

# Identifier-Constraints DIA: Towards Systematic Multi-Fault Identification in Geodetic Quality Control

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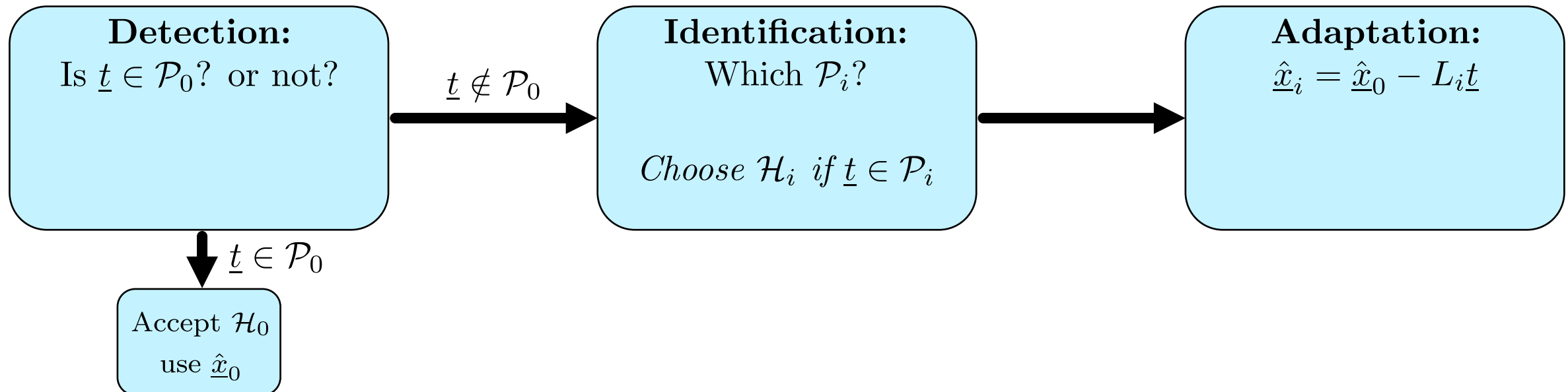
# DIA Framework

**Observation model:**  $\underline{y} = Ax + \underline{e} \in \mathbb{R}^m, \quad \underline{e} \sim \mathcal{N}(0, Q_{yy})$

**Null hypothesis:**  $\mathcal{H}_0 : E(\underline{y}) = Ax$

**Alternative hypotheses:**  $\mathcal{H}_i : E(\underline{y}) = Ax + C_i b_i, \quad i \in \{1, \dots, N\}$

**Misclosure vector:**  $\underline{t} = B^\top \underline{y} \in \mathbb{R}^r, \quad B^\top A = 0, \quad Q_{tt} = B^\top Q_{yy} B$



# Traditional Identification

How do we identify today? Maximize test statistics over user-defined alternatives

**Test statistic:**  $\underline{T}_j := \|\hat{b}_j(\underline{t})\|_{Q_{\hat{b}_j \hat{b}_j}}^2 \stackrel{\mathcal{H}_0}{\sim} \chi^2(q_j, 0)$

**Identification step:**  $i = \arg \max_{j \in \{1, \dots, N\}} (\text{CDF}_{\chi^2(q_j, 0)} [\underline{T}_j])$

$\downarrow$   
 # Hypotheses

**Adaptation:**  $\hat{\underline{x}}_i = \hat{\underline{x}}_0 - \underbrace{A^+ C_i \hat{b}_i(\underline{t})}_{L_i \underline{t}}$

✓ Handles any degrees of freedom

No restriction on  $q_j$

× Limited by the list

$\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_N$   
must be written in advance

# Identifier-Constraints: Key Idea

Define a **fault library** once:

$$C = [C_1, \dots, C_k] \in \mathbb{R}^{m \times q}, \quad q = \sum_{j=1}^k q_j$$

e.g.  $C = [I_m, C_{\text{iono}}, C_{\text{tropo}}]$

One fault library, many hypotheses

- ▶  $\mathcal{H}_i : E(\underline{y}) = Ax + CZ_i b$  with  $i \in \{1, \dots, N\}$
- ▶  $C \rightarrow$  **fault library**;  $Z_i \rightarrow$  **activation matrix**
- ▶  $N \leq 2^k - 1$  from  $k$  signatures, combinations need no new candidate

Identifier and **activation matrix**:

$$z^{(i)} = [z_1^{(i)}, \dots, z_k^{(i)}]^T \in \{0, 1\}^k$$

$$Z_i = \text{blkdiag} \left( z_1^{(i)} I_{q_1}, \dots, z_k^{(i)} I_{q_k} \right)$$

$$z^{(i)} \mapsto Z_i \mapsto \mathcal{H}_i$$

Example: one vector, one hypothesis

| Signature      | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $C_6$ | ... |
|----------------|-------|-------|-------|-------|-------|-------|-----|
| Identifier $z$ | 0     | 1     | 1     | 0     | 1     | 0     | ... |
| Status         | off   | on    | on    | off   | on    | off   | ... |

Selected active model:  $E(\underline{y}) = Ax + C_2 b_2 + C_3 b_3 + C_5 b_5$

$z_j^{(i)} = 0 \Rightarrow b_j^{(i)} = 0$   
Inactive  $\Rightarrow C_j$  discarded

$z_j^{(i)} = 1 \Rightarrow b_j^{(i)}$  is *estimated*  
Active  $\Rightarrow C_j$  included

# Identification as *Estimation*

After **Detection** ( $t \notin \mathcal{P}_0$ )  $\rightarrow$  estimate **active pattern** and **bias vector** jointly:

$$(\bar{Z}(t), \bar{b}(t)) = \arg \min_{\substack{i \in \{1, \dots, N\} \\ b \in \mathbb{R}^q}} \left[ \underbrace{\| \underline{t} - B^\top C Z_i b \|^2_{Q_{tt}}}_{\text{goodness of fit}} + \underbrace{\rho(Z_i)}_{\text{dimension penalty}} \right]$$

subject to  $z_j^{(i)} = 0 \Rightarrow b_j^{(i)} = 0, \quad z_j^{(i)} \in \{0, 1\} \quad (\textit{trivial})$

+ non-trivial identifier constraints

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+ non-trivial identifier constraints

- ▶ Corresponding adaptation:  $\bar{x} = \hat{x}_0 - A^+ C \bar{Z}(\underline{t}) \bar{b}(\underline{t})$
- ▶ The DIA-estimator combines all  $\hat{x}_i$  into one:  $\bar{x} = \sum_{i=0}^N \hat{x}_i \mathbb{1}_{\mathcal{P}_i}(\underline{t}), \quad \mathbb{1}_{\mathcal{P}_i}(\underline{t}) = \begin{cases} 1, & \underline{t} \in \mathcal{P}_i \\ 0, & \underline{t} \notin \mathcal{P}_i \end{cases}$
- ▶ Distribution:  $f_{\bar{x}}(x) = \sum_{i=0}^N \mathbb{E} [f_{\hat{x}_0}(x + A^+ C \bar{Z}(\underline{t}) \bar{b}(\underline{t})) \mathbb{1}_{\mathcal{P}_i}(\underline{t})]$

# Non-Trivial Identifier-Constraints

## Capacity

$$\sum_{j=1}^k z_j q_j \leq q_{\max} < r$$

At most  $q_{\max}$  bias parameters at once, and  $q_{\max} < r$  since  $\underline{t} \in \mathbb{R}^r$

## Rank

$$\text{rank}(B^T C Z_i) = \sum_{j=1}^k z_j q_j$$

Identifier constraints can be written for *estimability* and *separability*

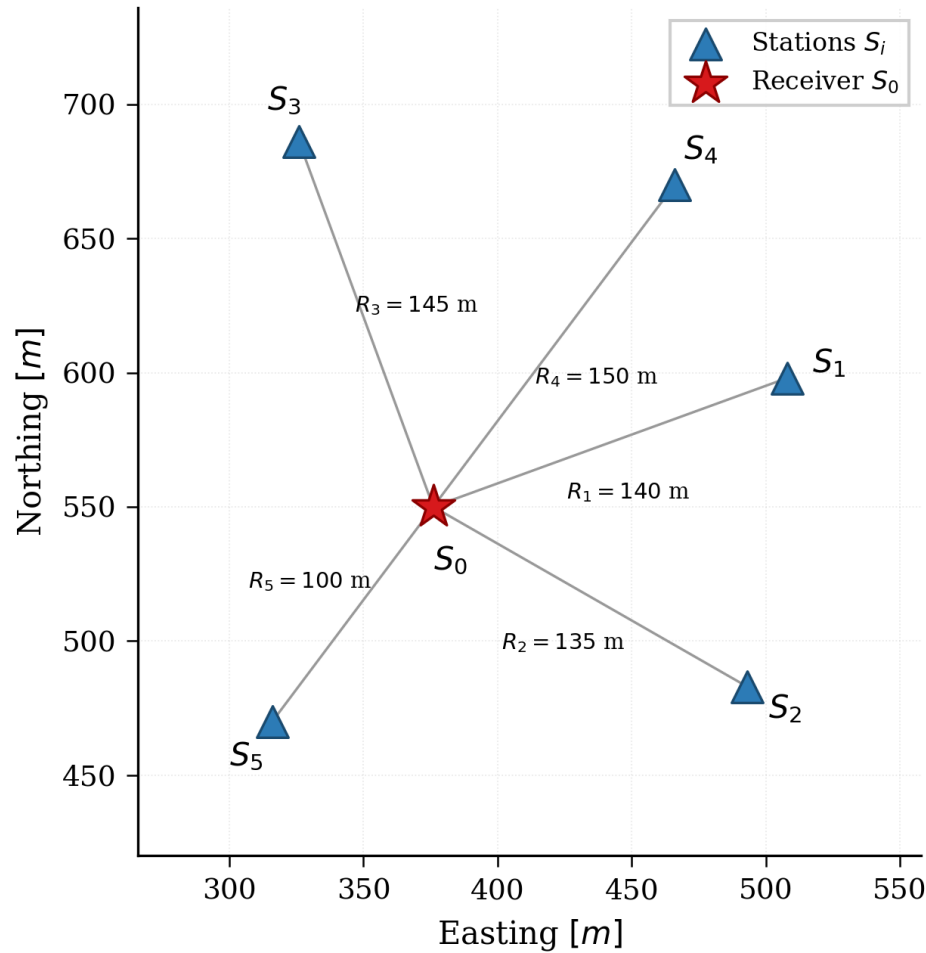
## Physical

$$z_N = z_E \leq z_{\text{ZTD}}$$

e.g. two tropospheric gradients form one physical quantity, they enter together only when zenith delay does

**Non-trivial identifier constraints** define admissible set  $\mathcal{Z} \subseteq \{0, 1\}^k$   
each  $z \in \mathcal{Z}$  is one alternative hypothesis  $\mathcal{H}_i$ , so  $N = |\mathcal{Z}|$

# Example



## ► Model:

$m = 5$  ranges,  $n = 2$  coordinates, redundancy  $r = 3$

## ► Fault library:

$C = [C_1, \dots, C_5] = I_5$ , one outlier per range,  $q_j = 1$

Collinearity of  $S_5 - S_0 - S_4 \rightarrow \text{rank}(B^T[C_1, C_2, C_3]) = 2$

## ► Identifier constraints:

1. *Capacity*
2. *Rank*

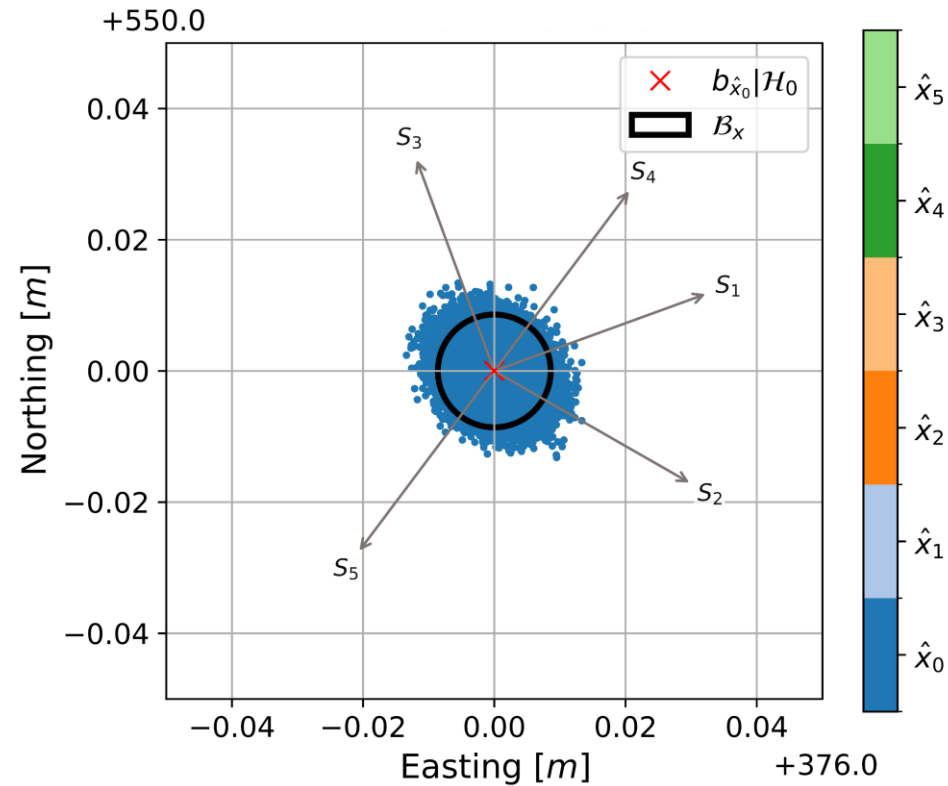
5 ranges observed from  $S_0$  to  $S_1, \dots, S_5$  with  $\sigma = 5$  mm

# Example: Capacity-only

► Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 \mathbf{z}_j \leq 1 \quad \left. \vphantom{\sum_{j=1}^5} \right\} \text{Datasnooping}$$

$N = 5$



Correct acceptance

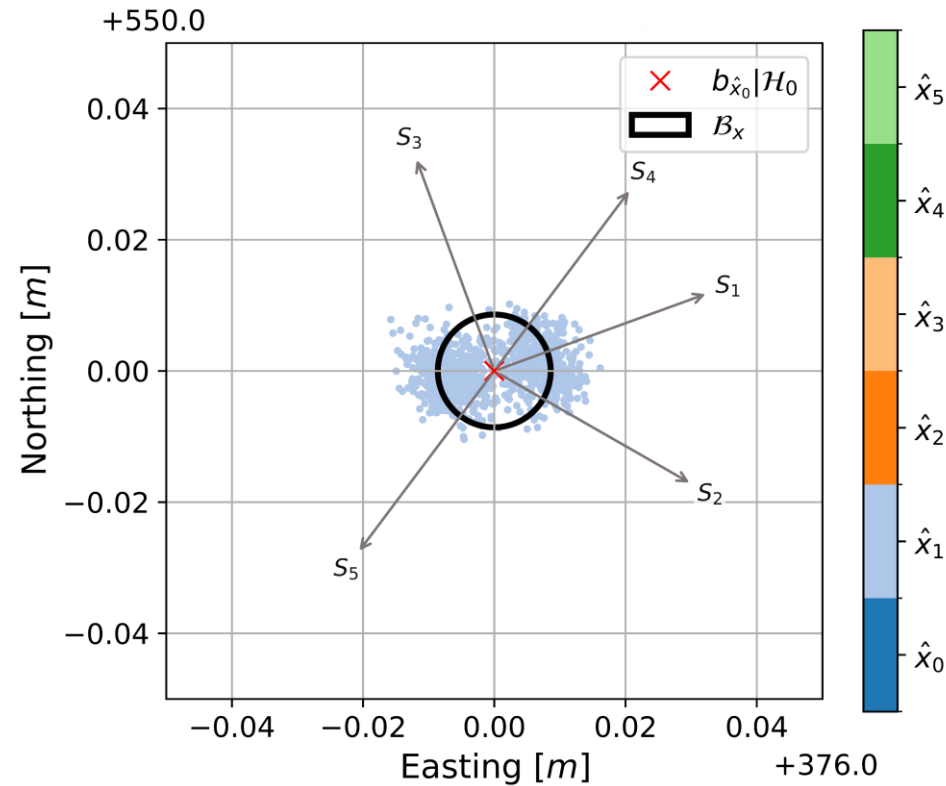
$$\begin{aligned} &\rightarrow \mathcal{P}_5 \rightarrow \underline{\hat{x}}_5 \\ &\rightarrow \mathcal{P}_4 \rightarrow \underline{\hat{x}}_4 \\ &\rightarrow \mathcal{P}_3 \rightarrow \underline{\hat{x}}_3 \\ &\rightarrow \mathcal{P}_2 \rightarrow \underline{\hat{x}}_2 \\ &\rightarrow \mathcal{P}_1 \rightarrow \underline{\hat{x}}_1 \\ &\rightarrow \mathcal{P}_0 \rightarrow \underline{\hat{x}}_0 \end{aligned}$$

$\underline{t} \in$

# Example: Capacity-only

► Under null hypothesis  $\mathcal{H}_0$ :

$$\left. \begin{array}{l} \text{IC: } \sum_{j=1}^5 \mathbf{z}_j \leq 1 \\ N = 5 \end{array} \right\} \text{Datasnooping}$$



$\underline{t} \in$

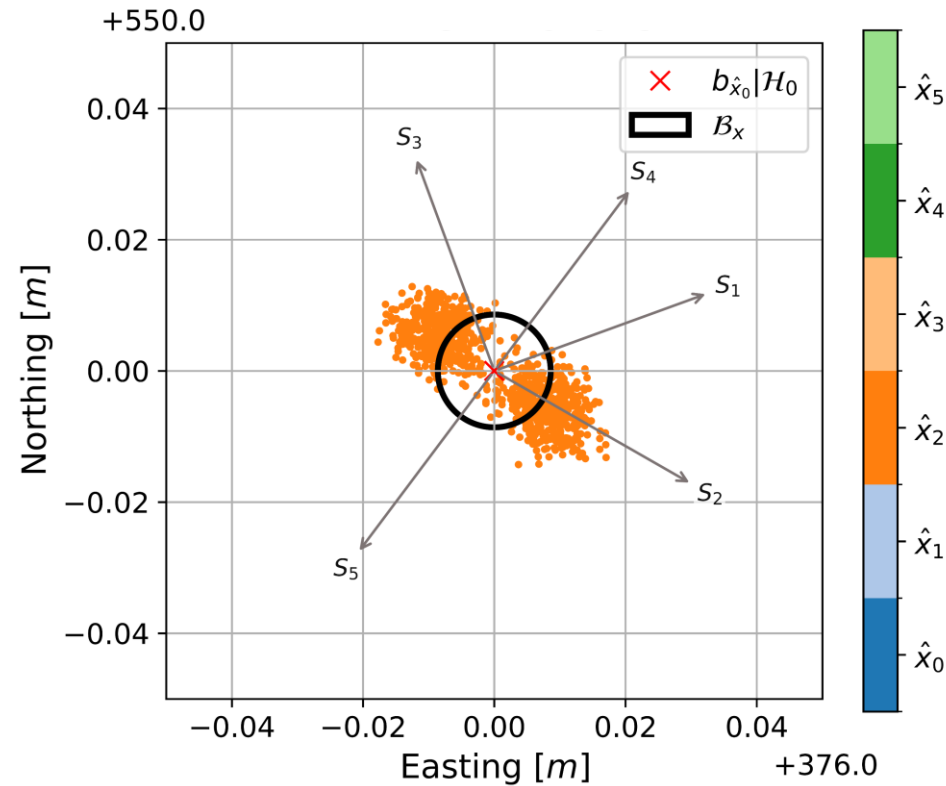
- $\rightarrow \mathcal{P}_5 \rightarrow \underline{\hat{x}}_5$
- $\rightarrow \mathcal{P}_4 \rightarrow \underline{\hat{x}}_4$
- $\rightarrow \mathcal{P}_3 \rightarrow \underline{\hat{x}}_3$
- $\rightarrow \mathcal{P}_2 \rightarrow \underline{\hat{x}}_2$
- $\rightarrow \mathcal{P}_1 \rightarrow \underline{\hat{x}}_1$
- $\rightarrow \mathcal{P}_0 \rightarrow \underline{\hat{x}}_0$

# Example: Capacity-only

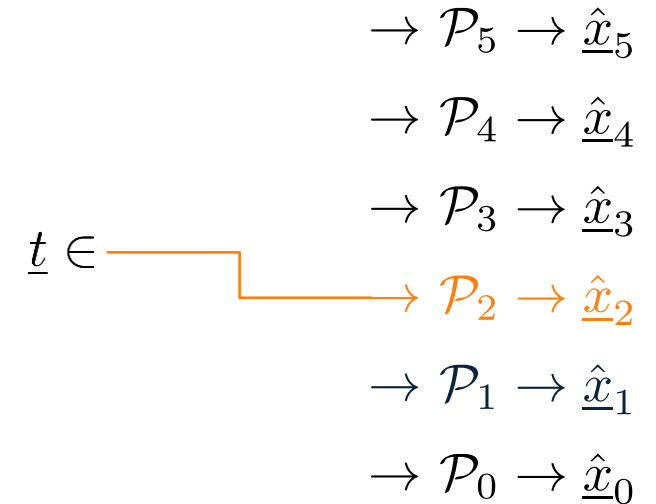
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$$\text{IC: } \sum_{j=1}^5 \mathbf{z}_j \leq 1 \quad \left. \vphantom{\sum_{j=1}^5} \right\} \text{ Datasnooping}$$

$N = 5$



False alarm due to  $\mathcal{H}_2$  ( $S_2$ )

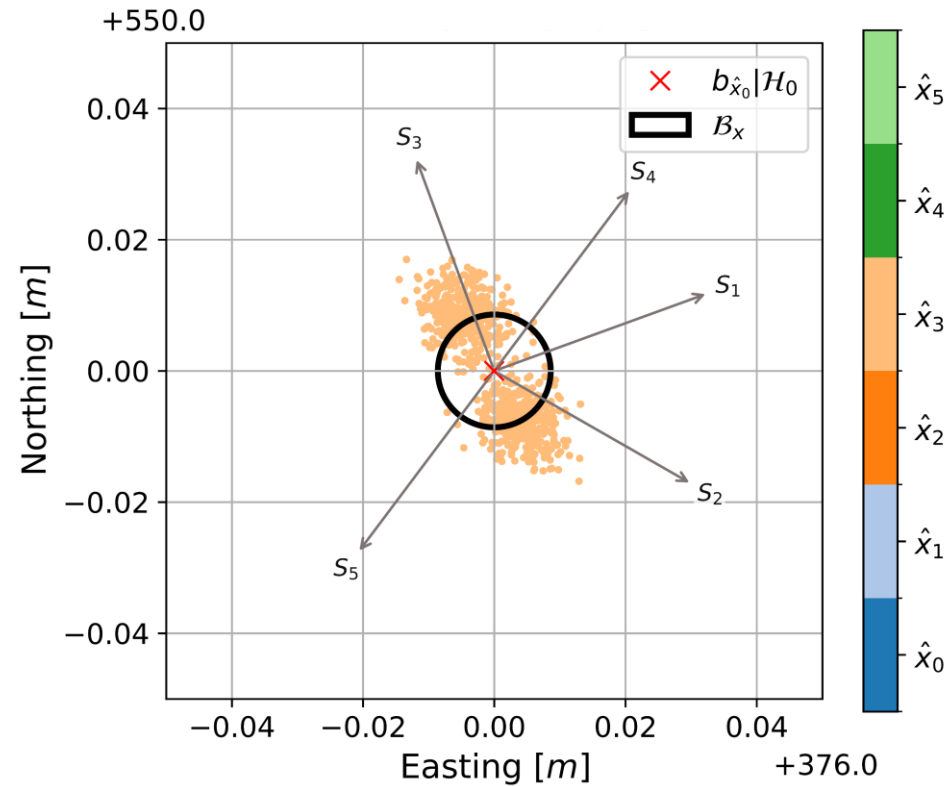


# Example: Capacity-only

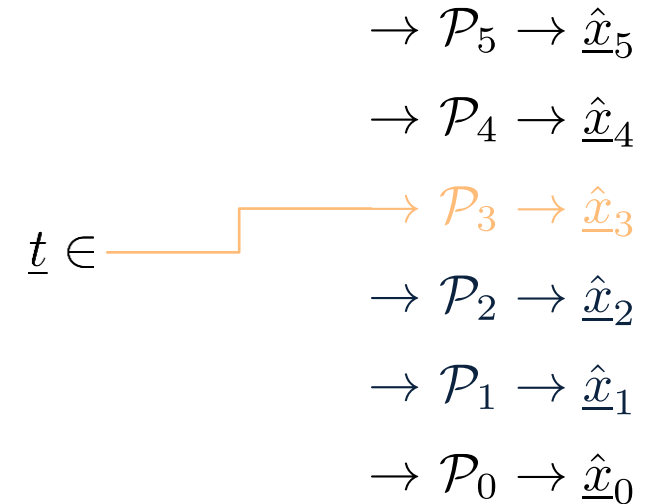
► Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 \mathbf{z}_j \leq 1 \quad \left. \vphantom{\sum_{j=1}^5} \right\} \text{Datasnooping}$$

