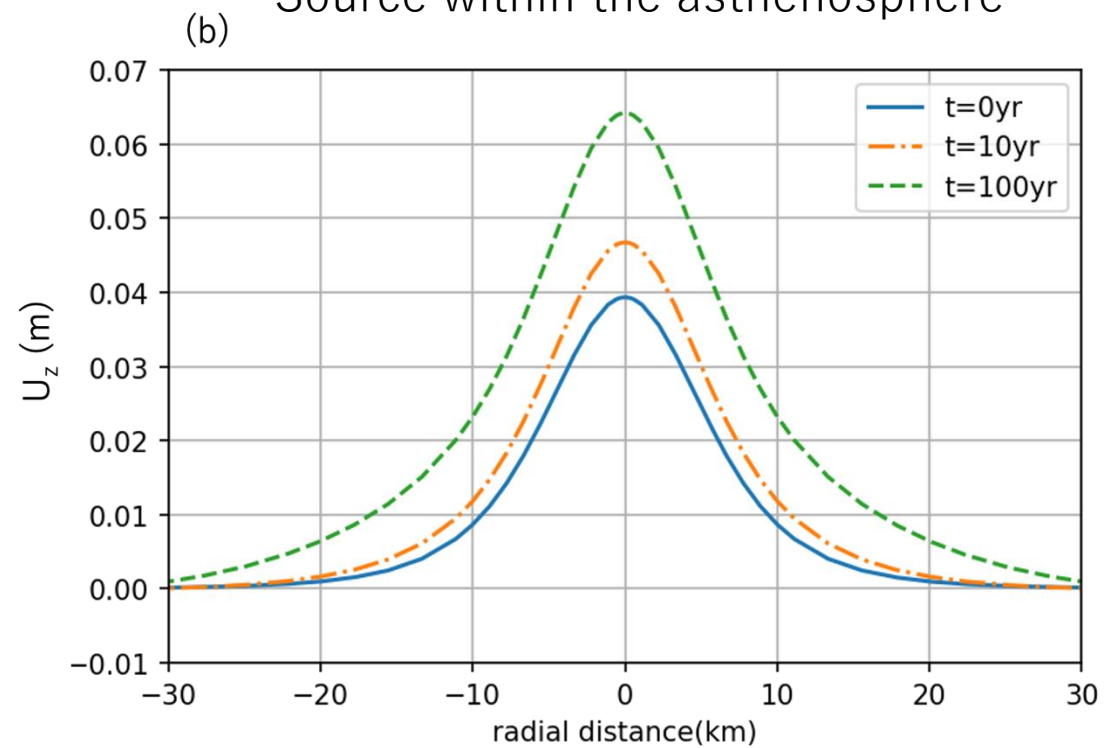
- In the purely elastic case, the model is adjusted so that release and accumulation of mass balance.
- For **inflation/deflation**, the surface gradually subsides/uplifts with time.
- Inflation occurs slowly, whereas deflation occurs instantaneously. Consequently, the uplift due to deflation is larger.
- If the source is placed **within the viscous layer**, the behavior is reversed.