$N = 5$



False alarm due to  $\mathcal{H}_3$  ( $S_3$ )

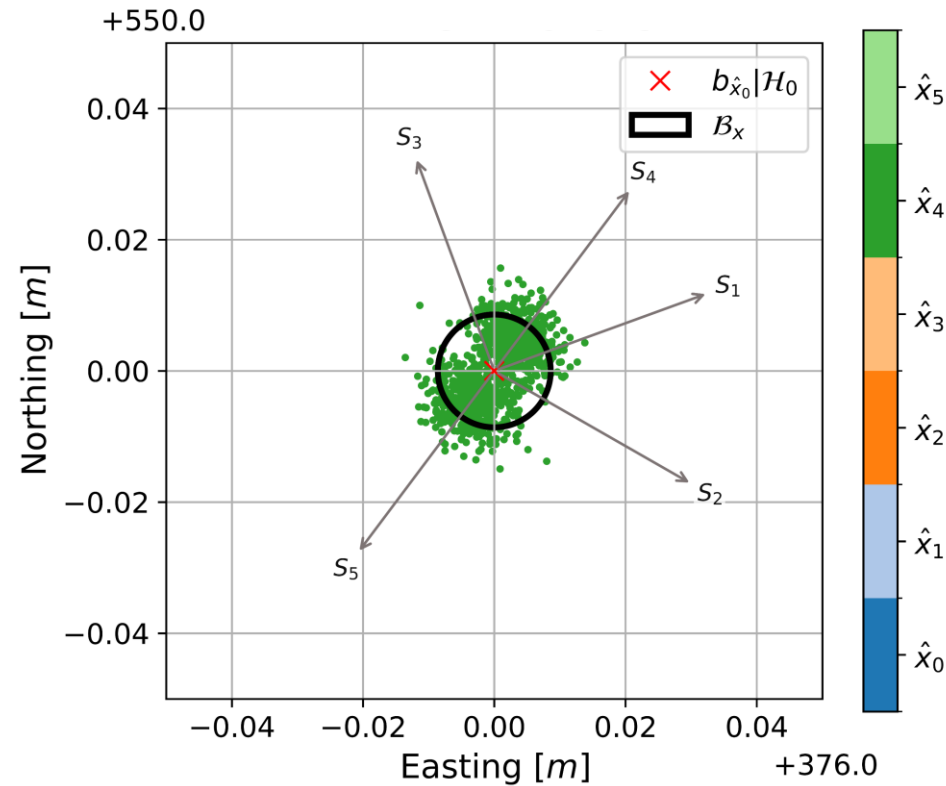


# Example: Capacity-only

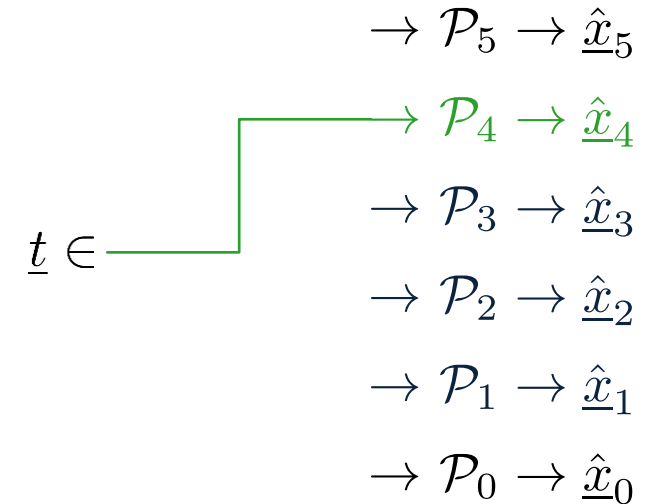
► Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 \mathbf{z}_j \leq 1 \quad \left. \vphantom{\sum_{j=1}^5} \right\} \text{Dat snooping}$$

$N = 5$



False alarm due to  $\mathcal{H}_4$  ( $S_4$ )

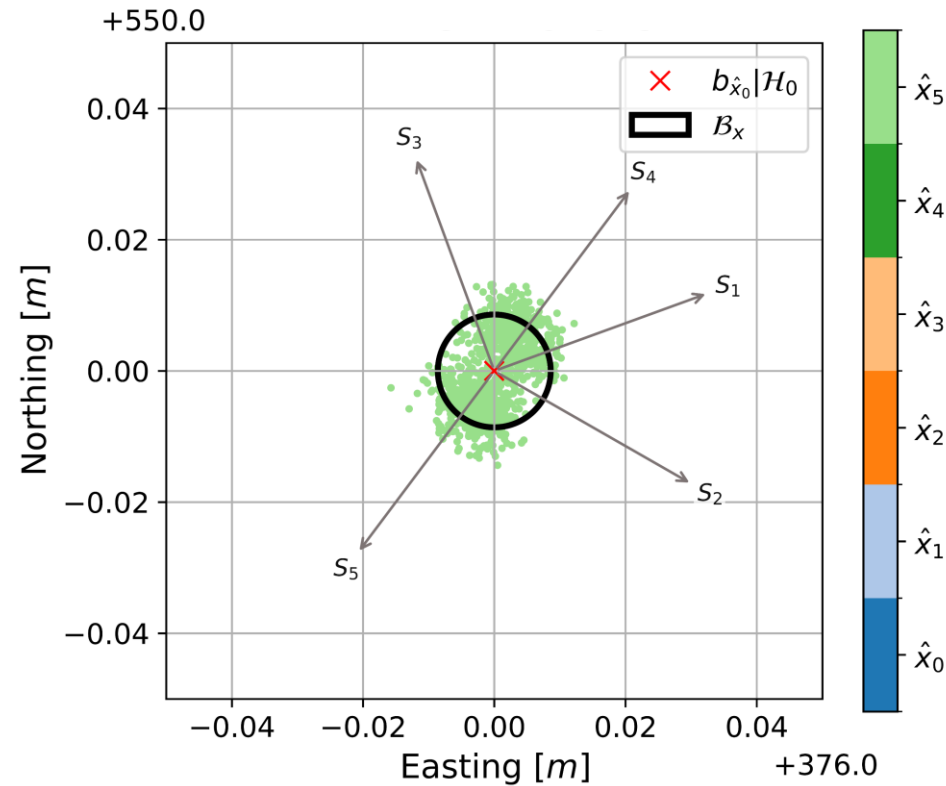


# Example: Capacity-only

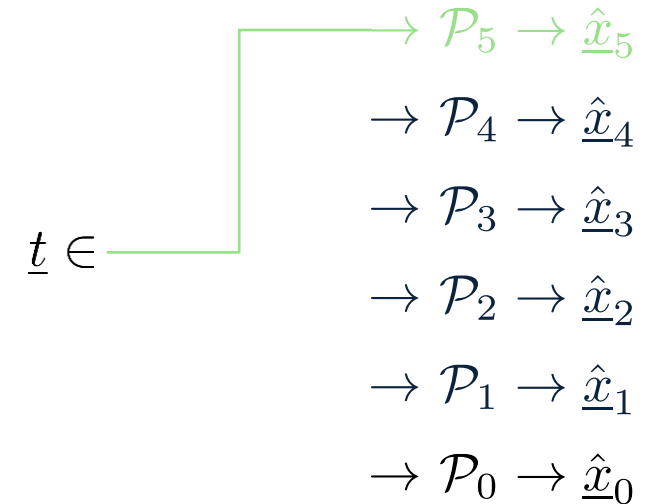
► Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 \mathbf{z}_j \leq 1 \quad \left. \vphantom{\sum_{j=1}^5} \right\} \text{Datasnooping}$$

$N = 5$



False alarm due to  $\mathcal{H}_5$  ( $S_5$ )



# Example: Capacity-only

- Under null hypothesis  $\mathcal{H}_0$ :

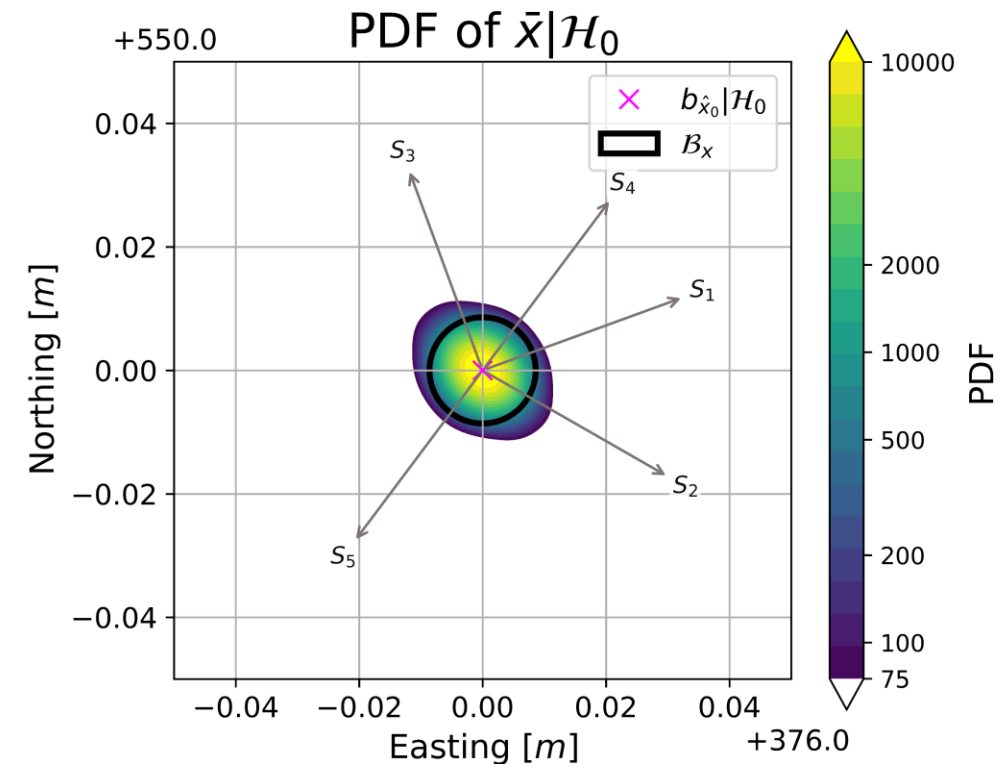
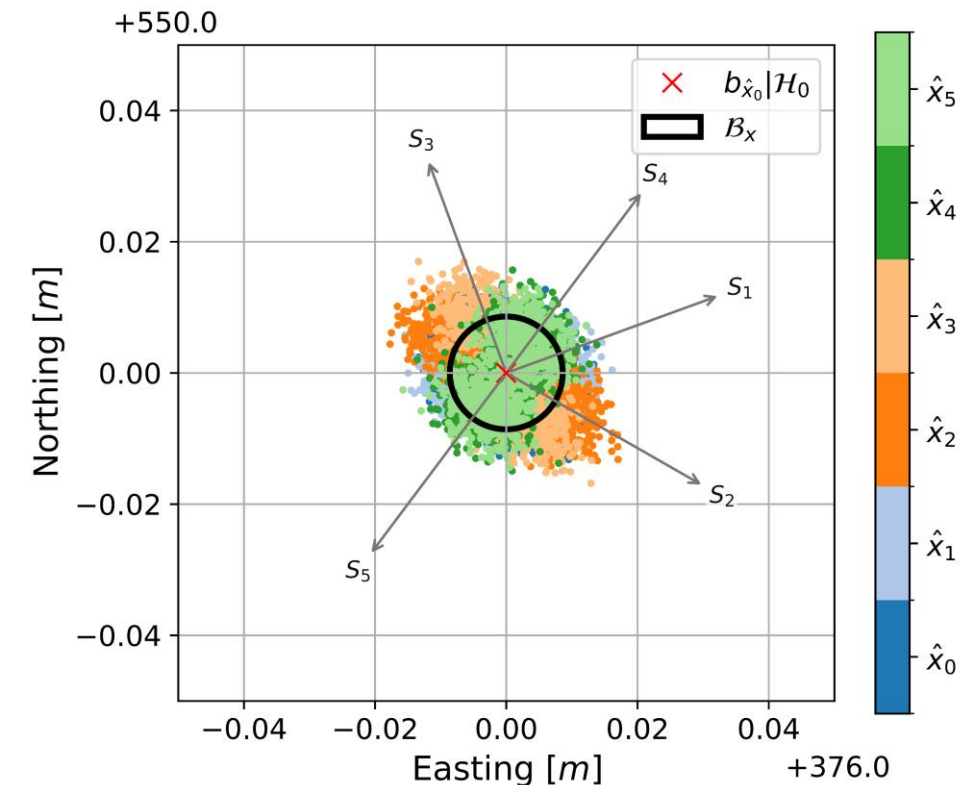
$$\bar{x} = \sum_{i=0}^N \hat{x}_i \mathbf{1}_{\mathcal{P}_i}(\underline{t})$$

$$\text{IC: } \sum_{j=1}^5 \mathbf{z}_j \leq 1$$

$$N = 5$$

*Datasnooping*

$$f_{\bar{x}}(x) = \sum_{i=0}^N \mathbb{E}[f_{\hat{x}_0}(x + A^+ C \bar{Z}(t) \bar{b}(t)) \mathbf{1}_{\mathcal{P}_i}(t)]$$

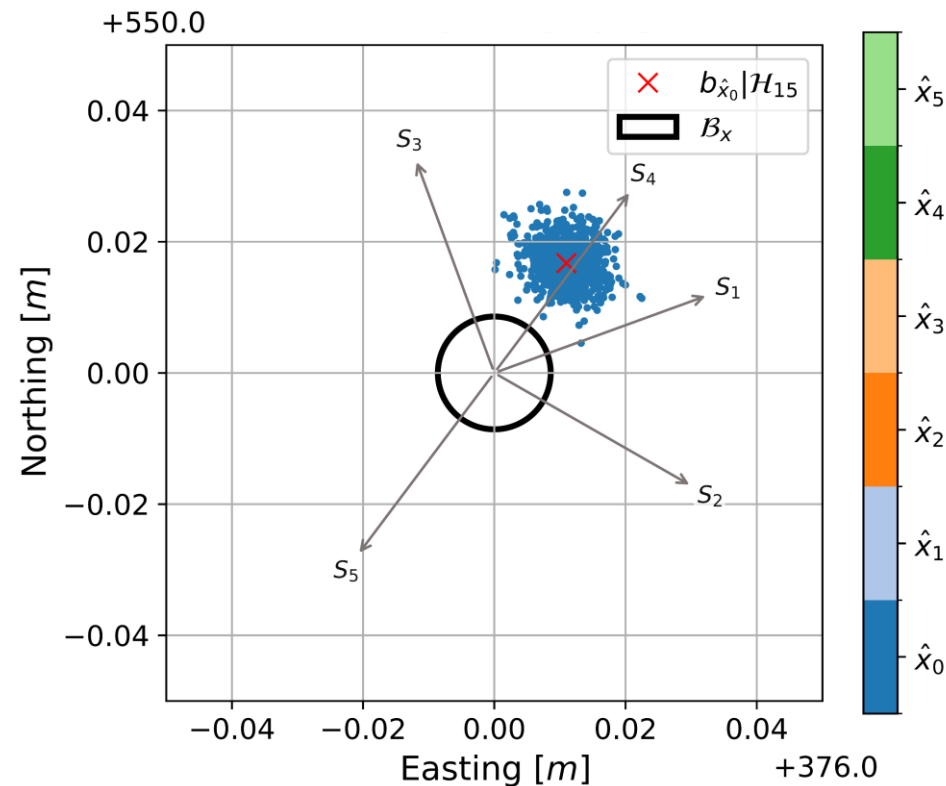


# Example: Capacity-only

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$

$$\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$$

$$\left. \begin{array}{l} \text{IC: } \sum_{j=1}^5 z_j \leq 1 \\ N = 5 \end{array} \right\} \text{Datasnooping}$$



Missed detection

$$\begin{array}{l} \rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5 \\ \rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4 \\ \rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3 \\ \rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2 \\ \rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1 \\ \rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0 \end{array}$$

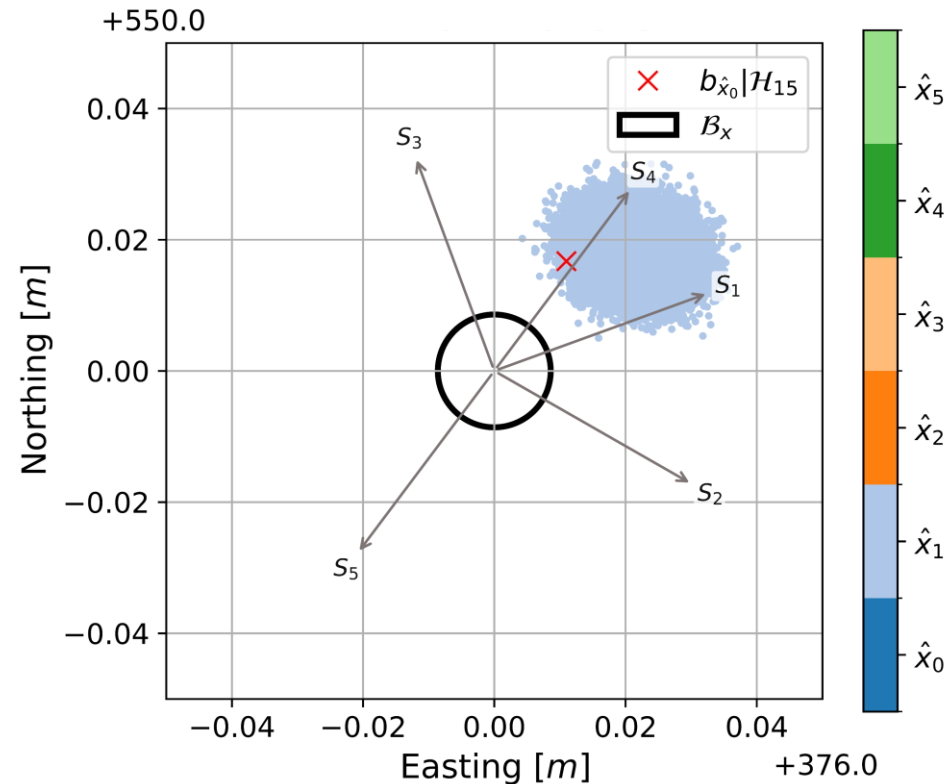
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Wrong identification due to  $\mathcal{H}_1$  ( $S_1$ )



$$\begin{array}{l} \rightarrow \mathcal{P}_5 \rightarrow \underline{\hat{x}}_5 \\ \rightarrow \mathcal{P}_4 \rightarrow \underline{\hat{x}}_4 \\ \rightarrow \mathcal{P}_3 \rightarrow \underline{\hat{x}}_3 \\ \rightarrow \mathcal{P}_2 \rightarrow \underline{\hat{x}}_2 \\ \rightarrow \mathcal{P}_1 \rightarrow \underline{\hat{x}}_1 \\ \rightarrow \mathcal{P}_0 \rightarrow \underline{\hat{x}}_0 \end{array}$$

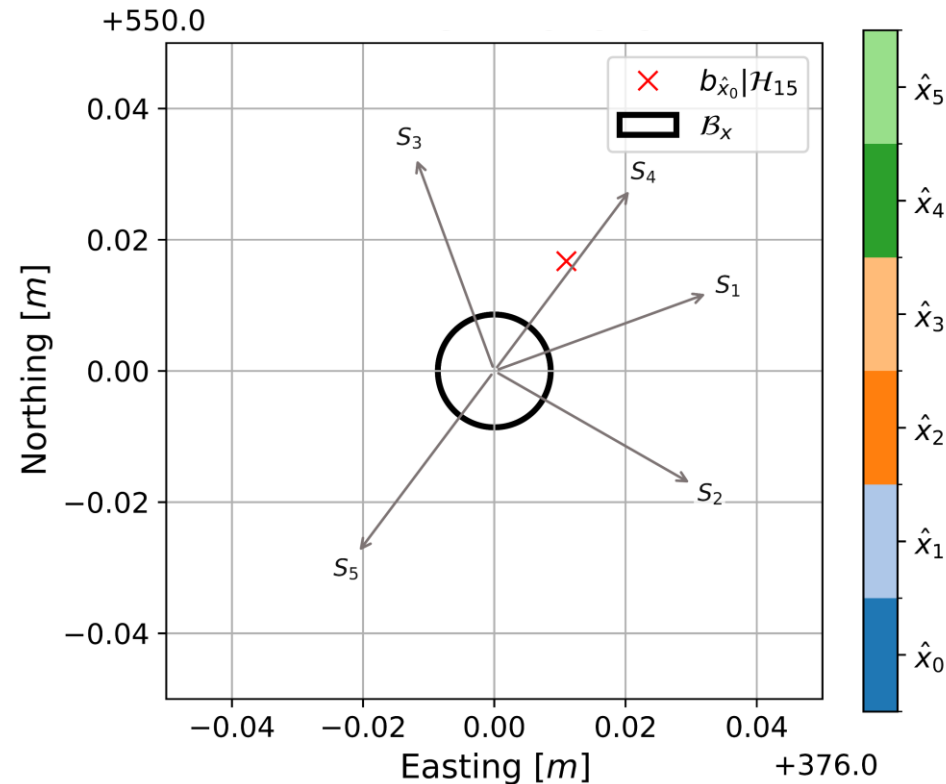
# Example: Capacity-only

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$$\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$$

$$\left. \begin{array}{l} \text{IC: } \sum_{j=1}^5 z_j \leq 1 \\ N = 5 \end{array} \right\} \text{Datasnooping}$$

Wrong identification due to  $\mathcal{H}_2$  ( $S_2$ )



$$\begin{array}{l} \rightarrow \mathcal{P}_5 \rightarrow \underline{\hat{x}}_5 \\ \rightarrow \mathcal{P}_4 \rightarrow \underline{\hat{x}}_4 \\ \rightarrow \mathcal{P}_3 \rightarrow \underline{\hat{x}}_3 \\ \rightarrow \mathcal{P}_2 \rightarrow \underline{\hat{x}}_2 \\ \rightarrow \mathcal{P}_1 \rightarrow \underline{\hat{x}}_1 \\ \rightarrow \mathcal{P}_0 \rightarrow \underline{\hat{x}}_0 \end{array}$$

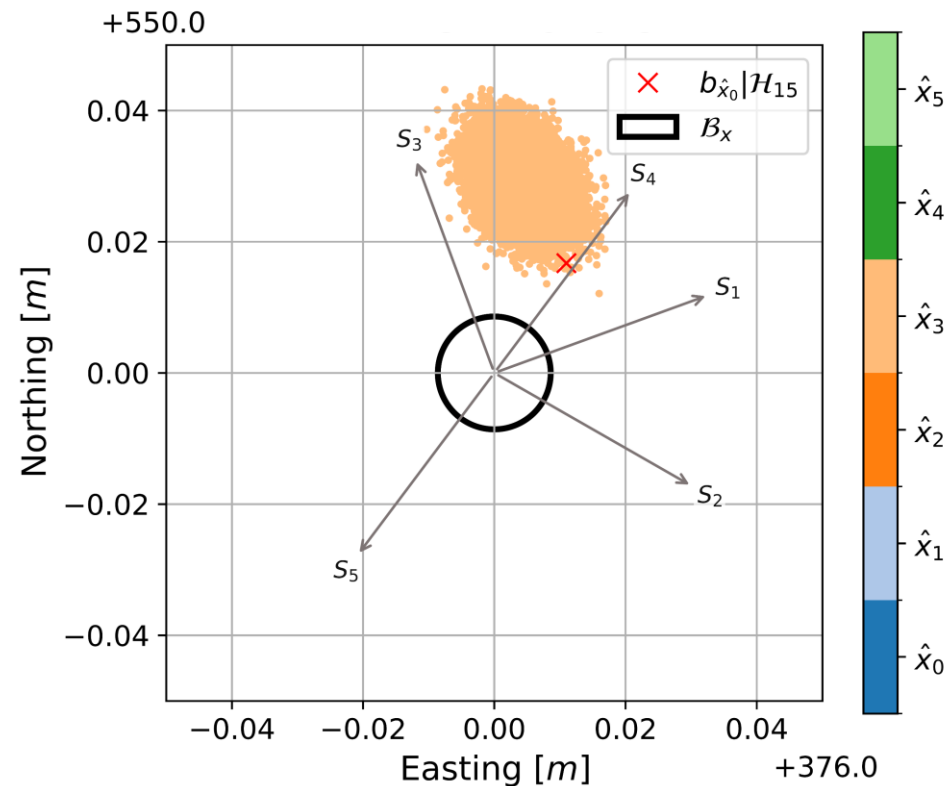
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Wrong identification due to  $\mathcal{H}_3$  ( $S_3$ )



$$\begin{array}{l} \rightarrow \mathcal{P}_5 \rightarrow \underline{\hat{x}}_5 \\ \rightarrow \mathcal{P}_4 \rightarrow \underline{\hat{x}}_4 \\ \rightarrow \mathcal{P}_3 \rightarrow \underline{\hat{x}}_3 \\ \rightarrow \mathcal{P}_2 \rightarrow \underline{\hat{x}}_2 \\ \rightarrow \mathcal{P}_1 \rightarrow \underline{\hat{x}}_1 \\ \rightarrow \mathcal{P}_0 \rightarrow \underline{\hat{x}}_0 \end{array}$$