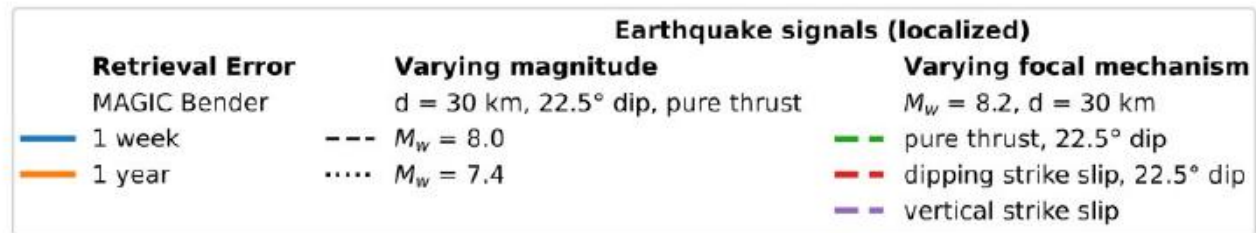
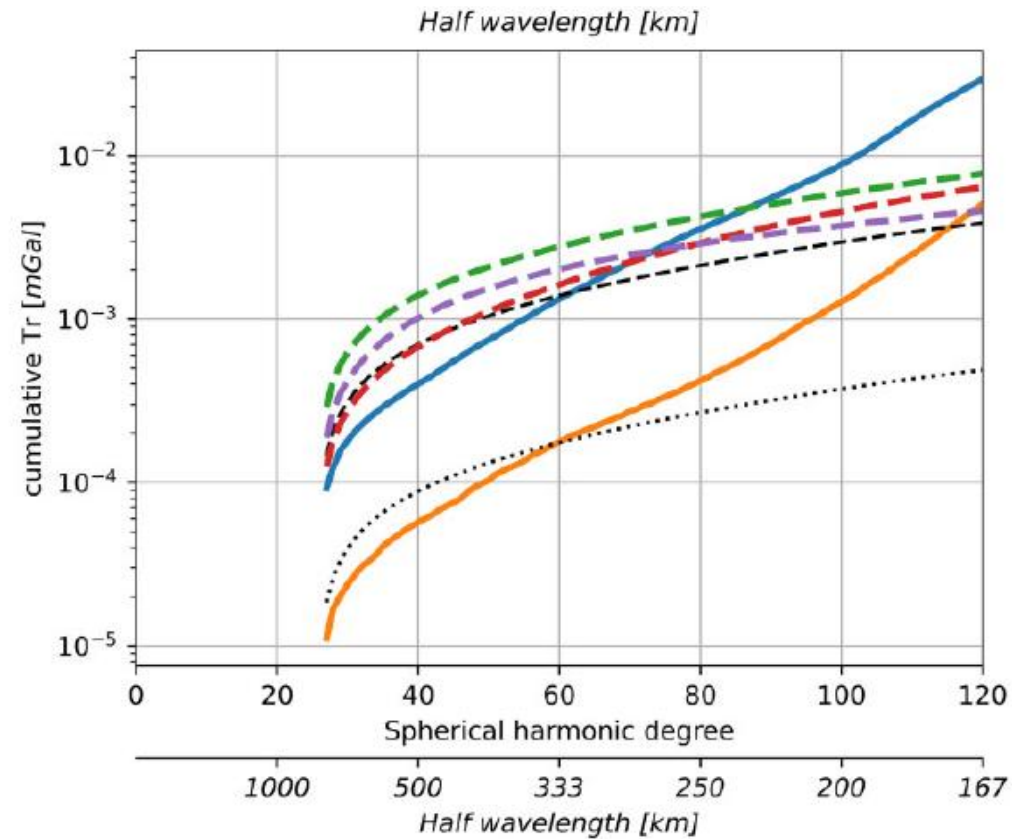
Response to step-like inflation