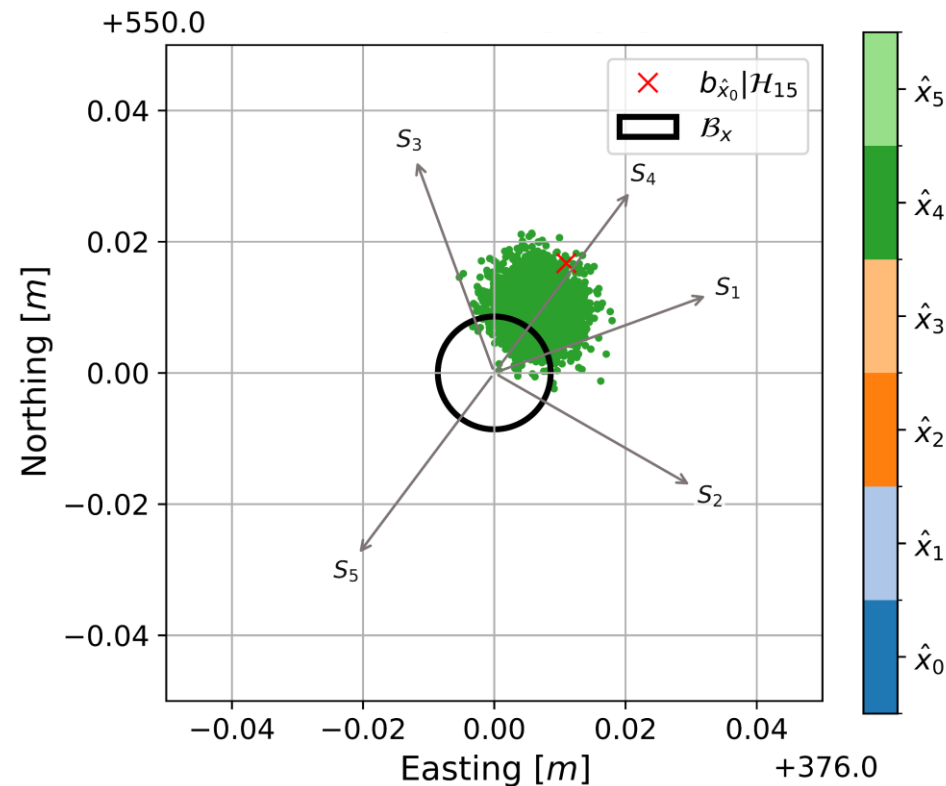
# Example: Capacity-only

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$$\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$$

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Wrong identification due to  $\mathcal{H}_4$  ( $S_4$ )



$$\begin{array}{l} \rightarrow \mathcal{P}_5 \rightarrow \underline{\hat{x}}_5 \\ \rightarrow \mathcal{P}_4 \rightarrow \underline{\hat{x}}_4 \\ \rightarrow \mathcal{P}_3 \rightarrow \underline{\hat{x}}_3 \\ \rightarrow \mathcal{P}_2 \rightarrow \underline{\hat{x}}_2 \\ \rightarrow \mathcal{P}_1 \rightarrow \underline{\hat{x}}_1 \\ \rightarrow \mathcal{P}_0 \rightarrow \underline{\hat{x}}_0 \end{array}$$

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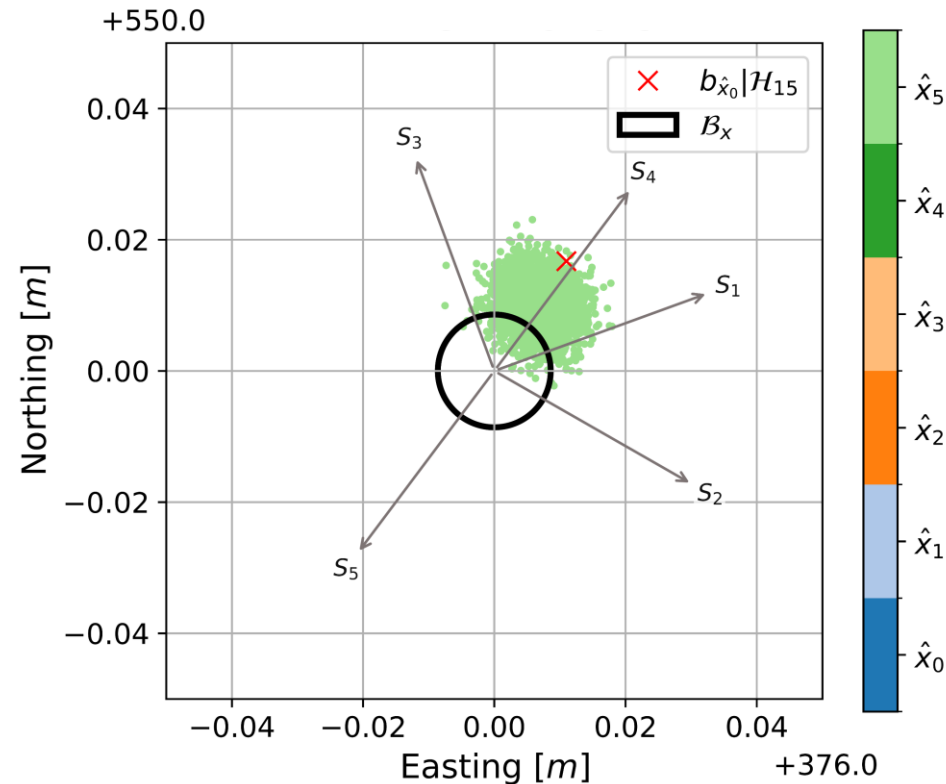
$$\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 1$$

$$N = 5$$

*Datasnooping*

Wrong identification due to  $\mathcal{H}_5$  ( $S_5$ )



$$\underline{t} \in \begin{matrix} \rightarrow \mathcal{P}_5 \rightarrow \underline{\hat{x}}_5 \\ \rightarrow \mathcal{P}_4 \rightarrow \underline{\hat{x}}_4 \\ \rightarrow \mathcal{P}_3 \rightarrow \underline{\hat{x}}_3 \\ \rightarrow \mathcal{P}_2 \rightarrow \underline{\hat{x}}_2 \\ \rightarrow \mathcal{P}_1 \rightarrow \underline{\hat{x}}_1 \\ \rightarrow \mathcal{P}_0 \rightarrow \underline{\hat{x}}_0 \end{matrix}$$

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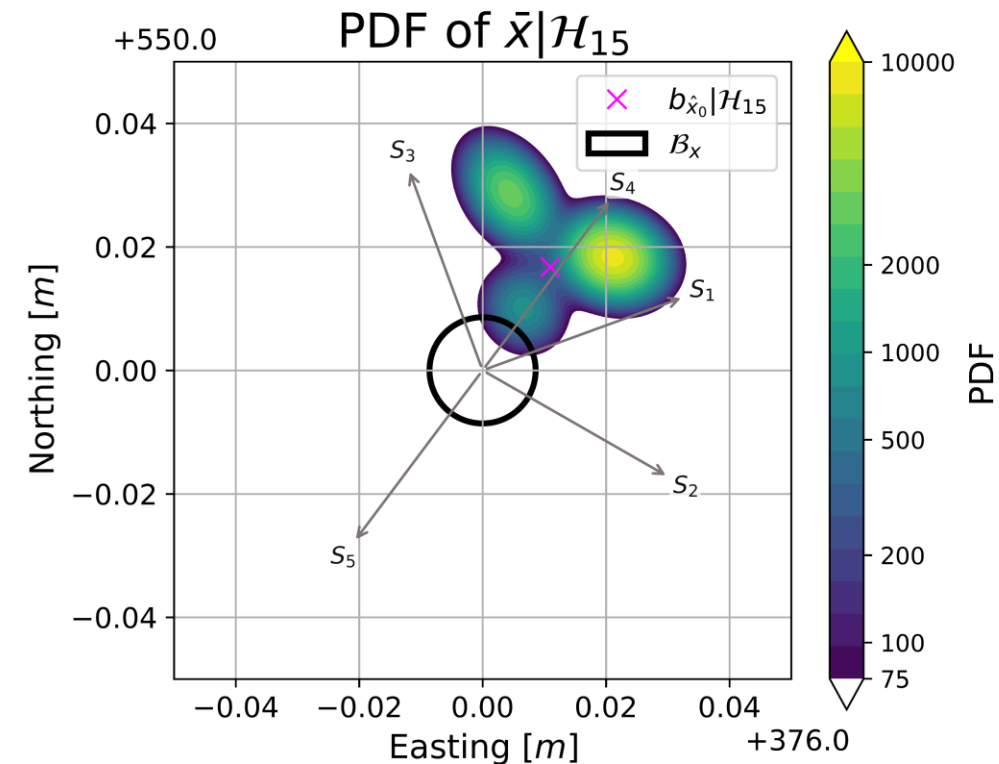
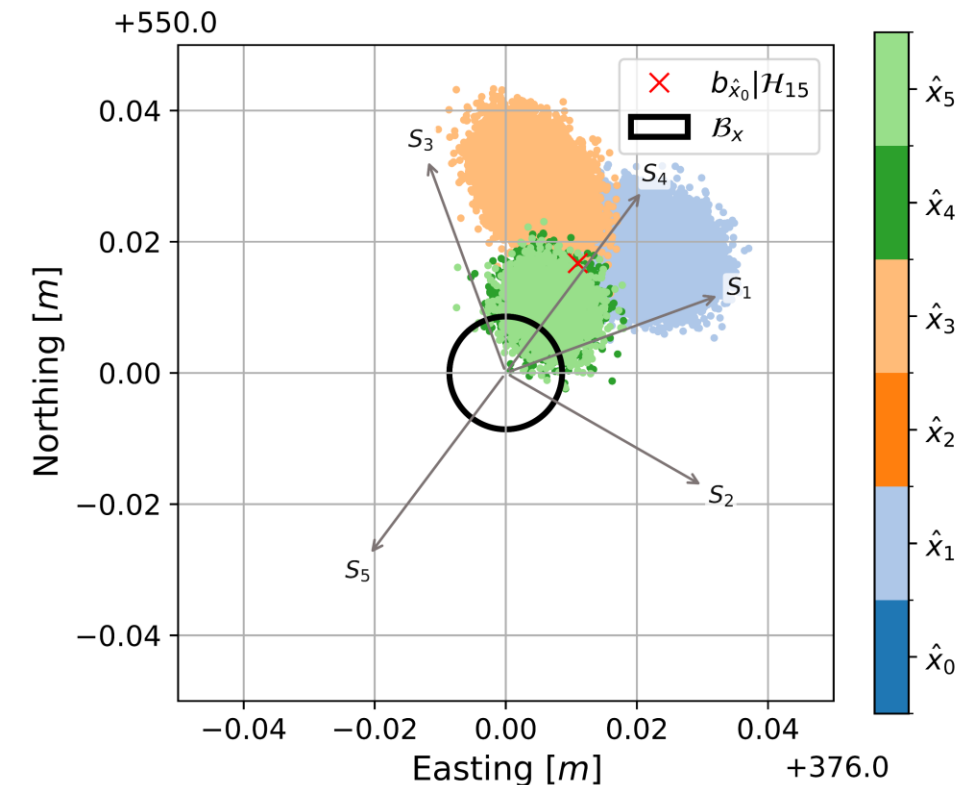
$$\bar{x} = \sum_{i=0}^N \hat{x}_i \mathbf{1}_{\mathcal{P}_i}(\underline{t})$$

$$\text{IC: } \sum_{j=1}^5 \mathbf{z}_j \leq 1$$

$$N = 5$$

*Datasnooping*

$$f_{\bar{x}}(x) = \sum_{i=0}^N \mathbb{E}[f_{\hat{x}_0}(x + A^+ C \bar{Z}(t) \bar{b}(t)) \mathbf{1}_{\mathcal{P}_i}(t)]$$



# Table: Testing Probabilities

$$\left. \begin{array}{l} \text{IC: } \sum_{j=1}^5 \mathbf{z}_j \leq \mathbf{1} \\ N = 5 \end{array} \right\} \text{Datasnooping}$$

► Under  $\mathcal{H}_0$ :

|                                                              | $\mathcal{P}_0$ | $\mathcal{P}_1$ | $\mathcal{P}_2$ | $\mathcal{P}_3$ | $\mathcal{P}_4$ | $\mathcal{P}_5$ |
|--------------------------------------------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $P(\underline{t} \in \mathcal{P}_i \mid \mathcal{H}_0) [\%]$ | 94.89           | 1.14            | 0.97            | 0.88            | 1.03            | 1.08            |

► Under  $\mathcal{H}_{15}$ :

|                                                                 | $\mathcal{P}_0$ | $\mathcal{P}_1$ | $\mathcal{P}_2$ | $\mathcal{P}_3$ | $\mathcal{P}_4$ | $\mathcal{P}_5$ |
|-----------------------------------------------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $P(\underline{t} \in \mathcal{P}_i \mid \mathcal{H}_{15}) [\%]$ | 0.89            | 66.76           | 0.00            | 26.08           | 3.12            | 3.12            |

# Example: Capacity + Rank

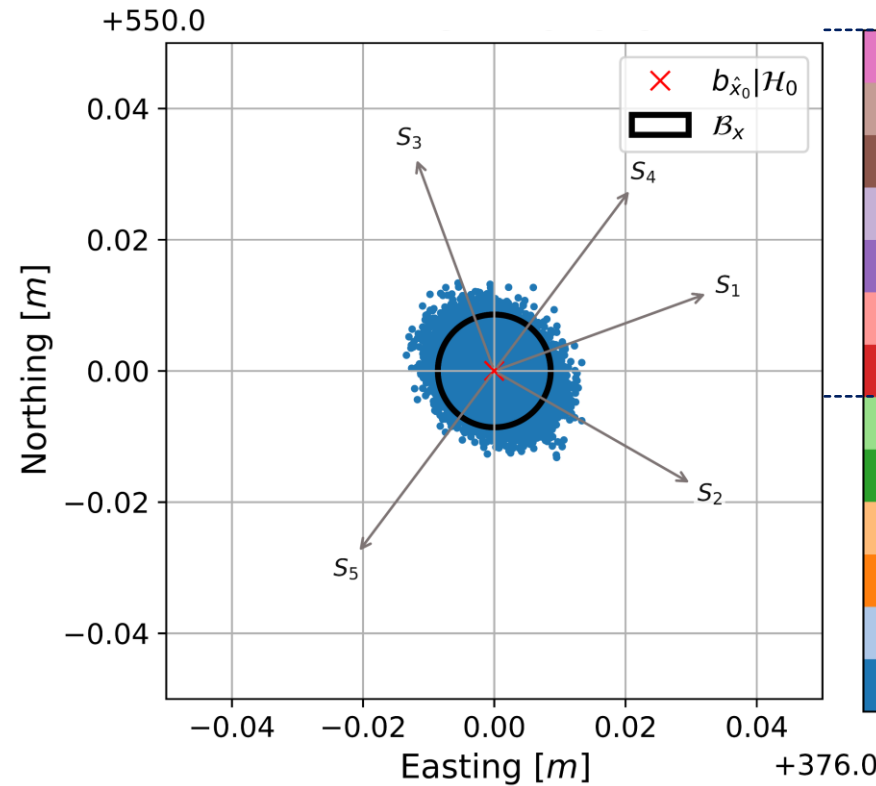
removes  $\hat{x}_6: \{z_1=1, z_2=1\}$   
 $\hat{x}_7: \{z_1=1, z_3=1\}$   
 $\hat{x}_{10}: \{z_2=1, z_3=1\}$

► Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2 \text{ and } z_1 + z_2 + z_3 \leq 1$$

$N = 12$

due to  
non-separability



Correct acceptance

$\underline{t} \in$

$\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 $\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 $\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 $\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 $\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 $\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9$   
 $\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8$   
 $\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5$   
 $\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4$   
 $\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3$   
 $\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2$   
 $\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1$   
 $\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0$

# Example: Capacity + Rank

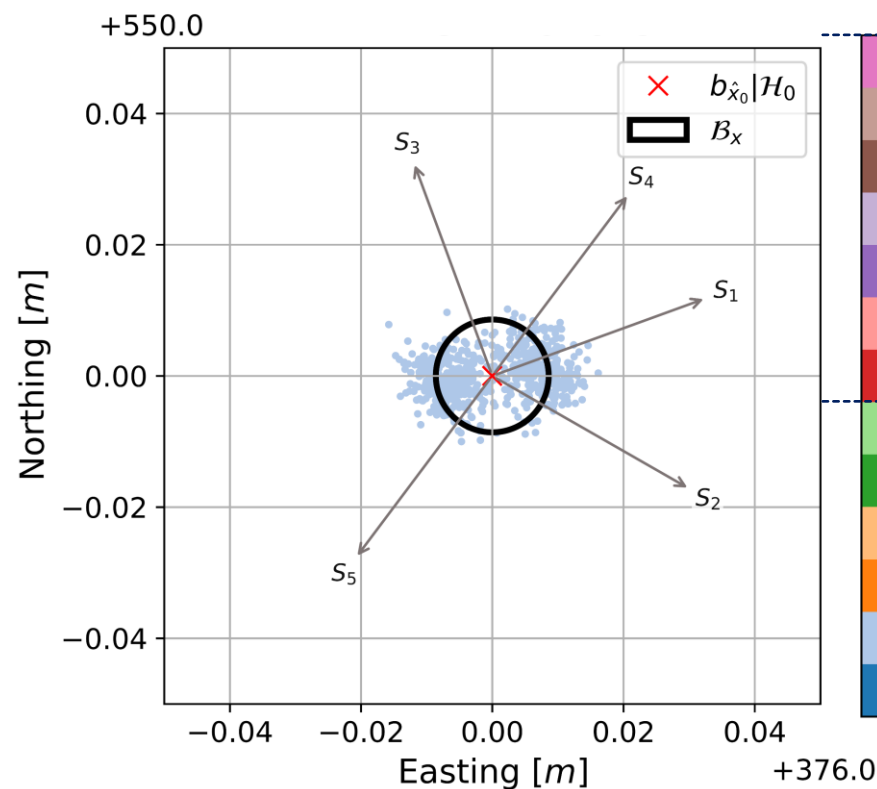
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$N = 12$

due to  
non-separability



False alarm due to  $\mathcal{H}_1$  ( $S_1$ )

$\underline{t} \in$

→  $\mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 →  $\mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 →  $\mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 →  $\mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 →  $\mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 →  $\mathcal{P}_9 \rightarrow \hat{x}_9$   
 →  $\mathcal{P}_8 \rightarrow \hat{x}_8$   
 →  $\mathcal{P}_5 \rightarrow \hat{x}_5$   
 →  $\mathcal{P}_4 \rightarrow \hat{x}_4$   
 →  $\mathcal{P}_3 \rightarrow \hat{x}_3$   
 →  $\mathcal{P}_2 \rightarrow \hat{x}_2$   
 →  $\mathcal{P}_1 \rightarrow \hat{x}_1$   
 →  $\mathcal{P}_0 \rightarrow \hat{x}_0$

# Example: Capacity + Rank

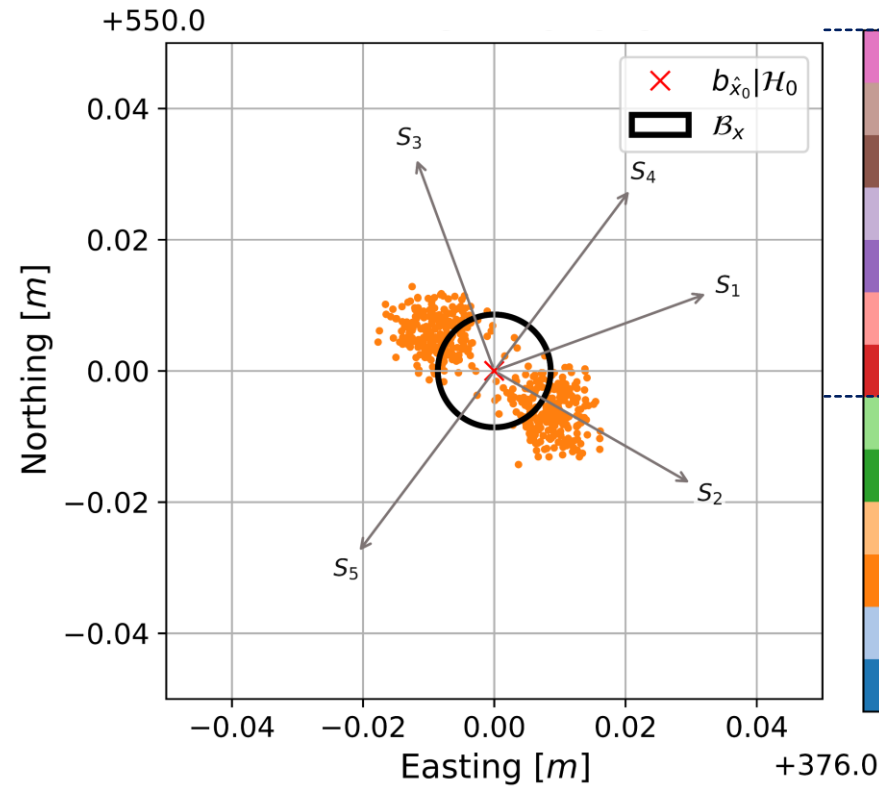
removes  $\hat{x}_6: \{z_1=1, z_2=1\}$   
 $\hat{x}_7: \{z_1=1, z_3=1\}$   
 $\hat{x}_{10}: \{z_2=1, z_3=1\}$

► Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2 \text{ and } \boxed{z_1 + z_2 + z_3 \leq 1}$$

$N = 12$

due to  
non-separability



False alarm due to  $\mathcal{H}_2$  ( $S_2$ )

$\underline{t} \in$

$\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 $\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 $\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 $\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 $\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 $\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9$   
 $\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8$   
 $\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5$   
 $\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4$   
 $\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3$   
 $\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2$   
 $\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1$   
 $\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0$

# Example: Capacity + Rank

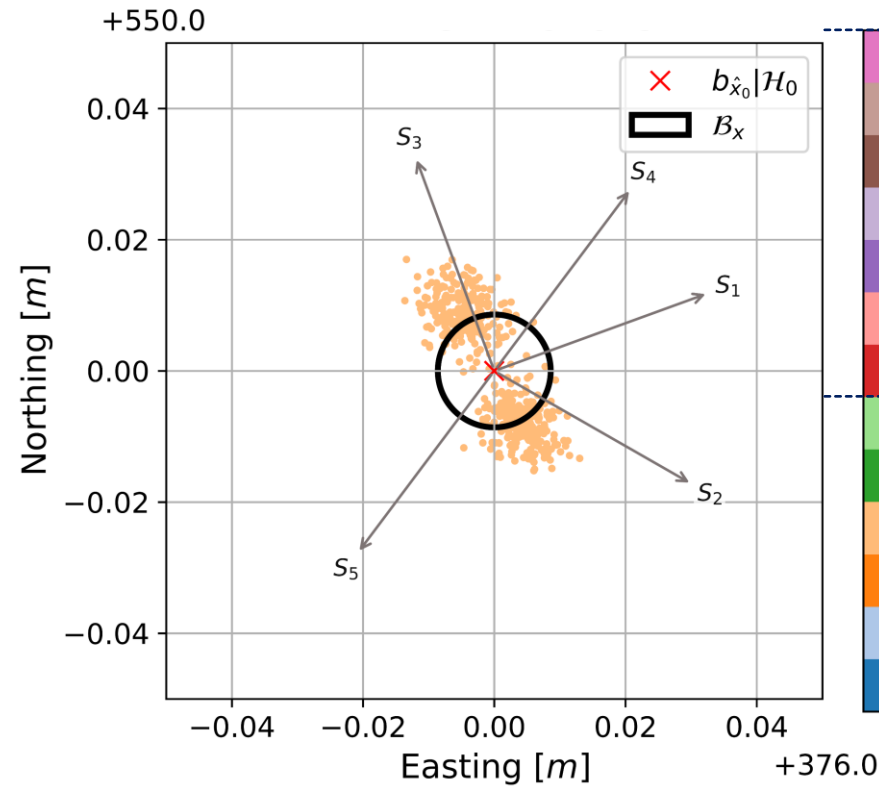
removes  $\hat{x}_6: \{z_1=1, z_2=1\}$   
 $\hat{x}_7: \{z_1=1, z_3=1\}$   
 $\hat{x}_{10}: \{z_2=1, z_3=1\}$

► Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2 \text{ and } \boxed{z_1 + z_2 + z_3 \leq 1}$$

$N = 12$

due to  
non-separability



+

$\underline{t} \in$

→  $\mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 →  $\mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 →  $\mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 →  $\mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 →  $\mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 →  $\mathcal{P}_9 \rightarrow \hat{x}_9$   
 →  $\mathcal{P}_8 \rightarrow \hat{x}_8$   
 →  $\mathcal{P}_5 \rightarrow \hat{x}_5$   
 →  $\mathcal{P}_4 \rightarrow \hat{x}_4$   
 →  $\mathcal{P}_3 \rightarrow \hat{x}_3$   
 →  $\mathcal{P}_2 \rightarrow \hat{x}_2$   
 →  $\mathcal{P}_1 \rightarrow \hat{x}_1$   
 →  $\mathcal{P}_0 \rightarrow \hat{x}_0$

False alarm due to  $\mathcal{H}_3$  ( $S_3$ )

# Example: Capacity + Rank

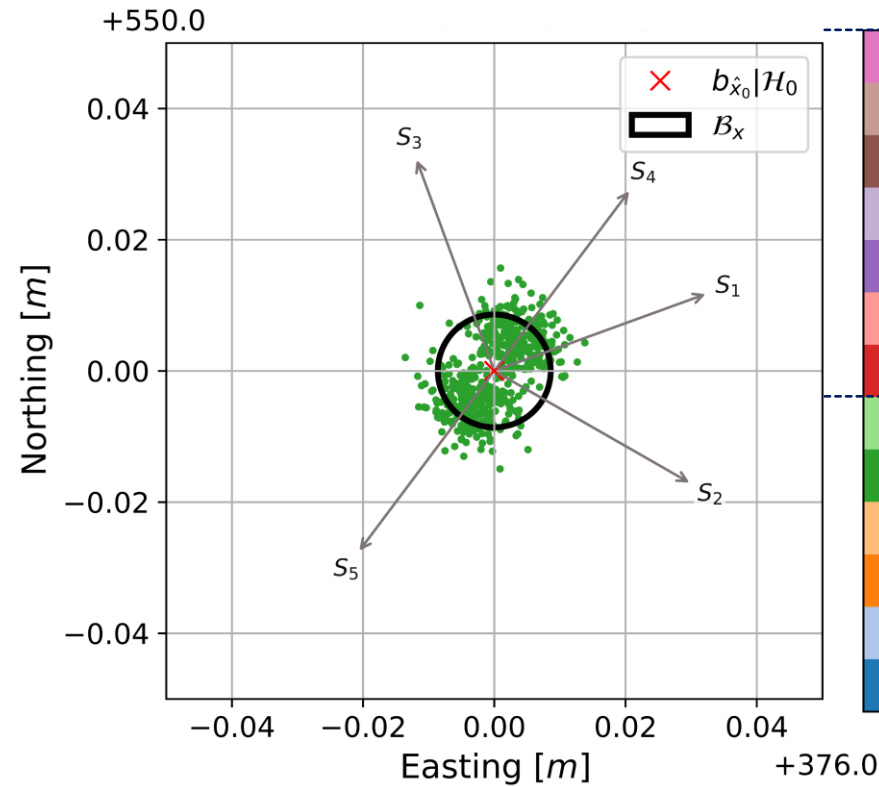
removes  $\hat{x}_6: \{z_1=1, z_2=1\}$   
 $\hat{x}_7: \{z_1=1, z_3=1\}$   
 $\hat{x}_{10}: \{z_2=1, z_3=1\}$

► Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2 \text{ and } \boxed{z_1 + z_2 + z_3 \leq 1}$$

$N = 12$

due to  
non-separability



False alarm due to  $\mathcal{H}_4$  ( $S_4$ )

$\underline{t} \in$

→  $\mathcal{P}_{15}$  →  $\hat{x}_{15}$   
 →  $\mathcal{P}_{14}$  →  $\hat{x}_{14}$   
 →  $\mathcal{P}_{13}$  →  $\hat{x}_{13}$   
 →  $\mathcal{P}_{12}$  →  $\hat{x}_{12}$   
 →  $\mathcal{P}_{11}$  →  $\hat{x}_{11}$   
 →  $\mathcal{P}_9$  →  $\hat{x}_9$   
 →  $\mathcal{P}_8$  →  $\hat{x}_8$   
 →  $\mathcal{P}_5$  →  $\hat{x}_5$   
 →  $\mathcal{P}_4$  →  $\hat{x}_4$   
 →  $\mathcal{P}_3$  →  $\hat{x}_3$   
 →  $\mathcal{P}_2$  →  $\hat{x}_2$   
 →  $\mathcal{P}_1$  →  $\hat{x}_1$   
 →  $\mathcal{P}_0$  →  $\hat{x}_0$

# Example: Capacity + Rank

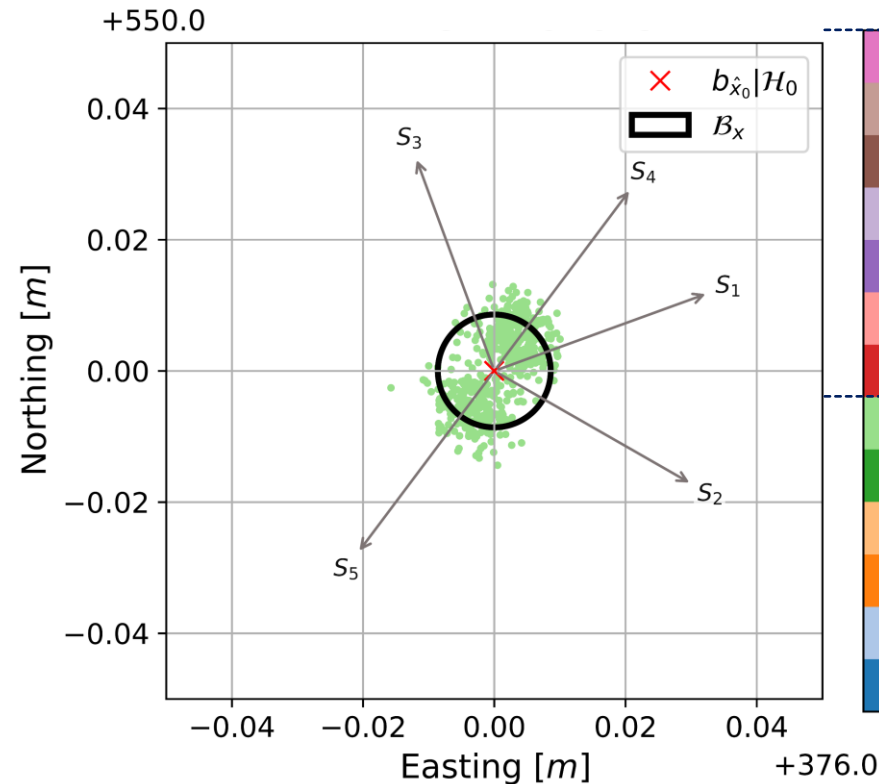
removes  $\hat{x}_6: \{z_1=1, z_2=1\}$   
 $\hat{x}_7: \{z_1=1, z_3=1\}$   
 $\hat{x}_{10}: \{z_2=1, z_3=1\}$

► Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2 \text{ and } \boxed{z_1 + z_2 + z_3 \leq 1}$$

$N = 12$

due to  
non-separability



False alarm due to  $\mathcal{H}_5$  ( $S_5$ )

$\underline{t} \in$

→  $\mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 →  $\mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 →  $\mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 →  $\mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 →  $\mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 →  $\mathcal{P}_9 \rightarrow \hat{x}_9$   
 →  $\mathcal{P}_8 \rightarrow \hat{x}_8$   
 →  $\mathcal{P}_5 \rightarrow \hat{x}_5$   
 →  $\mathcal{P}_4 \rightarrow \hat{x}_4$   
 →  $\mathcal{P}_3 \rightarrow \hat{x}_3$   
 →  $\mathcal{P}_2 \rightarrow \hat{x}_2$   
 →  $\mathcal{P}_1 \rightarrow \hat{x}_1$   
 →  $\mathcal{P}_0 \rightarrow \hat{x}_0$

# Example: Capacity + Rank

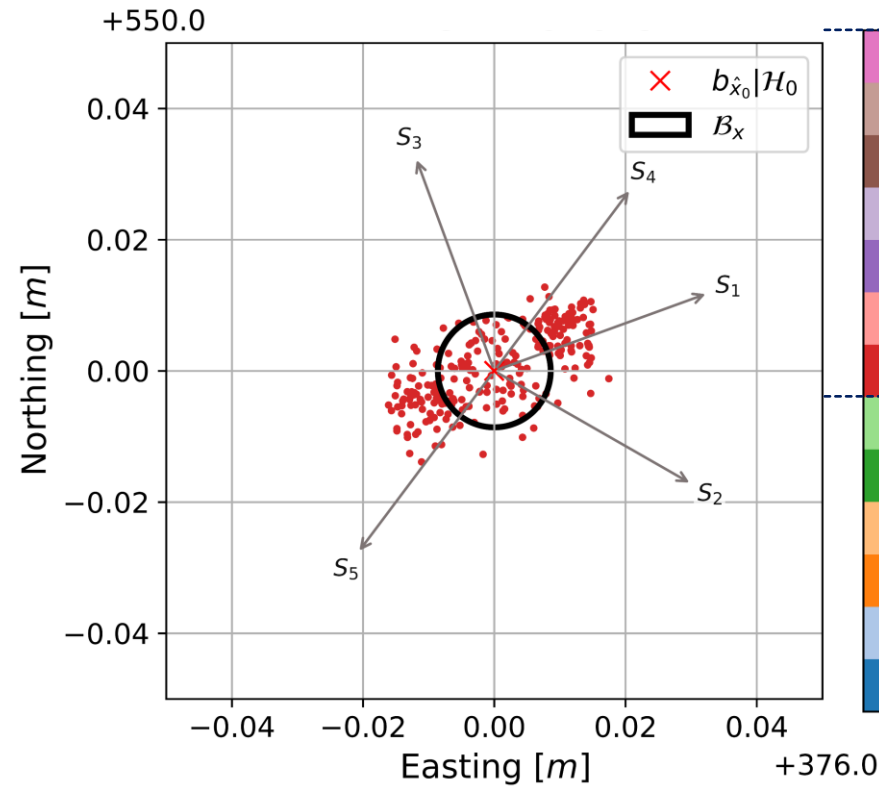
removes  $\hat{x}_6: \{z_1=1, z_2=1\}$   
 $\hat{x}_7: \{z_1=1, z_3=1\}$   
 $\hat{x}_{10}: \{z_2=1, z_3=1\}$

► Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2 \text{ and } \boxed{z_1 + z_2 + z_3 \leq 1}$$

$N = 12$

due to  
non-separability



False alarm due to  $\mathcal{H}_8$  ( $S_1, S_4$ )

$\underline{t} \in$

→  $\mathcal{P}_{15}$  →  $\hat{x}_{15}$   
 →  $\mathcal{P}_{14}$  →  $\hat{x}_{14}$   
 →  $\mathcal{P}_{13}$  →  $\hat{x}_{13}$   
 →  $\mathcal{P}_{12}$  →  $\hat{x}_{12}$   
 →  $\mathcal{P}_{11}$  →  $\hat{x}_{11}$   
 →  $\mathcal{P}_9$  →  $\hat{x}_9$   
 →  $\mathcal{P}_8$  →  $\hat{x}_8$   
 →  $\mathcal{P}_5$  →  $\hat{x}_5$   
 →  $\mathcal{P}_4$  →  $\hat{x}_4$   
 →  $\mathcal{P}_3$  →  $\hat{x}_3$   
 →  $\mathcal{P}_2$  →  $\hat{x}_2$   
 →  $\mathcal{P}_1$  →  $\hat{x}_1$   
 →  $\mathcal{P}_0$  →  $\hat{x}_0$

# Example: Capacity + Rank

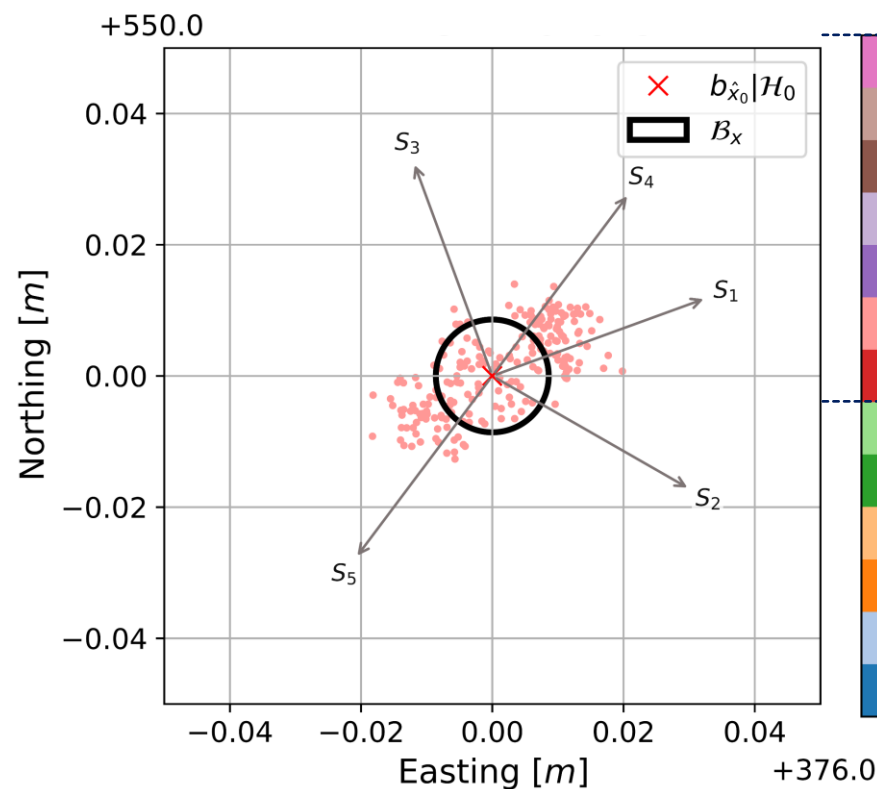
removes  $\hat{x}_6: \{z_1=1, z_2=1\}$   
 $\hat{x}_7: \{z_1=1, z_3=1\}$   
 $\hat{x}_{10}: \{z_2=1, z_3=1\}$

► Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2 \text{ and } \boxed{z_1 + z_2 + z_3 \leq 1}$$

$N = 12$

due to  
non-separability



False alarm due to  $\mathcal{H}_9$  ( $S_1, S_5$ )

$\underline{t} \in$

$\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 $\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 $\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 $\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 $\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 $\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9$   
 $\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8$   
 $\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5$   
 $\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4$   
 $\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3$   
 $\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2$   
 $\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1$   
 $\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0$

# Example: Capacity + Rank

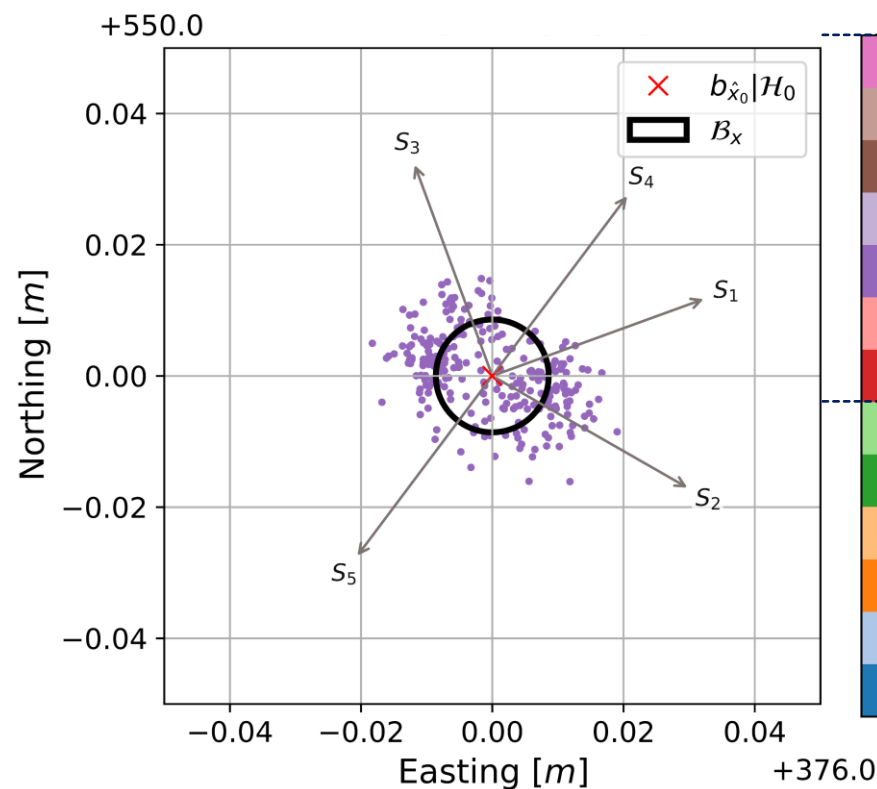
removes  $\hat{x}_6: \{z_1=1, z_2=1\}$   
 $\hat{x}_7: \{z_1=1, z_3=1\}$   
 $\hat{x}_{10}: \{z_2=1, z_3=1\}$

► Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2 \text{ and } \boxed{z_1 + z_2 + z_3 \leq 1}$$

$N = 12$

due to  
non-separability



False alarm due to  $\mathcal{H}_{11}$  ( $S_2, S_4$ )

$\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 $\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 $\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 $\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 $\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 $\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9$   
 $\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8$   
 $\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5$   
 $\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4$   
 $\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3$   
 $\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2$   
 $\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1$   
 $\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0$

# Example: Capacity + Rank

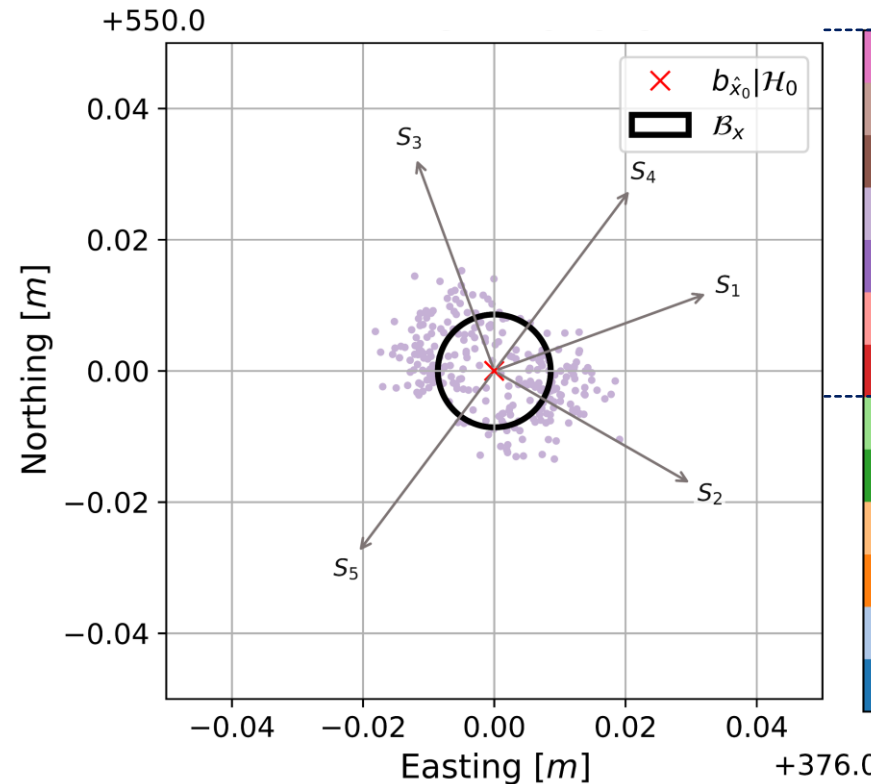
removes  $\hat{x}_6: \{z_1=1, z_2=1\}$   
 $\hat{x}_7: \{z_1=1, z_3=1\}$   
 $\hat{x}_{10}: \{z_2=1, z_3=1\}$

► Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2 \text{ and } \boxed{z_1 + z_2 + z_3 \leq 1}$$

$N = 12$

due to  
non-separability



False alarm due to  $\mathcal{H}_{12} (S_2, S_5)$

$\underline{t} \in$

→  $\mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 →  $\mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 →  $\mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 →  $\mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 →  $\mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 →  $\mathcal{P}_9 \rightarrow \hat{x}_9$   
 →  $\mathcal{P}_8 \rightarrow \hat{x}_8$   
 →  $\mathcal{P}_5 \rightarrow \hat{x}_5$   
 →  $\mathcal{P}_4 \rightarrow \hat{x}_4$   
 →  $\mathcal{P}_3 \rightarrow \hat{x}_3$   
 →  $\mathcal{P}_2 \rightarrow \hat{x}_2$   
 →  $\mathcal{P}_1 \rightarrow \hat{x}_1$   
 →  $\mathcal{P}_0 \rightarrow \hat{x}_0$

# Example: Capacity + Rank

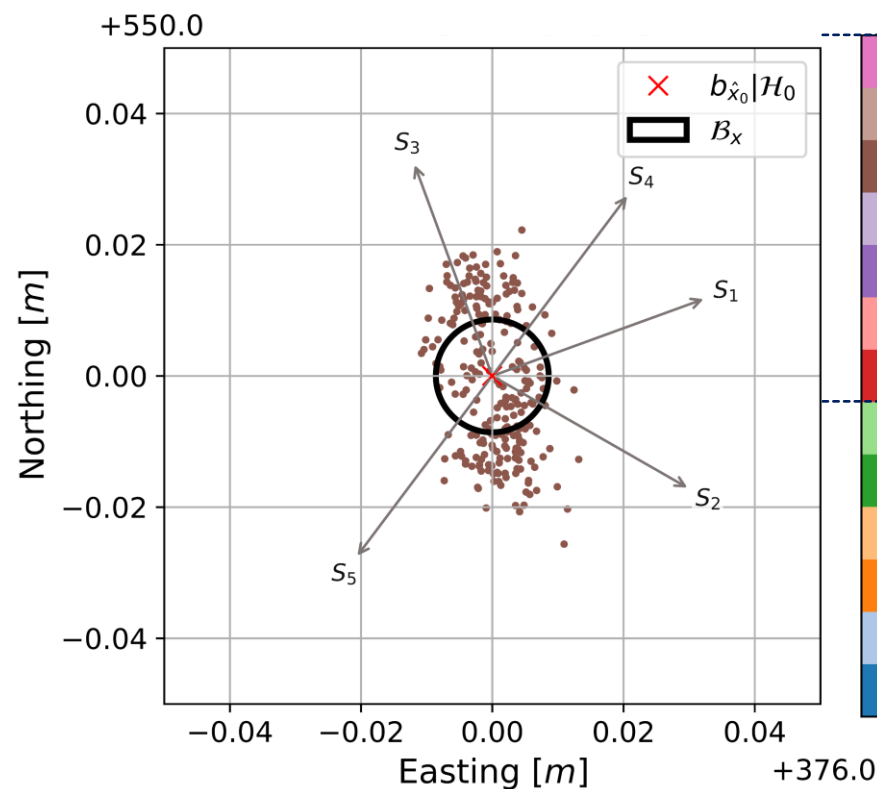
removes  $\hat{x}_6: \{z_1=1, z_2=1\}$   
 $\hat{x}_7: \{z_1=1, z_3=1\}$   
 $\hat{x}_{10}: \{z_2=1, z_3=1\}$

► Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2 \text{ and } \boxed{z_1 + z_2 + z_3 \leq 1}$$

$N = 12$

due to  
non-separability



False alarm due to  $\mathcal{H}_{13}$  ( $S_3, S_4$ )

$\underline{t} \in$

$\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 $\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 $\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 $\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 $\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 $\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9$   
 $\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8$   
 $\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5$   
 $\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4$   
 $\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3$   
 $\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2$   
 $\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1$   
 $\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0$