(a) Source within the lithosphere



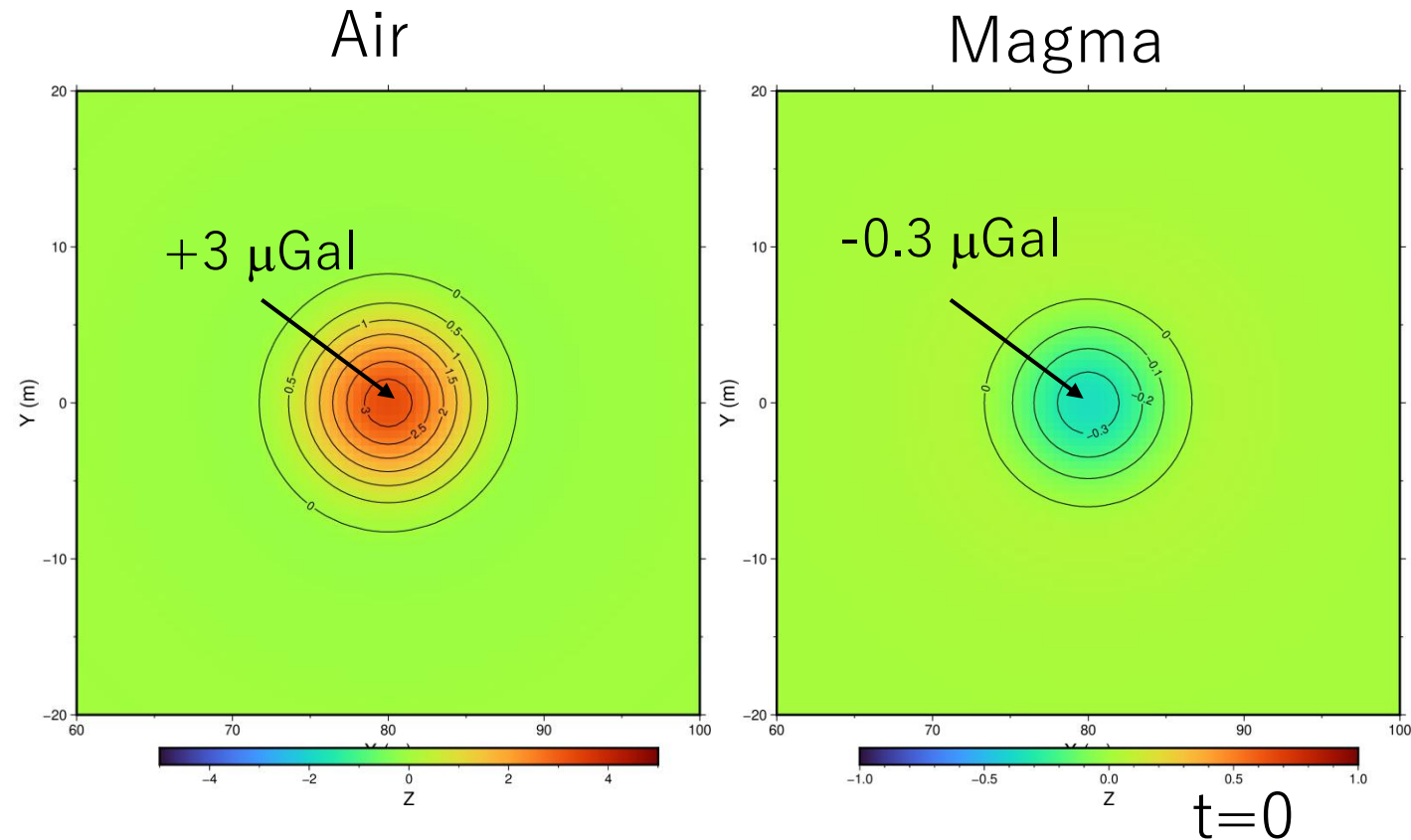
(b) Source within the asthenosphere



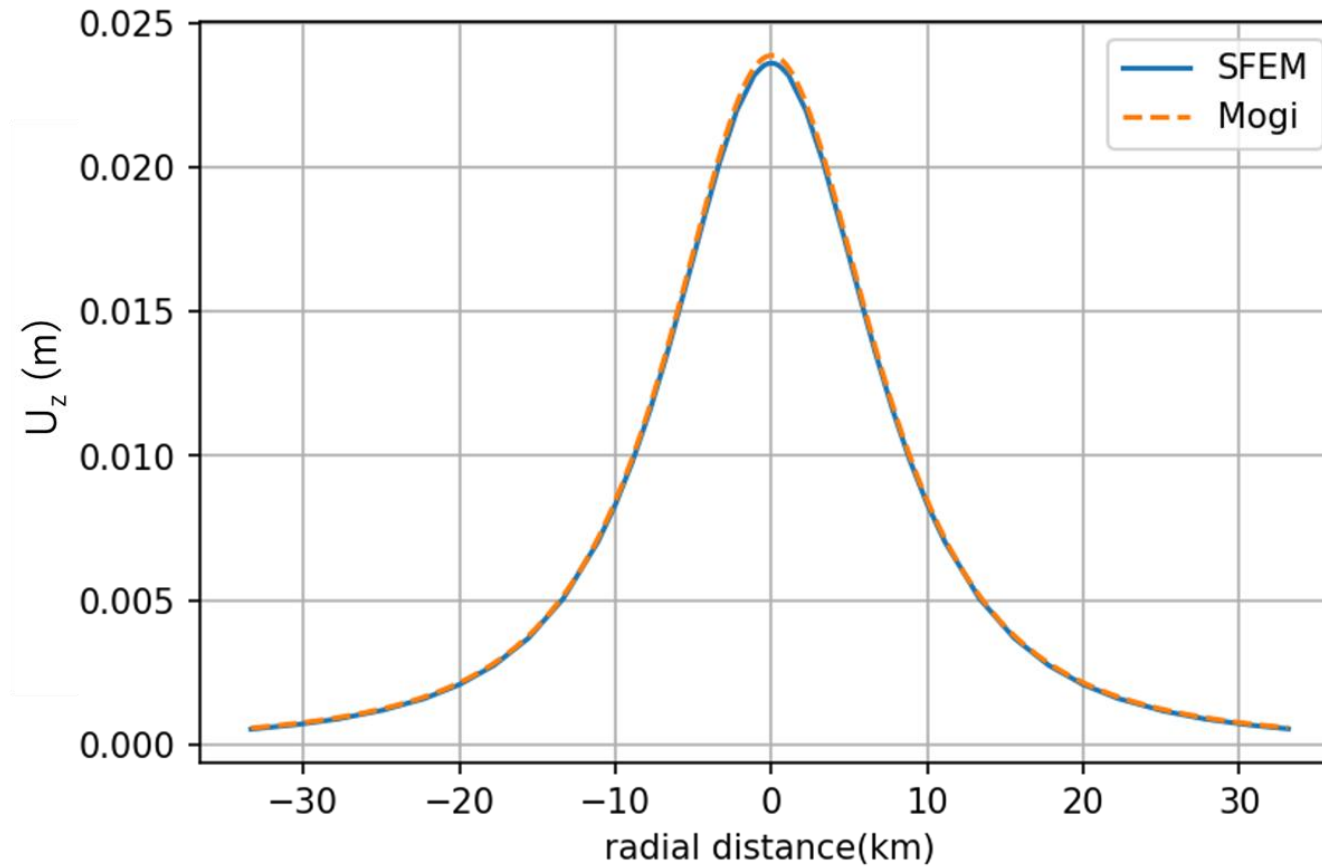


Density contrast

- Air is removed instead of magma
- 3 microGal is detectable by GRACE.
- No GRACE observations of gravity changes have been reported so far.
- The erupted material is thought to have a density similar to that of the crust.