# Example: Capacity + Rank

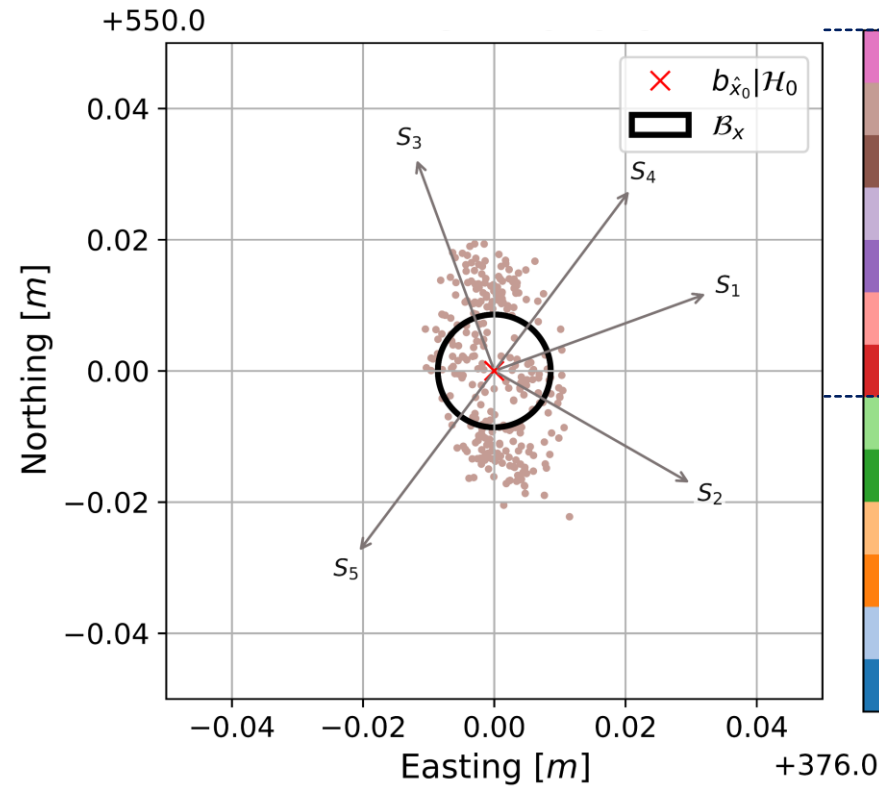
removes  $\hat{x}_6: \{z_1=1, z_2=1\}$   
 $\hat{x}_7: \{z_1=1, z_3=1\}$   
 $\hat{x}_{10}: \{z_2=1, z_3=1\}$

► Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2 \text{ and } \boxed{z_1 + z_2 + z_3 \leq 1}$$

$N = 12$

due to  
non-separability



False alarm due to  $\mathcal{H}_{14}$  ( $S_3, S_5$ )

$\underline{t} \in$

→  $\mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 →  $\mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 →  $\mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 →  $\mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 →  $\mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 →  $\mathcal{P}_9 \rightarrow \hat{x}_9$   
 →  $\mathcal{P}_8 \rightarrow \hat{x}_8$   
 →  $\mathcal{P}_5 \rightarrow \hat{x}_5$   
 →  $\mathcal{P}_4 \rightarrow \hat{x}_4$   
 →  $\mathcal{P}_3 \rightarrow \hat{x}_3$   
 →  $\mathcal{P}_2 \rightarrow \hat{x}_2$   
 →  $\mathcal{P}_1 \rightarrow \hat{x}_1$   
 →  $\mathcal{P}_0 \rightarrow \hat{x}_0$

# Example: Capacity + Rank

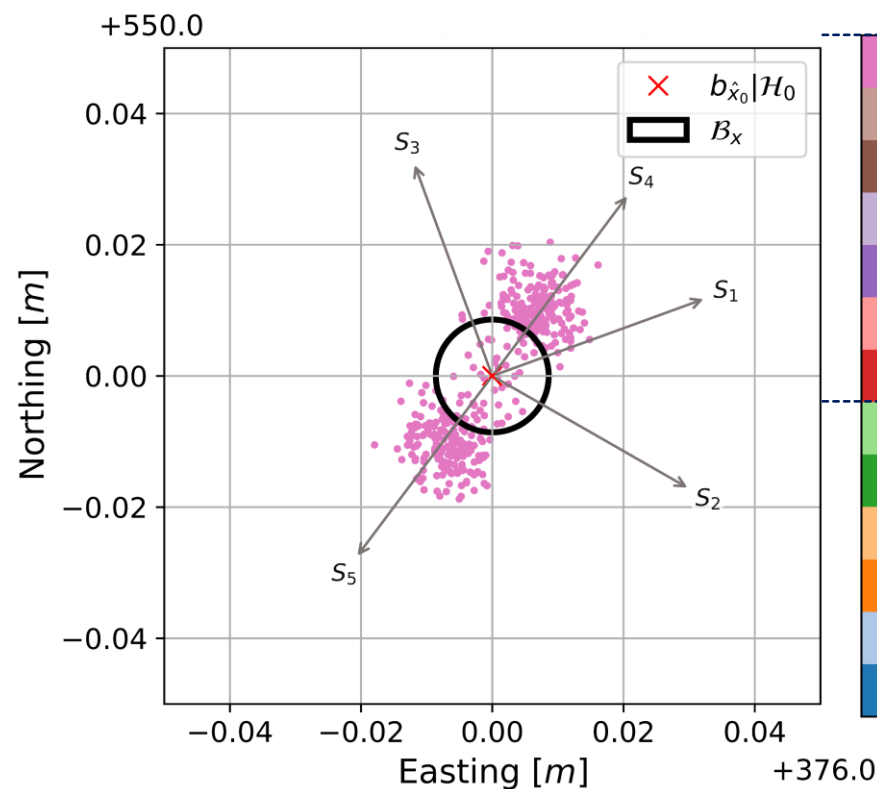
removes  $\hat{x}_6: \{z_1=1, z_2=1\}$   
 $\hat{x}_7: \{z_1=1, z_3=1\}$   
 $\hat{x}_{10}: \{z_2=1, z_3=1\}$

► Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2 \text{ and } \boxed{z_1 + z_2 + z_3 \leq 1}$$

$N = 12$

due to  
non-separability



False alarm due to  $\mathcal{H}_{15}$  ( $S_4, S_5$ )

$\underline{t} \in$

$\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 $\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 $\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 $\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 $\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 $\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9$   
 $\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8$   
 $\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5$   
 $\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4$   
 $\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3$   
 $\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2$   
 $\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1$   
 $\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0$

# Example: Capacity + Rank

removes  $\hat{x}_6: \{z_1=1, z_2=1\}$   
 $\hat{x}_7: \{z_1=1, z_3=1\}$   
 $\hat{x}_{10}: \{z_2=1, z_3=1\}$

► Under null hypothesis  $\mathcal{H}_0$ :

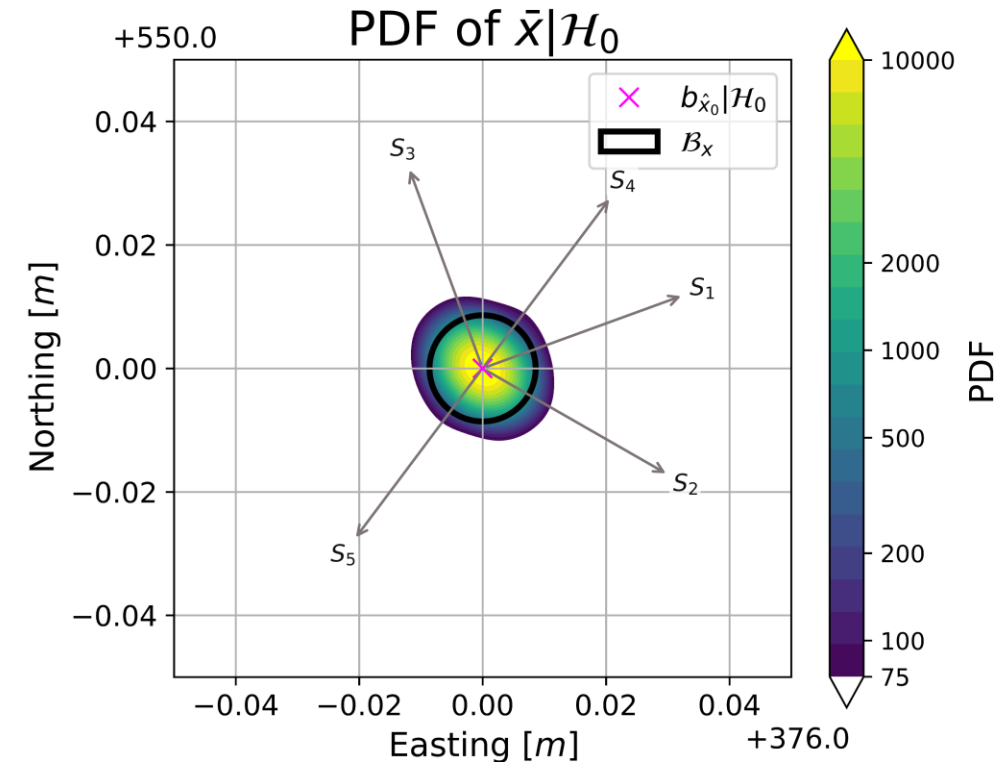
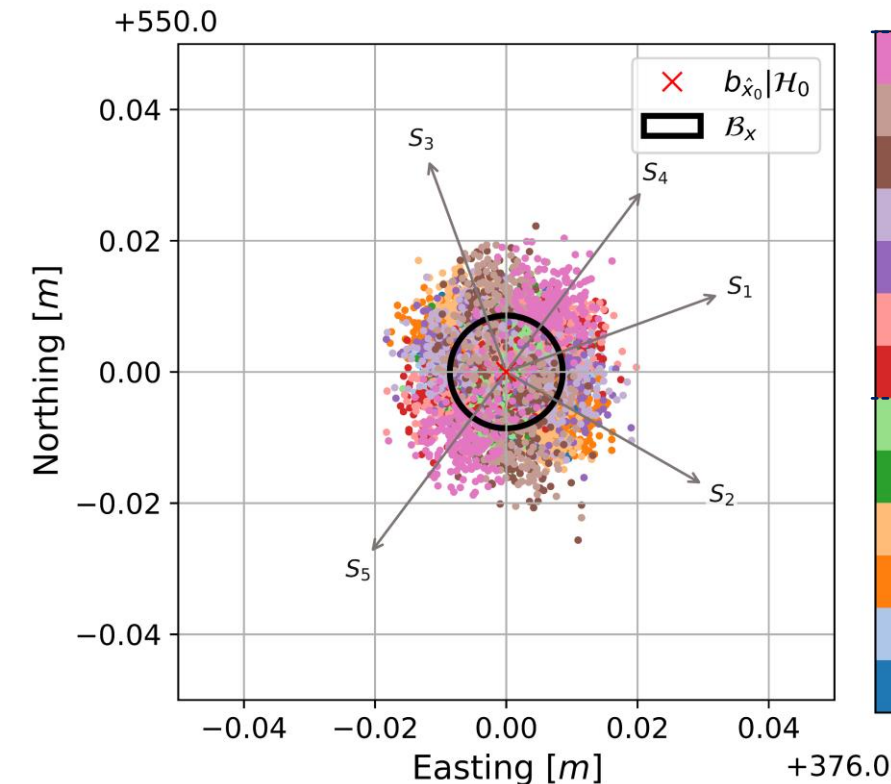
$$\bar{x} = \sum_{i=0}^N \hat{x}_i \mathbf{1}_{\mathcal{P}_i}(t)$$

$$\text{IC: } \sum_{j=1}^5 \mathbf{z}_j \leq 2 \text{ and } \boxed{\mathbf{z}_1 + \mathbf{z}_2 + \mathbf{z}_3 \leq 1}$$

$N = 12$

$$f_{\bar{x}}(x) = \sum_{i=0}^N \mathbb{E}[f_{\hat{x}_i}(x + A^+ C \bar{Z}(t) \bar{b}(t)) \mathbf{1}_{\mathcal{P}_i}(t)]$$

due to  
non-separability



# Example: Capacity + Rank

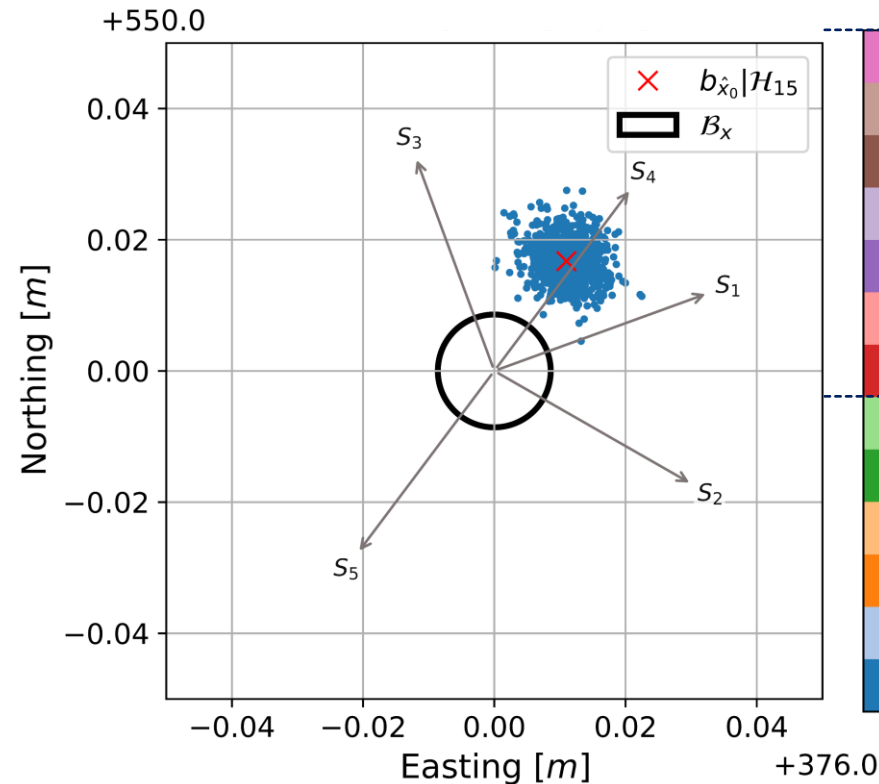
removes  $\hat{x}_6: \{z_1=1, z_2=1\}$   
 $\hat{x}_7: \{z_1=1, z_3=1\}$   
 $\hat{x}_{10}: \{z_2=1, z_3=1\}$

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$   
 $\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2 \text{ and } z_1 + z_2 + z_3 \leq 1$$

$N = 12$

due to  
non-separability



$t \in$

$\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 $\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 $\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 $\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 $\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 $\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9$   
 $\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8$   
 $\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5$   
 $\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4$   
 $\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3$   
 $\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2$   
 $\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1$   
 $\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0$

Missed detection

# Example: Capacity + Rank

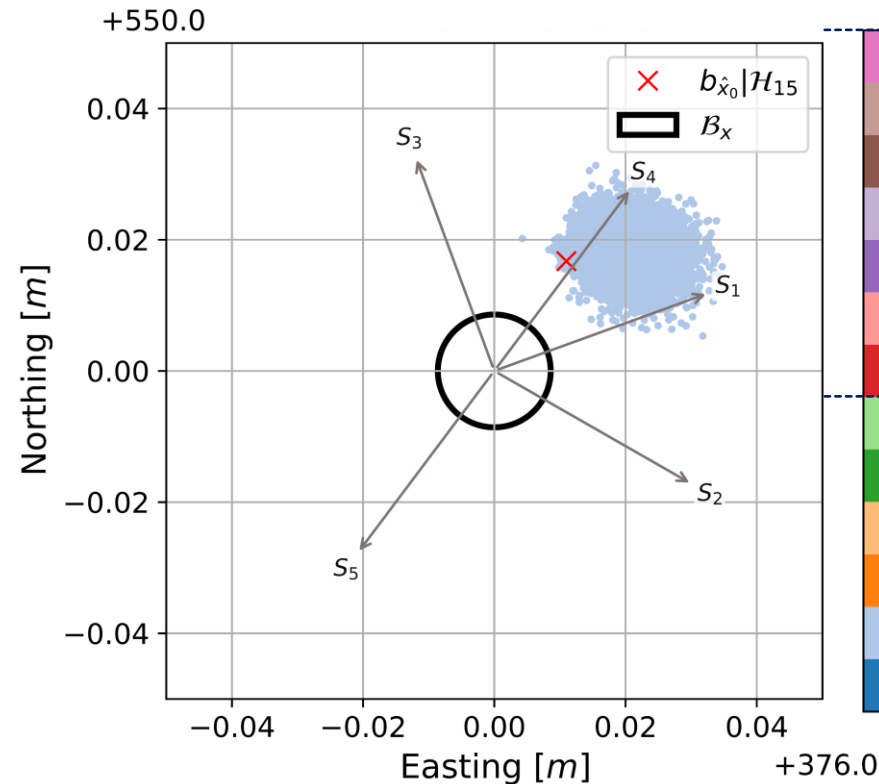
removes  $\hat{x}_6: \{z_1=1, z_2=1\}$   
 $\hat{x}_7: \{z_1=1, z_3=1\}$   
 $\hat{x}_{10}: \{z_2=1, z_3=1\}$

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$   
 $\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2 \text{ and } z_1 + z_2 + z_3 \leq 1$$

$N = 12$

due to  
non-separability



$\underline{t} \in$

$\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 $\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 $\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 $\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 $\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 $\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9$   
 $\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8$   
 $\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5$   
 $\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4$   
 $\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3$   
 $\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2$   
 $\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1$   
 $\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0$

# Example: Capacity + Rank

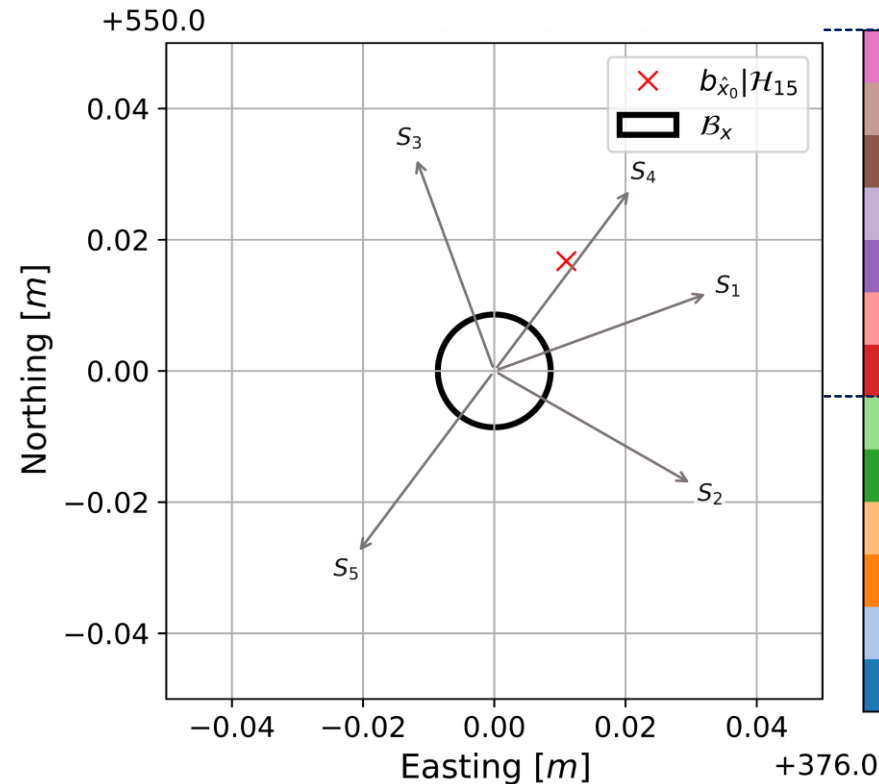
removes  $\hat{x}_6: \{z_1=1, z_2=1\}$   
 $\hat{x}_7: \{z_1=1, z_3=1\}$   
 $\hat{x}_{10}: \{z_2=1, z_3=1\}$

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$   
 $\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2 \text{ and } z_1 + z_2 + z_3 \leq 1$$

$N = 12$

due to  
non-separability



$\underline{t} \in$

$\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 $\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 $\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 $\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 $\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 $\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9$   
 $\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8$   
 $\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5$   
 $\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4$   
 $\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3$   
 $\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2$   
 $\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1$   
 $\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0$

Wrong identification due to  $\mathcal{H}_2 (S_2)$

# Example: Capacity + Rank

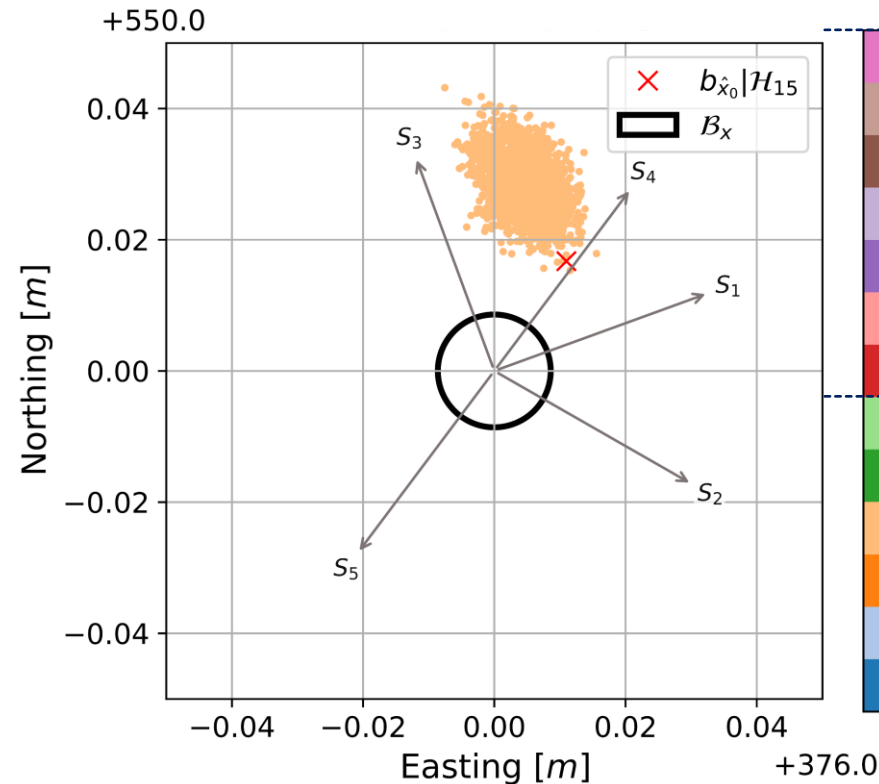
removes  $\hat{x}_6: \{z_1=1, z_2=1\}$   
 $\hat{x}_7: \{z_1=1, z_3=1\}$   
 $\hat{x}_{10}: \{z_2=1, z_3=1\}$

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$   
 $\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2 \text{ and } z_1 + z_2 + z_3 \leq 1$$

$N = 12$

due to  
non-separability



$\underline{t} \in$

$\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 $\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 $\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 $\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 $\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 $\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9$   
 $\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8$   
 $\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5$   
 $\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4$   
 $\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3$   
 $\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2$   
 $\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1$   
 $\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0$

Wrong identification due to  $\mathcal{H}_3 (S_3)$

# Example: Capacity + Rank

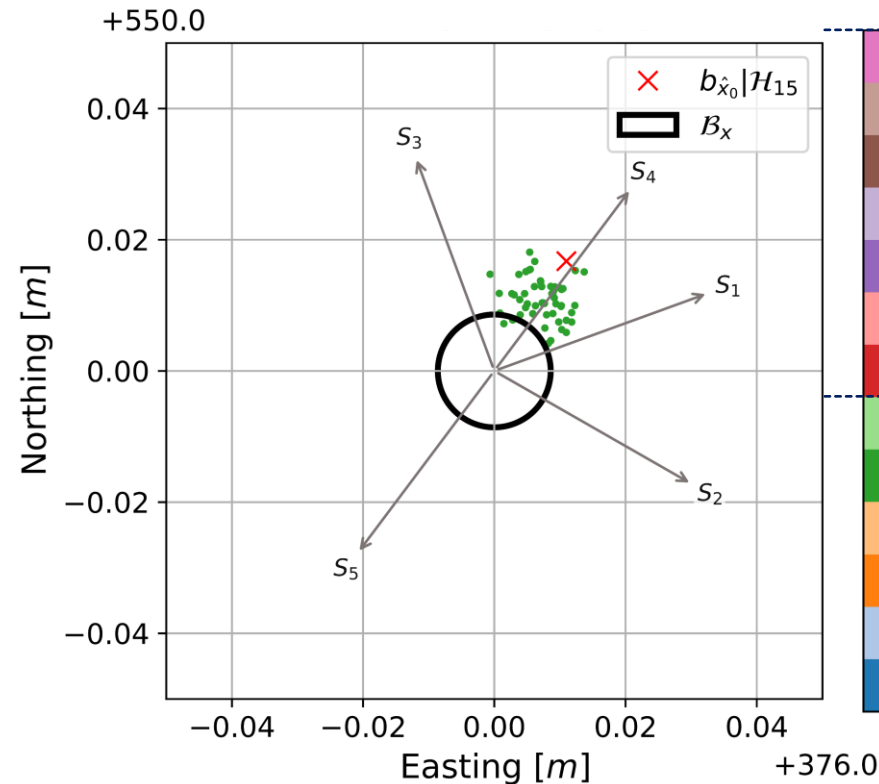
removes  $\hat{x}_6: \{z_1=1, z_2=1\}$   
 $\hat{x}_7: \{z_1=1, z_3=1\}$   
 $\hat{x}_{10}: \{z_2=1, z_3=1\}$

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$   
 $\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2 \text{ and } z_1 + z_2 + z_3 \leq 1$$

$N = 12$

due to  
non-separability



$\underline{t} \in$

→  $\mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 →  $\mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 →  $\mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 →  $\mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 →  $\mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 →  $\mathcal{P}_9 \rightarrow \hat{x}_9$   
 →  $\mathcal{P}_8 \rightarrow \hat{x}_8$   
 →  $\mathcal{P}_5 \rightarrow \hat{x}_5$   
 →  $\mathcal{P}_4 \rightarrow \hat{x}_4$   
 →  $\mathcal{P}_3 \rightarrow \hat{x}_3$   
 →  $\mathcal{P}_2 \rightarrow \hat{x}_2$   
 →  $\mathcal{P}_1 \rightarrow \hat{x}_1$   
 →  $\mathcal{P}_0 \rightarrow \hat{x}_0$

Wrong identification due to  $\mathcal{H}_4 (S_4)$

# Example: Capacity + Rank

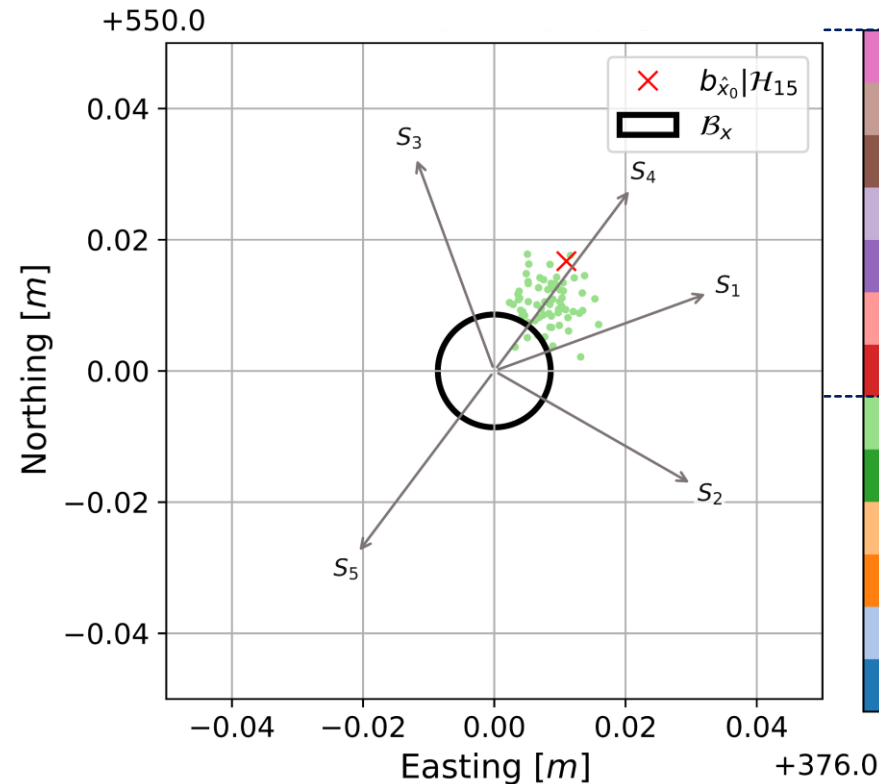
removes  $\hat{x}_6: \{z_1=1, z_2=1\}$   
 $\hat{x}_7: \{z_1=1, z_3=1\}$   
 $\hat{x}_{10}: \{z_2=1, z_3=1\}$

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$   
 $\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2 \text{ and } z_1 + z_2 + z_3 \leq 1$$

$N = 12$

due to  
non-separability



$\underline{t} \in$

→  $\mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 →  $\mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 →  $\mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 →  $\mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 →  $\mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 →  $\mathcal{P}_9 \rightarrow \hat{x}_9$   
 →  $\mathcal{P}_8 \rightarrow \hat{x}_8$   
 →  $\mathcal{P}_5 \rightarrow \hat{x}_5$   
 →  $\mathcal{P}_4 \rightarrow \hat{x}_4$   
 →  $\mathcal{P}_3 \rightarrow \hat{x}_3$   
 →  $\mathcal{P}_2 \rightarrow \hat{x}_2$   
 →  $\mathcal{P}_1 \rightarrow \hat{x}_1$   
 →  $\mathcal{P}_0 \rightarrow \hat{x}_0$

Wrong identification due to  $\mathcal{H}_5 (S_5)$

# Example: Capacity + Rank

removes  $\hat{x}_6: \{z_1=1, z_2=1\}$   
 $\hat{x}_7: \{z_1=1, z_3=1\}$   
 $\hat{x}_{10}: \{z_2=1, z_3=1\}$

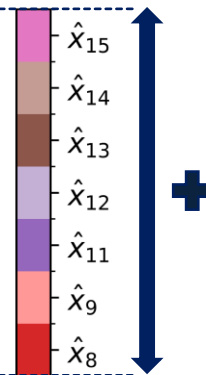
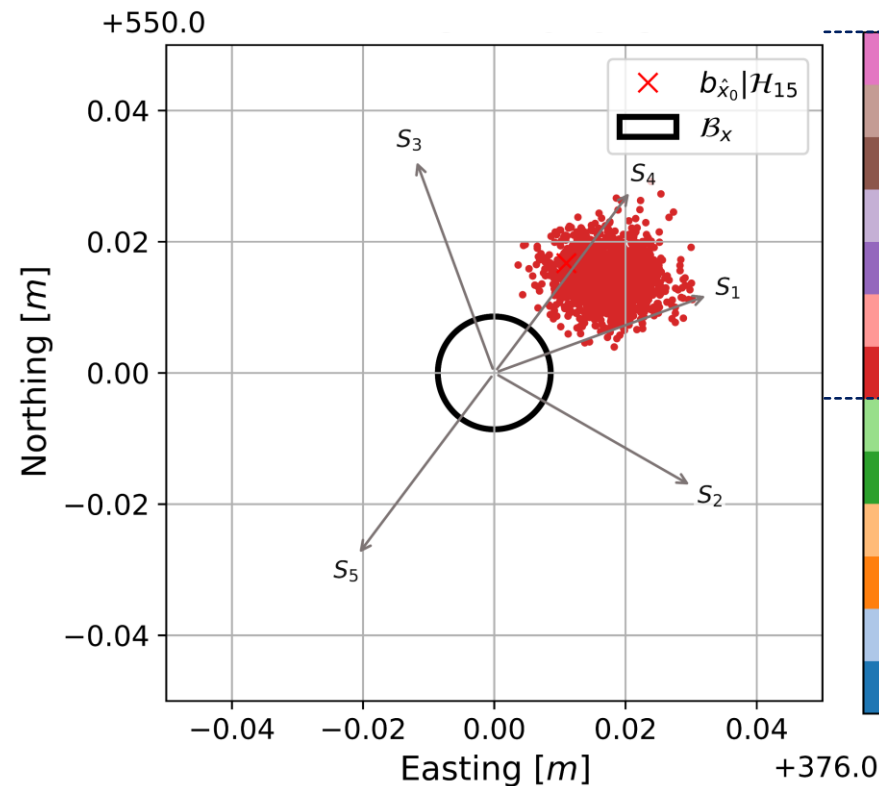
- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$

$$\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2 \text{ and } z_1 + z_2 + z_3 \leq 1$$

$N = 12$

due to  
non-separability



$\underline{t} \in$

$\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 $\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 $\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 $\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 $\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 $\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9$   
 $\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8$   
 $\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5$   
 $\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4$   
 $\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3$   
 $\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2$   
 $\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1$   
 $\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0$

Wrong identification due to  $\mathcal{H}_8 (S_1, S_4)$

# Example: Capacity + Rank

removes  $\hat{x}_6: \{z_1=1, z_2=1\}$   
 $\hat{x}_7: \{z_1=1, z_3=1\}$   
 $\hat{x}_{10}: \{z_2=1, z_3=1\}$

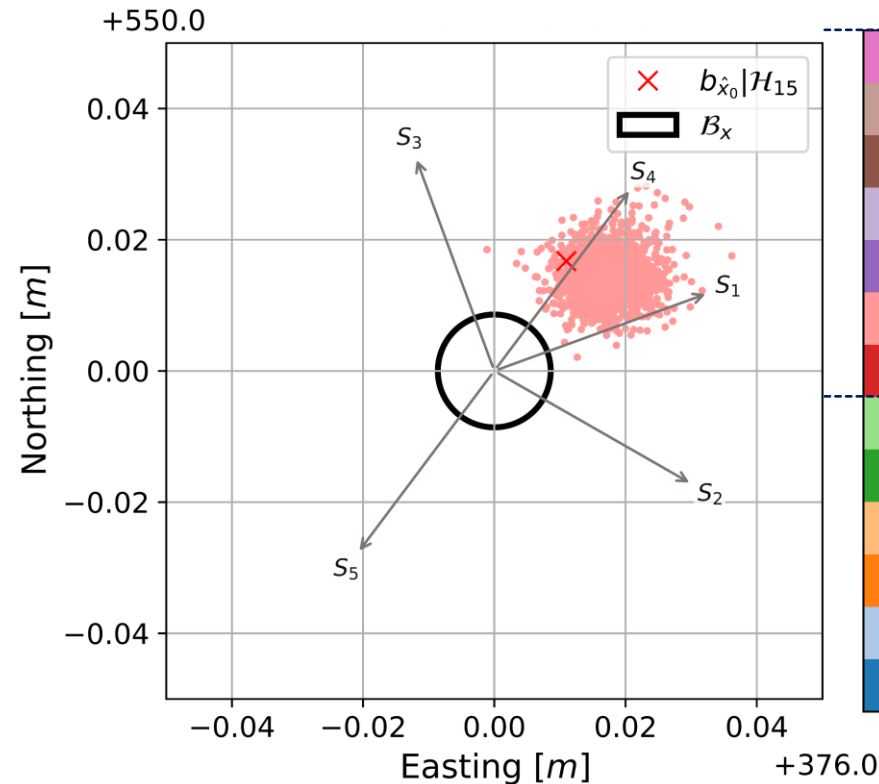
- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$

$$\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2 \text{ and } z_1 + z_2 + z_3 \leq 1$$

$N = 12$

due to  
non-separability



$\underline{t} \in$

$\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 $\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 $\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 $\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 $\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 $\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9$   
 $\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8$   
 $\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5$   
 $\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4$   
 $\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3$   
 $\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2$   
 $\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1$   
 $\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0$

Wrong identification due to  $\mathcal{H}_9 (S_1, S_5)$

# Example: Capacity + Rank

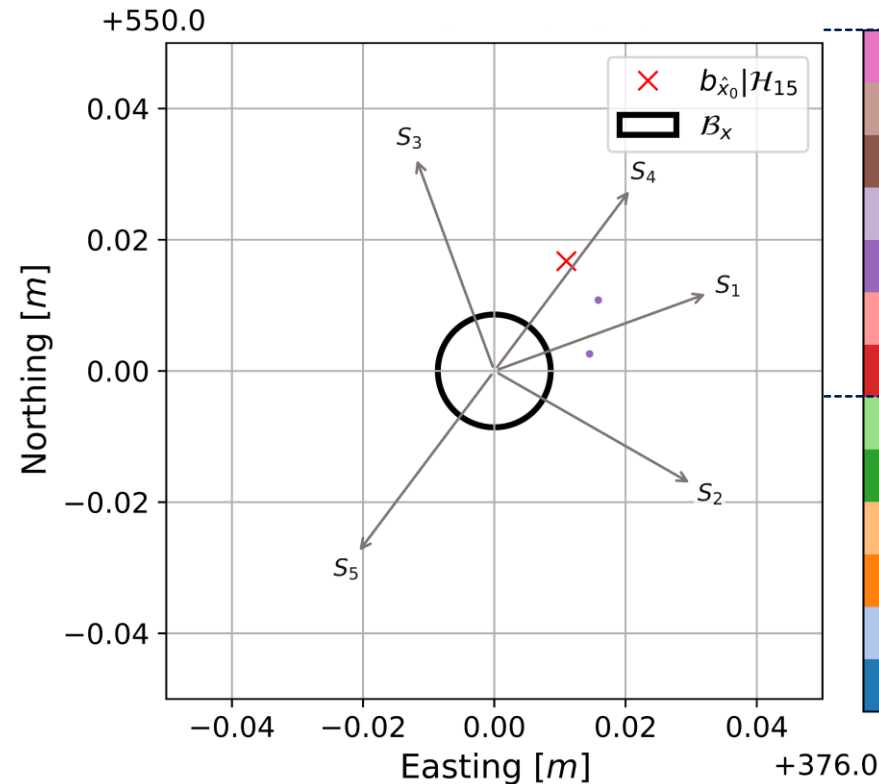
removes  $\hat{x}_6: \{z_1=1, z_2=1\}$   
 $\hat{x}_7: \{z_1=1, z_3=1\}$   
 $\hat{x}_{10}: \{z_2=1, z_3=1\}$

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$   
 $\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2 \text{ and } z_1 + z_2 + z_3 \leq 1$$

$N = 12$

due to  
non-separability



$\underline{t} \in$

→  $\mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 →  $\mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 →  $\mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 →  $\mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 →  $\mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 →  $\mathcal{P}_9 \rightarrow \hat{x}_9$   
 →  $\mathcal{P}_8 \rightarrow \hat{x}_8$   
 →  $\mathcal{P}_5 \rightarrow \hat{x}_5$   
 →  $\mathcal{P}_4 \rightarrow \hat{x}_4$   
 →  $\mathcal{P}_3 \rightarrow \hat{x}_3$   
 →  $\mathcal{P}_2 \rightarrow \hat{x}_2$   
 →  $\mathcal{P}_1 \rightarrow \hat{x}_1$   
 →  $\mathcal{P}_0 \rightarrow \hat{x}_0$

# Example: Capacity + Rank

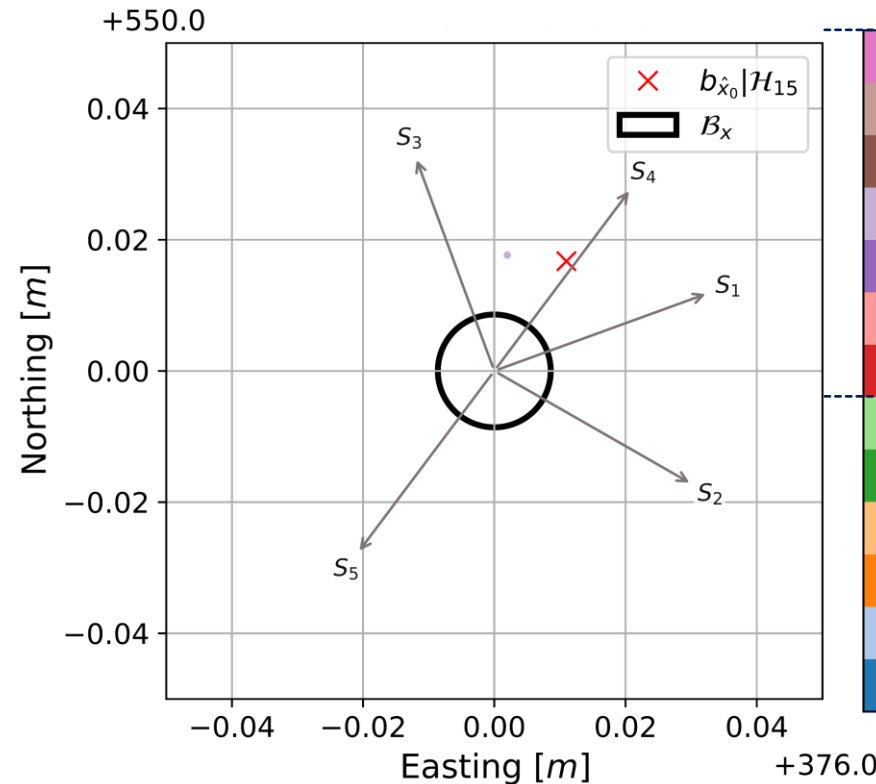
removes  $\hat{x}_6: \{z_1=1, z_2=1\}$   
 $\hat{x}_7: \{z_1=1, z_3=1\}$   
 $\hat{x}_{10}: \{z_2=1, z_3=1\}$

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$   
 $\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2 \text{ and } z_1 + z_2 + z_3 \leq 1$$

$N = 12$

due to  
non-separability



$\underline{t} \in$

- $\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15}$
- $\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14}$
- $\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13}$
- $\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12}$
- $\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11}$
- $\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9$
- $\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8$
- $\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5$
- $\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4$
- $\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3$
- $\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2$
- $\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1$
- $\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0$

# Example: Capacity + Rank

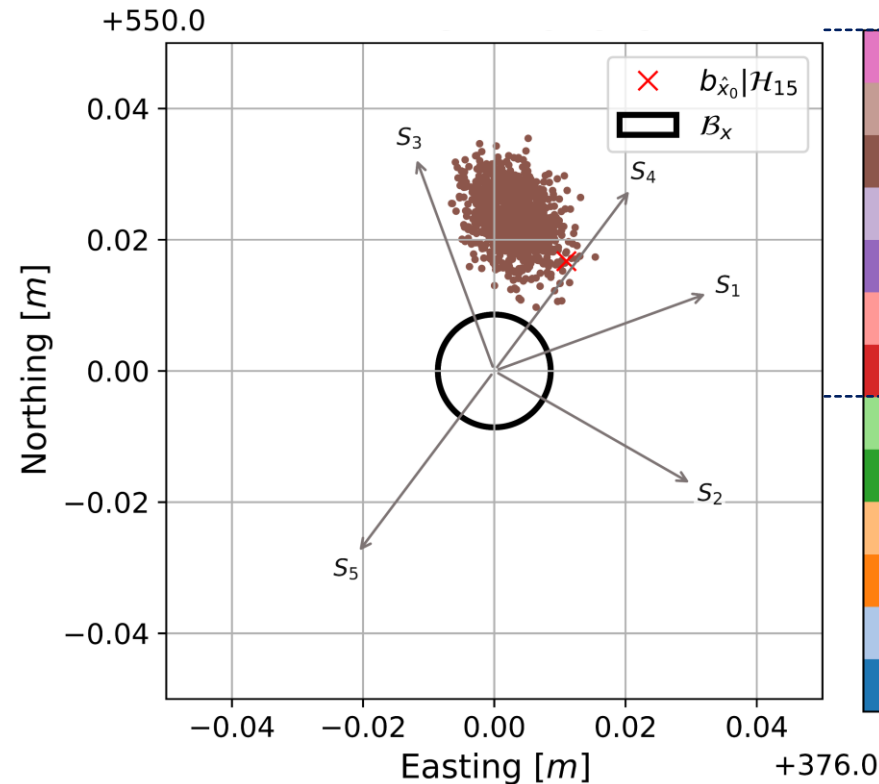
removes  $\hat{x}_6: \{z_1=1, z_2=1\}$   
 $\hat{x}_7: \{z_1=1, z_3=1\}$   
 $\hat{x}_{10}: \{z_2=1, z_3=1\}$

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$   
 $\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2 \text{ and } z_1 + z_2 + z_3 \leq 1$$

$N = 12$

due to  
non-separability



$\underline{t} \in$

→  $\mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 →  $\mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 →  $\mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 →  $\mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 →  $\mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 →  $\mathcal{P}_9 \rightarrow \hat{x}_9$   
 →  $\mathcal{P}_8 \rightarrow \hat{x}_8$   
 →  $\mathcal{P}_5 \rightarrow \hat{x}_5$   
 →  $\mathcal{P}_4 \rightarrow \hat{x}_4$   
 →  $\mathcal{P}_3 \rightarrow \hat{x}_3$   
 →  $\mathcal{P}_2 \rightarrow \hat{x}_2$   
 →  $\mathcal{P}_1 \rightarrow \hat{x}_1$   
 →  $\mathcal{P}_0 \rightarrow \hat{x}_0$

Wrong identification due to  $\mathcal{H}_{13} (S_3, S_4)$

# Example: Capacity + Rank

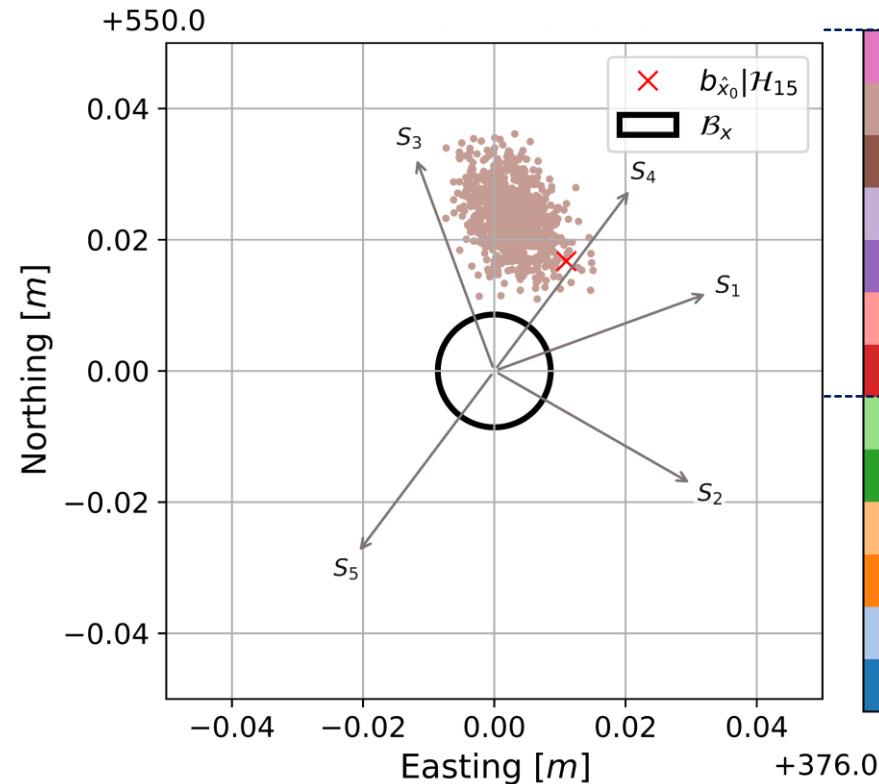
removes  $\hat{x}_6: \{z_1=1, z_2=1\}$   
 $\hat{x}_7: \{z_1=1, z_3=1\}$   
 $\hat{x}_{10}: \{z_2=1, z_3=1\}$

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$   
 $\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2 \text{ and } \boxed{z_1 + z_2 + z_3 \leq 1}$$

$N = 12$

due to  
non-separability



$\underline{t} \in$

$\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 $\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 $\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 $\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 $\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 $\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9$   
 $\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8$   
 $\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5$   
 $\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4$   
 $\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3$   
 $\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2$   
 $\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1$   
 $\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0$

Wrong identification due to  $\mathcal{H}_{14} (S_3, S_5)$

# Example: Capacity + Rank

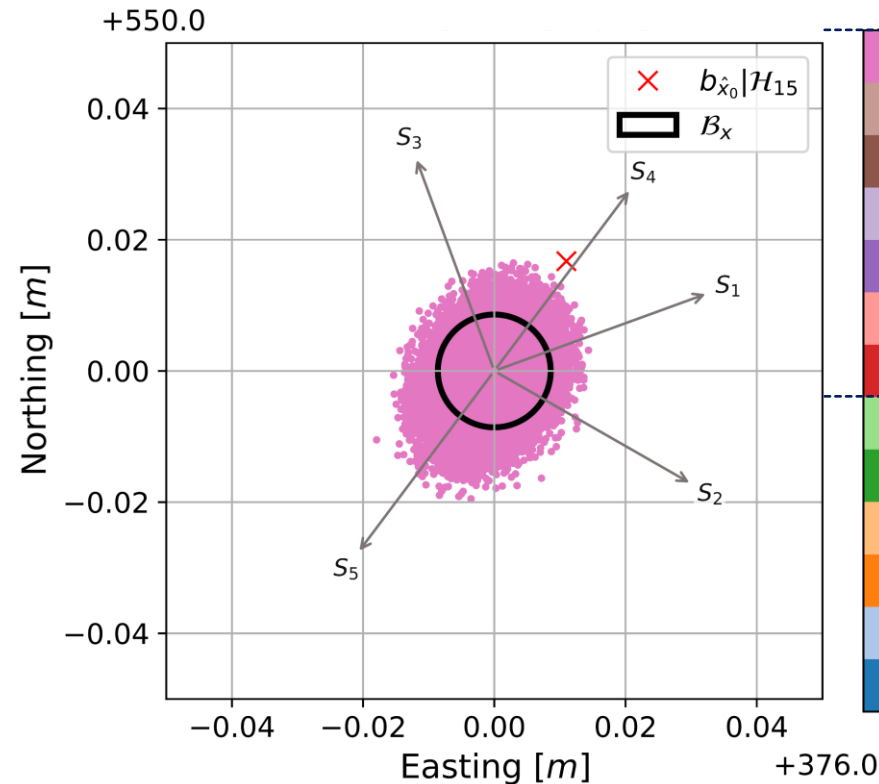
removes  $\hat{x}_6: \{z_1=1, z_2=1\}$   
 $\hat{x}_7: \{z_1=1, z_3=1\}$   
 $\hat{x}_{10}: \{z_2=1, z_3=1\}$