Validation



Analytical solution of Mogi model vs SFEM.

The difference is 1.3% above the source. Numerical error and curvature effects.

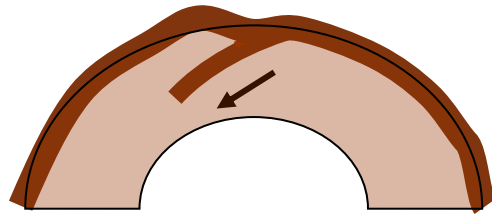
提案手法(1) – 動機と spectral FEM の導入

- M9 earthquakes in subduction zones → strong lateral viscosity contrast by the slab

and disadvantage

- Ordinary finite element methods have advantages
complex geometry & heterogeneities

Considering self-gravitation, forces don't balance.
Approximation of gravitational effects to cancel translation.



$$\bar{\sigma} \rightarrow \bar{\sigma} - \rho_0 g_0 u_r \bar{I}$$

$$\vec{u}(r, \theta, \varphi) = \sum_{jm} [\underline{U}_{jm}(r) S_{jm}^{(-1)}(\theta, \varphi) + \underline{V}_{jm}(r) S_{jm}^{(1)}(\theta, \varphi) + \underline{W}_{jm}(r) S_j^{(0)}(\theta, \varphi)]$$

- $\tau \phi_1(r, \theta, \varphi) = \sum_{jm} \underline{F}_{jm}(r) Y_{jm}(\theta, \varphi)$ ion more rigorously, we partly use analytical representations in FEM



Finite elements for the radial coordinate only.

Horizontal resolutions change with j and m .

Developed for surface loading (Martinec, 2000)

With a small number of 1-D elements, the whole sphere is modeled!

提案手法(2) – 弱形式を一部解析的に解く

- Time discretization of the governing eqs.

$$\boxed{\begin{aligned}\nabla \cdot \tau - \rho_0 \nabla \phi_1 + \nabla \cdot (\rho_0 \vec{u}) \nabla \phi_0 - \nabla (\rho_0 \vec{u} \cdot \nabla \phi_0) &= 0 \\ \nabla^2 \phi_1 + 4\pi G \nabla \cdot (\rho_0 \vec{u}) &= 0\end{aligned}} + \boxed{\begin{aligned}\dot{\tau} &= \dot{\tau}^E - \frac{\mu}{\eta} \left(\tau - \frac{1}{3} \text{tr}(\tau) I \right) \\ \dot{\tau}^E &= \lambda (\nabla \vec{u}) I + 2\mu \varepsilon\end{aligned}} \quad \text{Peltier (1974)}$$

$\longrightarrow \nabla \cdot \tau^{E,i+1} - \rho_0 \nabla \phi_1^{i+1} + \nabla \cdot (\rho_0 \vec{u}^{i+1}) \nabla \phi_0 - \nabla (\rho_0 \vec{u}^{i+1} \cdot \nabla \phi_0) = -\nabla \cdot \tau^{V,i} + \rho_0 \vec{f}^{i+1}$
elastic deformation at time $i+1$ = viscous dissipation at time i + boundary condition at time $i+1$

- Energy variation and Galerkin method

$$\delta \varepsilon(\vec{u}^{i+1}, \phi_1^{i+1}, \delta \vec{u}, \delta \phi_1) = \delta F_{diss}^i(\delta \vec{u}, \delta \phi_1) + \delta W_{source}^{i+1}(\delta \vec{u}, \delta \phi_1)$$

$$\delta \varepsilon_{shear+bulk}^{i+1} = 2 \int_V \mu(\varepsilon^{i+1} : \delta \varepsilon) dV + \int_V \lambda(\nabla u^{i+1} \nabla \delta u) dV$$

$$\forall (\delta \vec{u}, \delta \phi_1) \longrightarrow (\vec{u}^{i+1}, \phi_1^{i+1})$$

Only this term depends on 3-D viscosity

- We assume 1-D structure for density and elasticity parameters.

LHS: 1-D orthogonality relation RHS: 3-D numerical evaluation

Result II isostatic effect on long-term deformation