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$   
 $\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2 \text{ and } z_1 + z_2 + z_3 \leq 1$$

$N = 12$

due to  
non-separability



$\underline{t} \in$

$\mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 $\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 $\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 $\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 $\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 $\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9$   
 $\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8$   
 $\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5$   
 $\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4$   
 $\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3$   
 $\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2$   
 $\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1$   
 $\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0$

# Example: Capacity + Rank

removes  $\hat{x}_6: \{z_1=1, z_2=1\}$   
 $\hat{x}_7: \{z_1=1, z_3=1\}$   
 $\hat{x}_{10}: \{z_2=1, z_3=1\}$

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$

$$\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$$

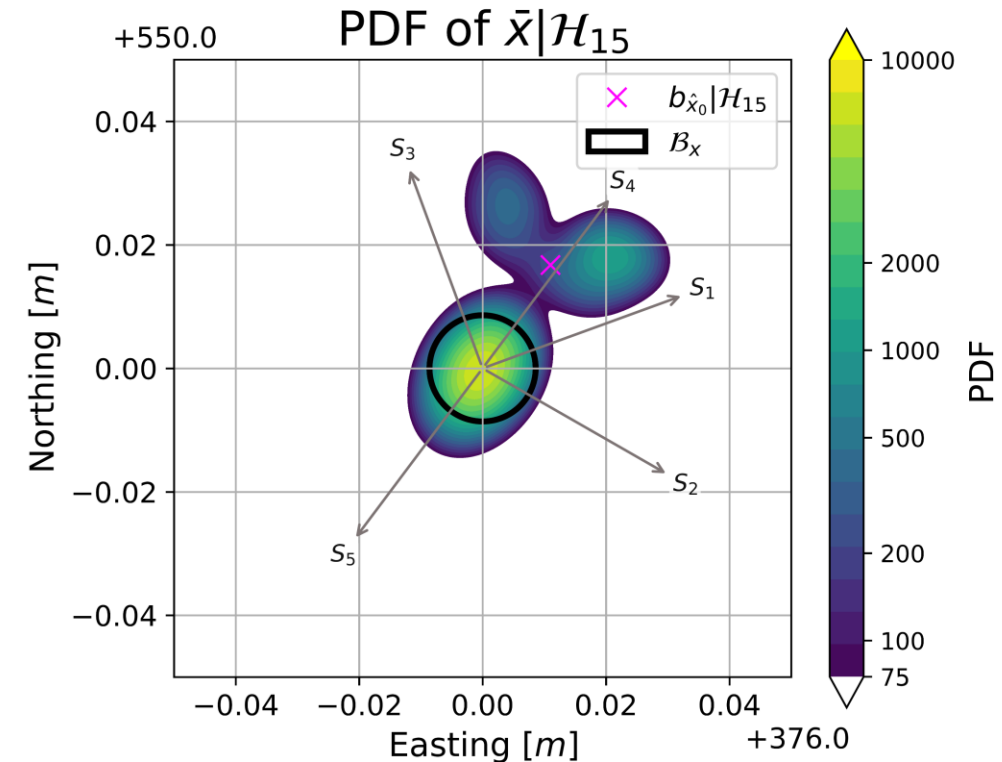
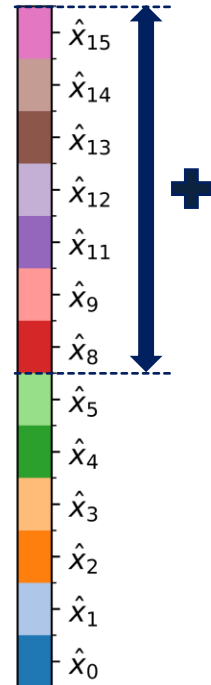
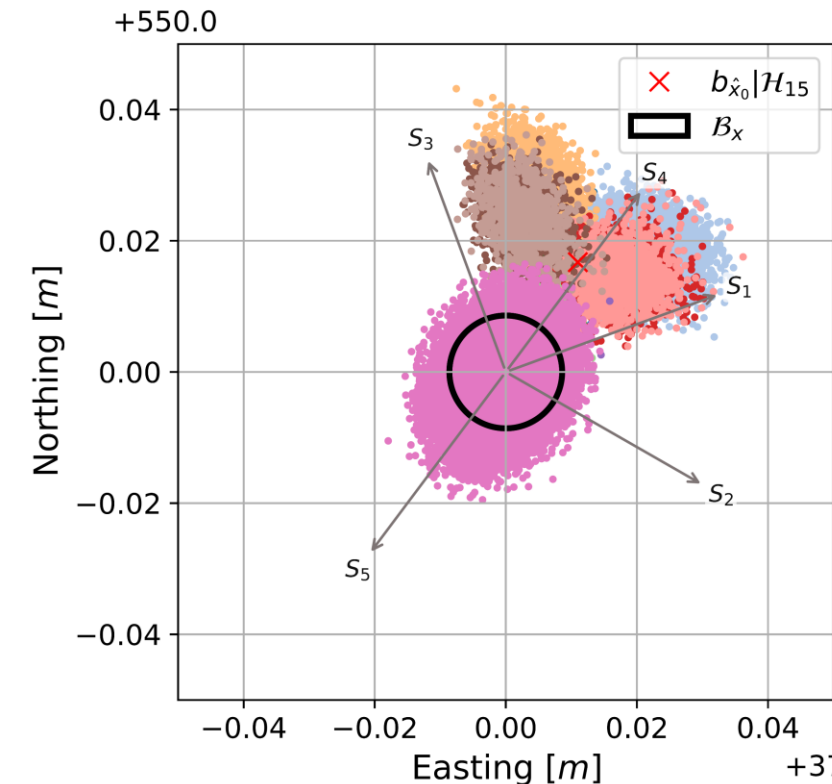
$$\bar{x} = \sum_{i=0}^N \hat{x}_i \mathbf{1}_{\mathcal{P}_i}(\underline{t})$$

$$\text{IC: } \sum_{j=1}^5 \mathbf{z}_j \leq 2 \text{ and } \boxed{\mathbf{z}_1 + \mathbf{z}_2 + \mathbf{z}_3 \leq 1}$$

$N = 12$

$$f_{\bar{x}}(x) = \sum_{i=0}^N \mathbb{E}[f_{\hat{x}_0}(x + A^+ C \bar{Z}(\underline{t}) \bar{b}(\underline{t})) \mathbf{1}_{\mathcal{P}_i}(\underline{t})]$$

due to  
non-separability



$$P(\bar{x} \notin \mathcal{B}_x | \mathcal{H}_{15}) = 99.3\% \Rightarrow P(\bar{x} \notin \mathcal{B}_x | \mathcal{H}_{15}) = 28.5\%$$

# Table: Testing Probabilities

removes  $\hat{x}_6: \{z_1=1, z_2=1\}$   
 $\hat{x}_7: \{z_1=1, z_3=1\}$   
 $\hat{x}_{10}: \{z_2=1, z_3=1\}$

due to  
non-separability

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2 \text{ and } \boxed{z_1 + z_2 + z_3 \leq 1}$$

$N = 12$

► Under  $\mathcal{H}_0$ :

|                                                              | $\mathcal{P}_0$ | $\mathcal{P}_1$ | $\mathcal{P}_2$ | $\mathcal{P}_3$ | $\mathcal{P}_4$ | $\mathcal{P}_5$ | $\mathcal{P}_8$ | $\mathcal{P}_9$ | $\mathcal{P}_{11}$ | $\mathcal{P}_{12}$ | $\mathcal{P}_{13}$ | $\mathcal{P}_{14}$ | $\mathcal{P}_{15}$ |
|--------------------------------------------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|--------------------|--------------------|--------------------|--------------------|--------------------|
| $P(\underline{t} \in \mathcal{P}_i \mid \mathcal{H}_0) [\%]$ | 94.89           | 0.70            | 0.54            | 0.55            | 0.57            | 0.59            | 0.28            | 0.23            | 0.29               | 0.29               | 0.27               | 0.31               | 0.47               |

► Under  $\mathcal{H}_{15}$ :

|                                                                 | $\mathcal{P}_0$ | $\mathcal{P}_1$ | $\mathcal{P}_2$ | $\mathcal{P}_3$ | $\mathcal{P}_4$ | $\mathcal{P}_5$ | $\mathcal{P}_8$ | $\mathcal{P}_9$ | $\mathcal{P}_{11}$ | $\mathcal{P}_{12}$ | $\mathcal{P}_{13}$ | $\mathcal{P}_{14}$ | $\mathcal{P}_{15}$ |
|-----------------------------------------------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|--------------------|--------------------|--------------------|--------------------|--------------------|
| $P(\underline{t} \in \mathcal{P}_i \mid \mathcal{H}_{15}) [\%]$ | 0.89            | 8.89            | 0.00            | 2.39            | 0.06            | 0.07            | 1.66            | 1.67            | 0.00               | 0.00               | 1.20               | 1.13               | 82.02              |

# Conclusion

1. **Detection:** is  $\underline{t} \in \mathcal{P}_0$ ? or not?

2. **Identification:**  $(\bar{Z}(\underline{t}), \bar{b}(\underline{t})) = \arg \min_{\substack{i \in \{1, \dots, N\} \\ b \in \mathbb{R}^q}} [||\underline{t} - B^\top C Z_i b||_{Q_{tt}}^2 + \rho(Z_i)]$

subject to **identifier constraints**

3. **Adaptation:**  $\underline{x} = \hat{x}_0 - A^+ C \bar{Z}(\underline{t}) \bar{b}(\underline{t})$

4. **DIA-estimator:**  $\underline{x} = \sum_{i=0}^N \hat{x}_i \mathbb{1}_{\mathcal{P}_i}(\underline{t}), \quad f_{\underline{x}}(x) = \sum_{i=0}^N \mathbb{E}[f_{\hat{x}_0}(x + A^+ C \bar{Z}(\underline{t}) \bar{b}(\underline{t})) \mathbb{1}_{\mathcal{P}_i}(\underline{t})]$

# Conclusion

1. **Detection:** is  $\underline{t} \in \mathcal{P}_0$ ? or not?

2. **Identification:**  $(\bar{Z}(\underline{t}), \bar{b}(\underline{t})) = \arg \min_{\substack{i \in \{1, \dots, N\} \\ b \in \mathbb{R}^q}} [||\underline{t} - B^\top C Z_i b||_{Q_{tt}}^2 + \rho(Z_i)]$

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3. **Adaptation:**  $\underline{x} = \hat{x}_0 - A^+ C \bar{Z}(\underline{t}) \bar{b}(\underline{t})$

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- ✓ **Flexible:** one library  $C$  generates every  $\mathcal{H}_i$ , single and combined
- ✓ **Automatic:** solver searches  $\mathcal{Z}$  by branching, combinations are never listed
- ✓ **Scalable:** a new fault is one more  $C_j$ , a new restriction one more constraint
- ✓ **Direct:** identified pattern  $\bar{Z}(\underline{t})$  prescribes the adaptation
- ✓ **Complete:** full distribution of  $\underline{x}$ , and probabilities of every testing decision

# Acknowledgements

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All support is gratefully appreciated!

**More math:**

DIA-estimator:  
(Teunissen, 2018)



**Thank you  
for your attention!**

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# References

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# Back-up Slides

# Impact of Testing on the DIA-estimator

| Decision \ Reality                                | $\mathcal{H}_0$ | $\mathcal{H}_1$   | $\mathcal{H}_2$   | ...      | $\mathcal{H}_N$   |
|---------------------------------------------------|-----------------|-------------------|-------------------|----------|-------------------|
| $\mathcal{H}_0 : \underline{t} \in \mathcal{P}_0$ | CA              | MD <sub>1</sub>   | MD <sub>2</sub>   | ...      | MD <sub>N</sub>   |
| $\mathcal{H}_1 : \underline{t} \in \mathcal{P}_1$ | FA <sub>1</sub> | CI <sub>1</sub>   | WI <sub>2,1</sub> | ...      | WI <sub>N,1</sub> |
| $\mathcal{H}_2 : \underline{t} \in \mathcal{P}_2$ | FA <sub>2</sub> | WI <sub>1,2</sub> | CI <sub>2</sub>   | ...      | WI <sub>N,2</sub> |
| $\vdots$                                          | $\vdots$        | $\vdots$          | $\vdots$          | $\ddots$ | $\vdots$          |
| $\mathcal{H}_N : \underline{t} \in \mathcal{P}_N$ | FA <sub>N</sub> | WI <sub>1,N</sub> | WI <sub>2,N</sub> | ...      | CI <sub>N</sub>   |

CA: Correct Acceptance

CI<sub>*i*</sub>: Correct Identification under  $\mathcal{H}_i$

FA<sub>*i*</sub>: False Alarm due to  $\mathcal{H}_i$

MD<sub>*i*</sub>: Missed Detection under  $\mathcal{H}_i$

WI<sub>*i,j*</sub>: Wrong Identification under  $\mathcal{H}_i$  due to  $\mathcal{H}_j$

# Impact of Testing on the DIA-estimator

► What if the true hypothesis is missing?

| Decision \ Reality                                | $\mathcal{H}_0$ | $\mathcal{H}_1$   | $\mathcal{H}_2$   | ...      | $\mathcal{H}_N$   | $\mathcal{H}_{N+1}$ |
|---------------------------------------------------|-----------------|-------------------|-------------------|----------|-------------------|---------------------|
| $\mathcal{H}_0 : \underline{t} \in \mathcal{P}_0$ | CA              | MD <sub>1</sub>   | MD <sub>2</sub>   | ...      | MD <sub>N</sub>   | MD <sub>N+1</sub>   |
| $\mathcal{H}_1 : \underline{t} \in \mathcal{P}_1$ | FA <sub>1</sub> | CI <sub>1</sub>   | WI <sub>2,1</sub> | ...      | WI <sub>N,1</sub> | WI <sub>N+1,1</sub> |
| $\mathcal{H}_2 : \underline{t} \in \mathcal{P}_2$ | FA <sub>2</sub> | WI <sub>1,2</sub> | CI <sub>2</sub>   | ...      | WI <sub>N,2</sub> | WI <sub>N+1,2</sub> |
| $\vdots$                                          | $\vdots$        | $\vdots$          | $\vdots$          | $\ddots$ | $\vdots$          | $\vdots$            |
| $\mathcal{H}_N : \underline{t} \in \mathcal{P}_N$ | FA <sub>N</sub> | WI <sub>1,N</sub> | WI <sub>2,N</sub> | ...      | CI <sub>N</sub>   | WI <sub>N+1,N</sub> |

CA: Correct Acceptance

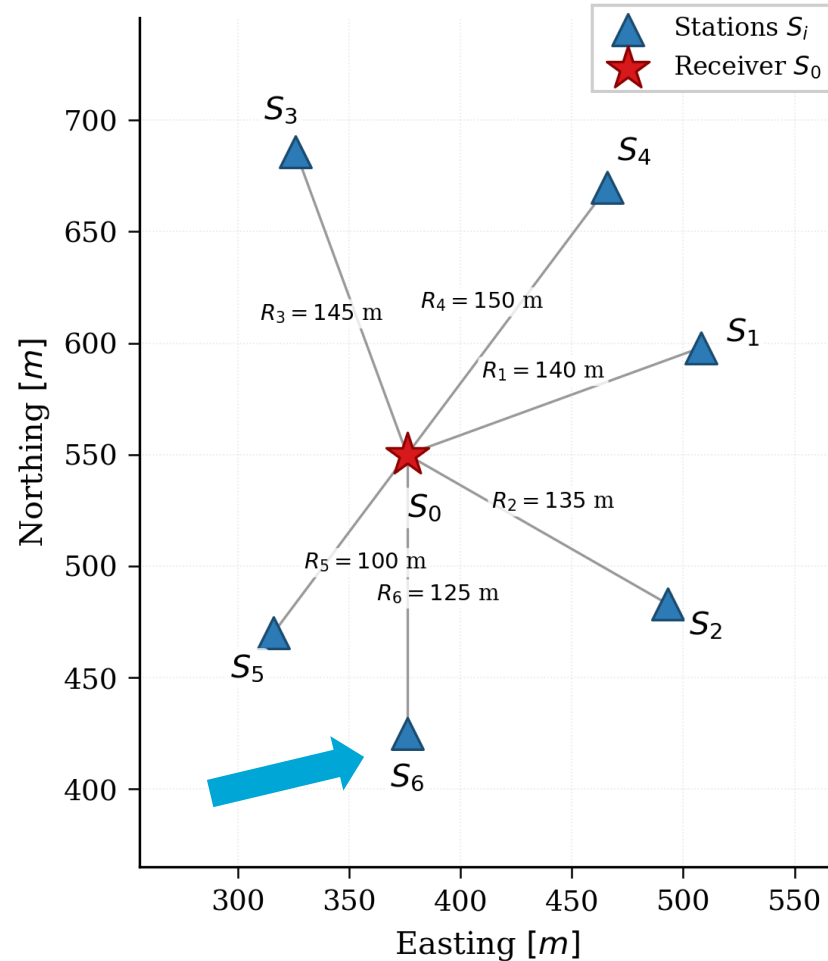
CI<sub>*i*</sub>: Correct Identification under  $\mathcal{H}_i$

FA<sub>*i*</sub>: False Alarm due to  $\mathcal{H}_i$

MD<sub>*i*</sub>: Missed Detection under  $\mathcal{H}_i$

WI<sub>*i,j*</sub>: Wrong Identification under  $\mathcal{H}_i$  due to  $\mathcal{H}_j$

# Adding a Sixth Station



## ► Model:

$m = 6$  ranges,  $n = 2$  coordinates, redundancy  $r = 4$

## ► Fault library:

$C = [C_1, \dots, C_5] \in \mathbb{R}^{6 \times 5}$ , one outlier per range except  $S_6$

$S_6$  adds redundancy, not a candidate

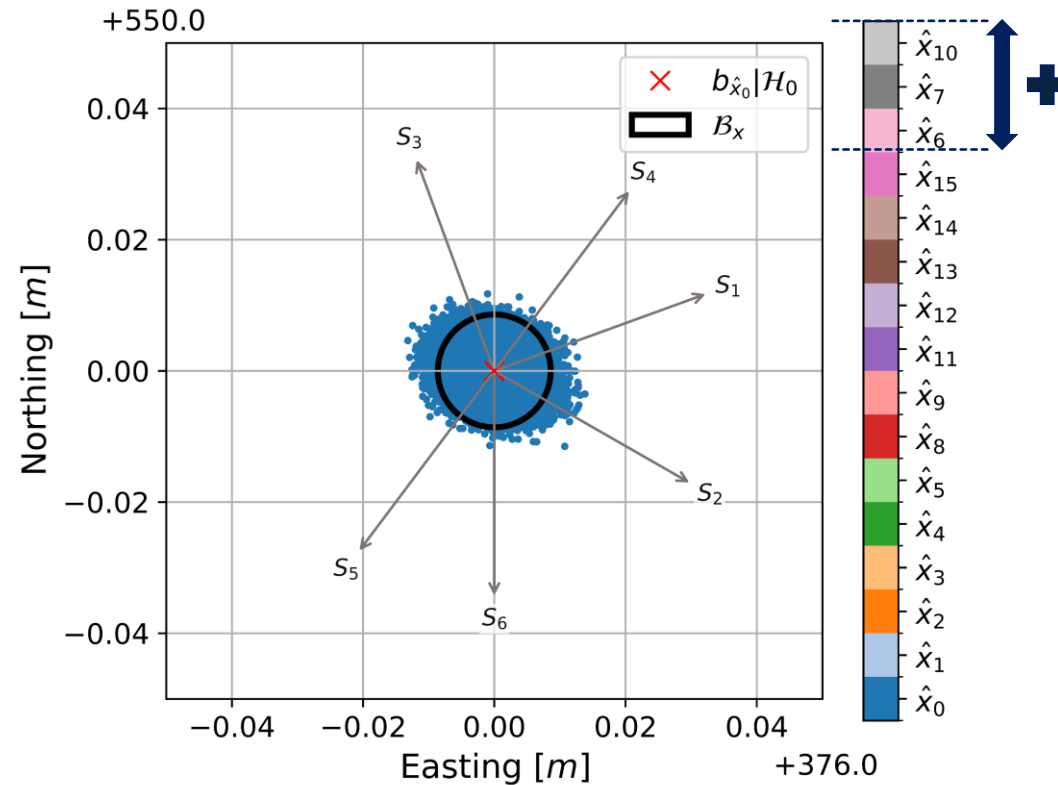
$\rightarrow \text{rank}(B^T[C_1, C_2, C_3]) = 3$  (not 2!)

# Adding a Sixth Station

- Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2$$

$$N = 15$$



Correct acceptance

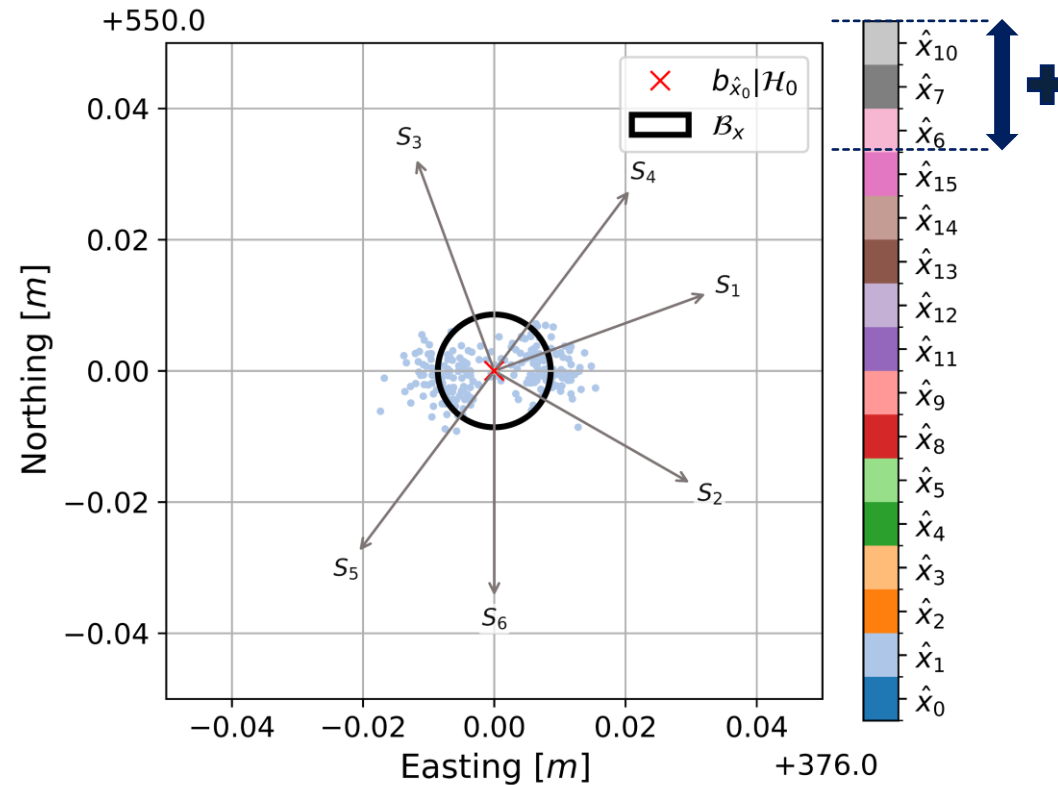
$\rightarrow \mathcal{P}_{10} \rightarrow \hat{x}_{10}$   
 $\rightarrow \mathcal{P}_7 \rightarrow \hat{x}_7$   
 $\rightarrow \mathcal{P}_6 \rightarrow \hat{x}_6$   
 $\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 $\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 $\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 $\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 $\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 $\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9$   
 $\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8$   
 $\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5$   
 $\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4$   
 $\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3$   
 $\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2$   
 $\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1$   
 $\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0$

# Adding a Sixth Station

- Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 \mathbf{z}_j \leq 2$$

$$N = 15$$



False alarm due to  $\mathcal{H}_1$  ( $S_1$ )

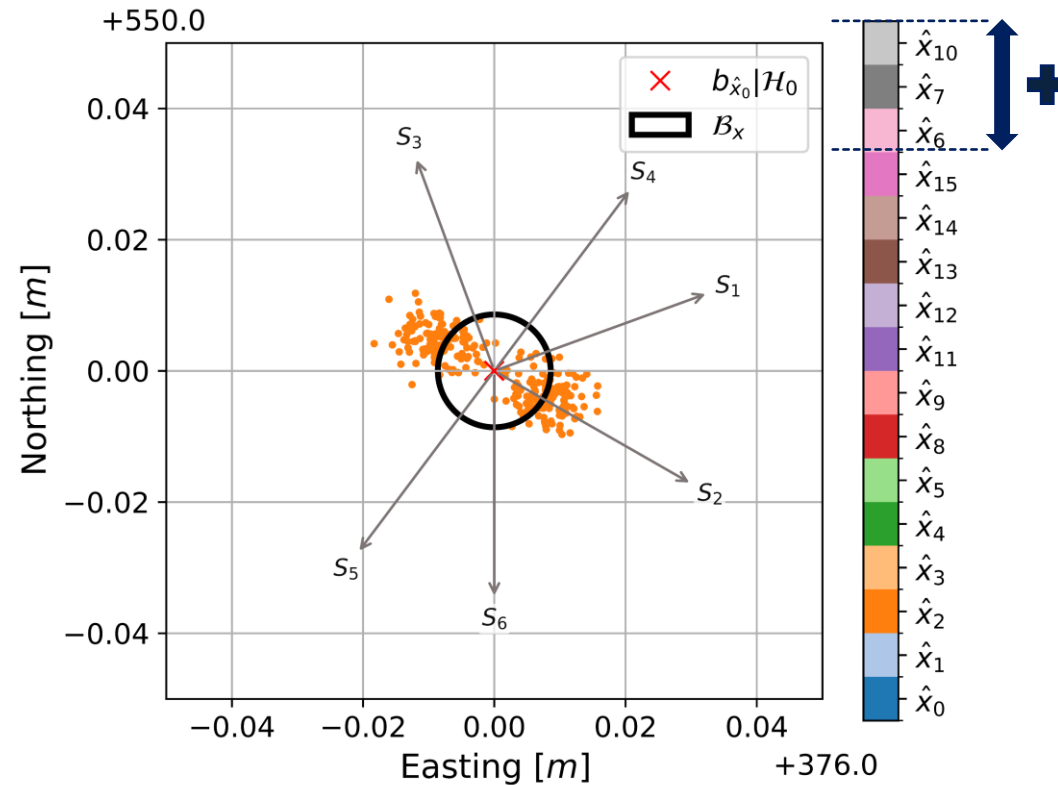
$$\begin{aligned} &\rightarrow \mathcal{P}_{10} \rightarrow \hat{x}_{10} \\ &\rightarrow \mathcal{P}_7 \rightarrow \hat{x}_7 \\ &\rightarrow \mathcal{P}_6 \rightarrow \hat{x}_6 \\ &\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15} \\ &\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14} \\ &\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13} \\ &\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12} \\ &\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11} \\ &\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9 \\ &\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8 \\ &\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5 \\ &\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4 \\ &\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3 \\ &\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2 \\ &\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1 \\ &\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0 \end{aligned}$$

# Adding a Sixth Station

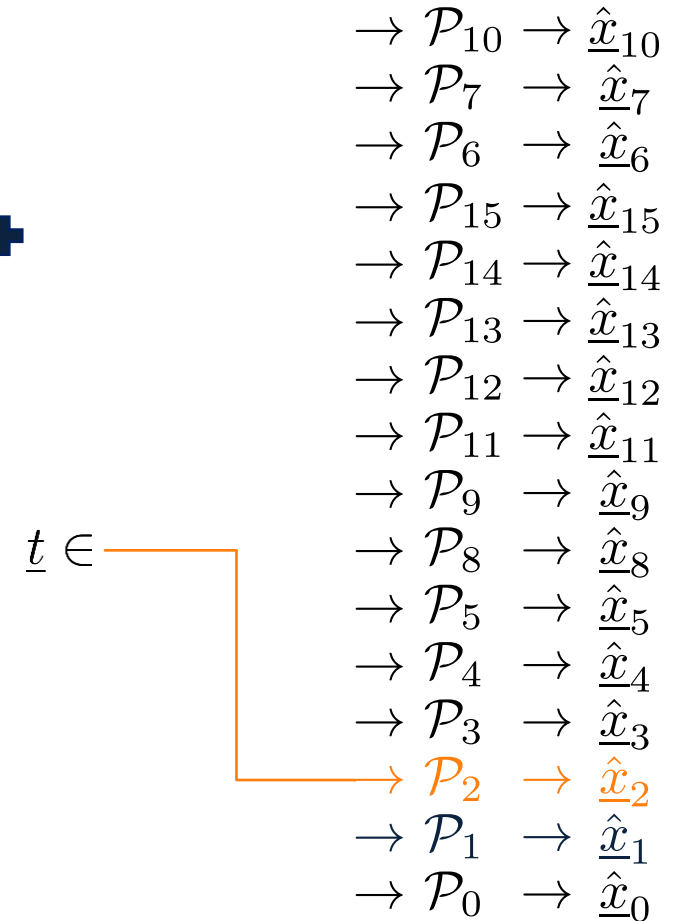
- Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 \mathbf{z}_j \leq 2$$

$$N = 15$$



False alarm due to  $\mathcal{H}_2$  ( $S_2$ )

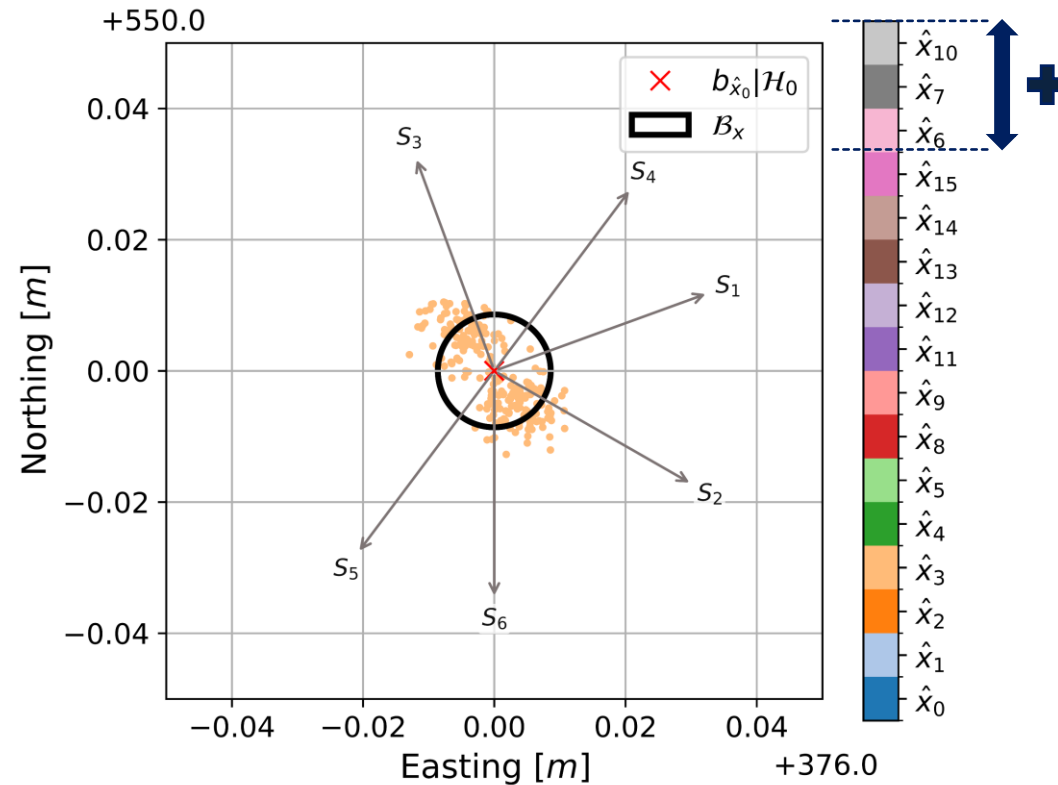


# Adding a Sixth Station

- Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 \mathbf{z}_j \leq 2$$

$$N = 15$$



False alarm due to  $\mathcal{H}_3$  ( $S_3$ )

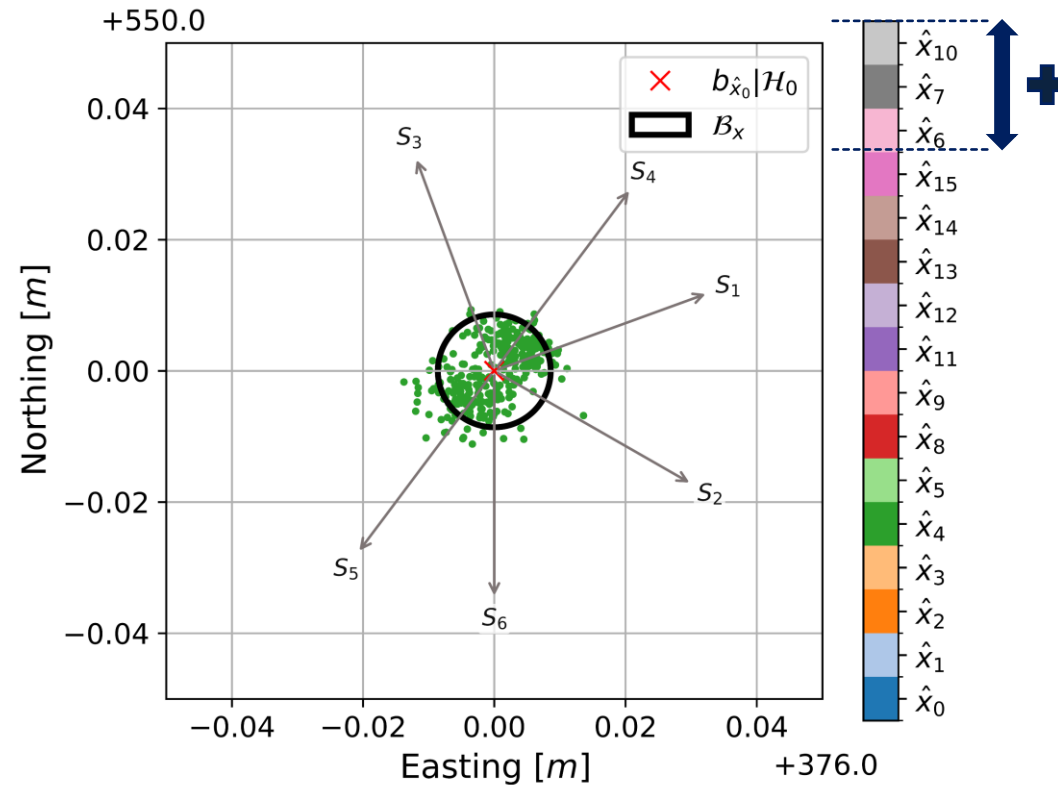
$\rightarrow \mathcal{P}_{10} \rightarrow \hat{x}_{10}$   
 $\rightarrow \mathcal{P}_7 \rightarrow \hat{x}_7$   
 $\rightarrow \mathcal{P}_6 \rightarrow \hat{x}_6$   
 $\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 $\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 $\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 $\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 $\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 $\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9$   
 $\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8$   
 $\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5$   
 $\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4$   
 $\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3$   
 $\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2$   
 $\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1$   
 $\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0$

# Adding a Sixth Station

- Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 \mathbf{z}_j \leq 2$$

$$N = 15$$



False alarm due to  $\mathcal{H}_4$  ( $S_4$ )

$$\begin{aligned} &\rightarrow \mathcal{P}_{10} \rightarrow \hat{\underline{x}}_{10} \\ &\rightarrow \mathcal{P}_7 \rightarrow \hat{\underline{x}}_7 \\ &\rightarrow \mathcal{P}_6 \rightarrow \hat{\underline{x}}_6 \\ &\rightarrow \mathcal{P}_{15} \rightarrow \hat{\underline{x}}_{15} \\ &\rightarrow \mathcal{P}_{14} \rightarrow \hat{\underline{x}}_{14} \\ &\rightarrow \mathcal{P}_{13} \rightarrow \hat{\underline{x}}_{13} \\ &\rightarrow \mathcal{P}_{12} \rightarrow \hat{\underline{x}}_{12} \\ &\rightarrow \mathcal{P}_{11} \rightarrow \hat{\underline{x}}_{11} \\ &\rightarrow \mathcal{P}_9 \rightarrow \hat{\underline{x}}_9 \\ &\rightarrow \mathcal{P}_8 \rightarrow \hat{\underline{x}}_8 \\ &\rightarrow \mathcal{P}_5 \rightarrow \hat{\underline{x}}_5 \\ &\rightarrow \mathcal{P}_4 \rightarrow \hat{\underline{x}}_4 \\ &\rightarrow \mathcal{P}_3 \rightarrow \hat{\underline{x}}_3 \\ &\rightarrow \mathcal{P}_2 \rightarrow \hat{\underline{x}}_2 \\ &\rightarrow \mathcal{P}_1 \rightarrow \hat{\underline{x}}_1 \\ &\rightarrow \mathcal{P}_0 \rightarrow \hat{\underline{x}}_0 \end{aligned}$$

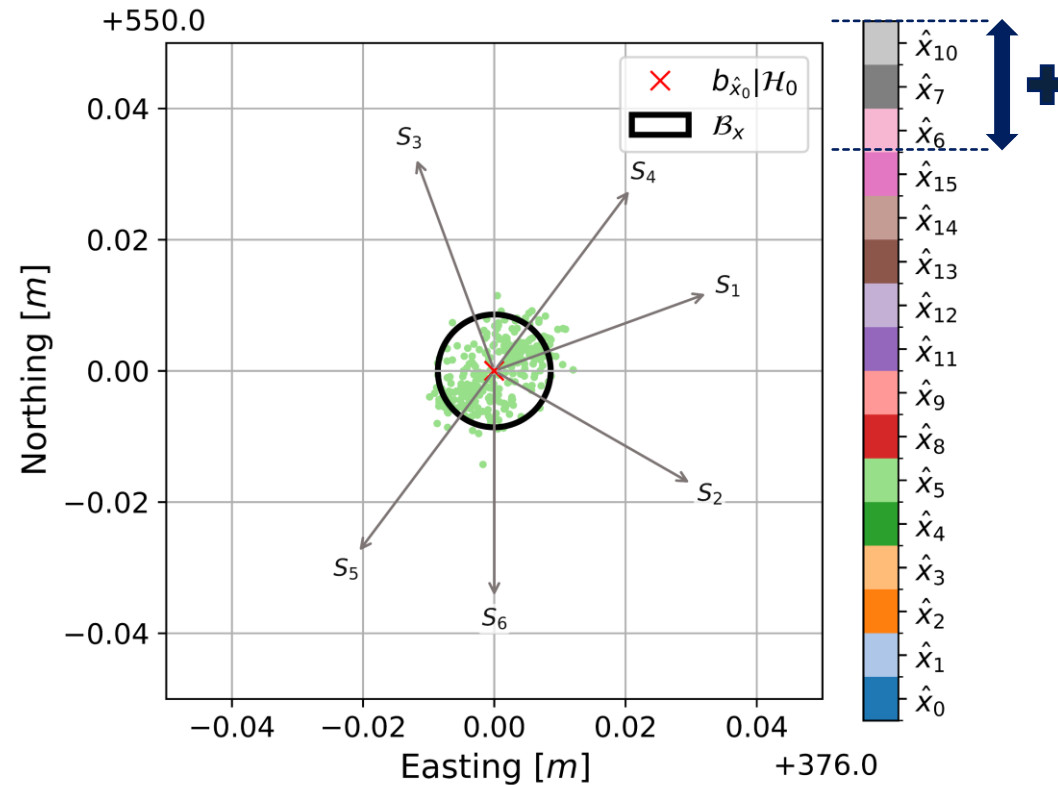
$\underline{t} \in$  (green line pointing to  $\mathcal{P}_4$ )

# Adding a Sixth Station

- Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2$$

$$N = 15$$



False alarm due to  $\mathcal{H}_5$  ( $S_5$ )

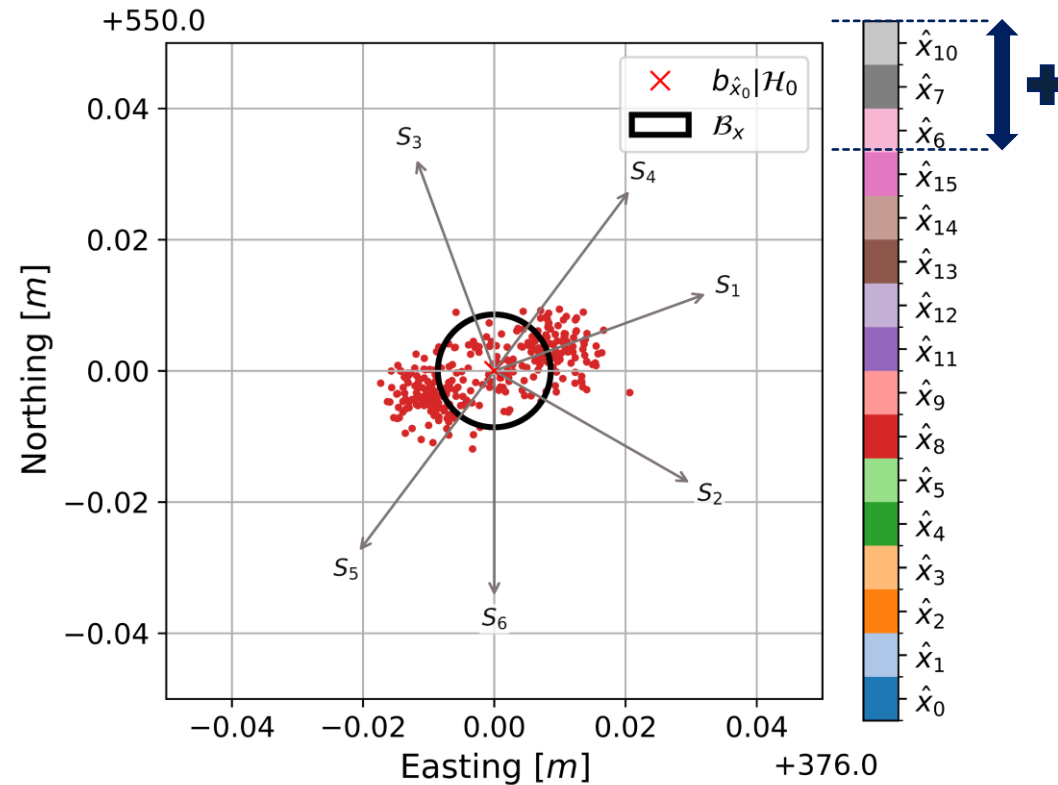
$$\begin{aligned} &\rightarrow \mathcal{P}_{10} \rightarrow \hat{x}_{10} \\ &\rightarrow \mathcal{P}_7 \rightarrow \hat{x}_7 \\ &\rightarrow \mathcal{P}_6 \rightarrow \hat{x}_6 \\ &\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15} \\ &\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14} \\ &\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13} \\ &\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12} \\ &\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11} \\ &\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9 \\ &\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8 \\ &\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5 \\ &\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4 \\ &\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3 \\ &\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2 \\ &\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1 \\ &\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0 \end{aligned}$$

# Adding a Sixth Station

- Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 \mathbf{z}_j \leq 2$$

$$N = 15$$



False alarm due to  $\mathcal{H}_8$  ( $S_1, S_4$ )

$$\begin{aligned} &\rightarrow \mathcal{P}_{10} \rightarrow \hat{x}_{10} \\ &\rightarrow \mathcal{P}_7 \rightarrow \hat{x}_7 \\ &\rightarrow \mathcal{P}_6 \rightarrow \hat{x}_6 \\ &\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15} \\ &\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14} \\ &\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13} \\ &\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12} \\ &\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11} \\ &\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9 \\ &\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8 \\ &\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5 \\ &\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4 \\ &\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3 \\ &\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2 \\ &\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1 \\ &\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0 \end{aligned}$$

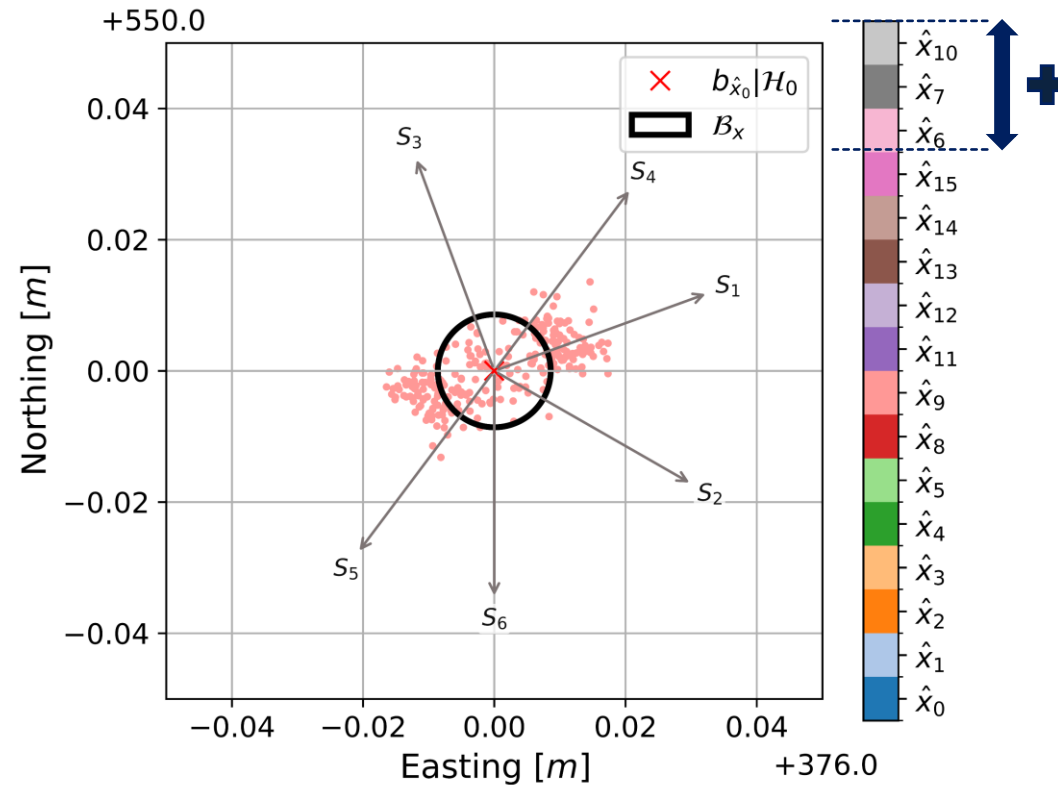
$\underline{t} \in$  —————→  $\mathcal{P}_8 \rightarrow \hat{x}_8$

# Adding a Sixth Station

- Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 \mathbf{z}_j \leq 2$$

$$N = 15$$



False alarm due to  $\mathcal{H}_9$  ( $S_1, S_5$ )

$$\begin{aligned} &\rightarrow \mathcal{P}_{10} \rightarrow \hat{\underline{x}}_{10} \\ &\rightarrow \mathcal{P}_7 \rightarrow \hat{\underline{x}}_7 \\ &\rightarrow \mathcal{P}_6 \rightarrow \hat{\underline{x}}_6 \\ &\rightarrow \mathcal{P}_{15} \rightarrow \hat{\underline{x}}_{15} \\ &\rightarrow \mathcal{P}_{14} \rightarrow \hat{\underline{x}}_{14} \\ &\rightarrow \mathcal{P}_{13} \rightarrow \hat{\underline{x}}_{13} \\ &\rightarrow \mathcal{P}_{12} \rightarrow \hat{\underline{x}}_{12} \\ &\rightarrow \mathcal{P}_{11} \rightarrow \hat{\underline{x}}_{11} \\ &\rightarrow \mathcal{P}_9 \rightarrow \hat{\underline{x}}_9 \\ &\rightarrow \mathcal{P}_8 \rightarrow \hat{\underline{x}}_8 \\ &\rightarrow \mathcal{P}_5 \rightarrow \hat{\underline{x}}_5 \\ &\rightarrow \mathcal{P}_4 \rightarrow \hat{\underline{x}}_4 \\ &\rightarrow \mathcal{P}_3 \rightarrow \hat{\underline{x}}_3 \\ &\rightarrow \mathcal{P}_2 \rightarrow \hat{\underline{x}}_2 \\ &\rightarrow \mathcal{P}_1 \rightarrow \hat{\underline{x}}_1 \\ &\rightarrow \mathcal{P}_0 \rightarrow \hat{\underline{x}}_0 \end{aligned}$$

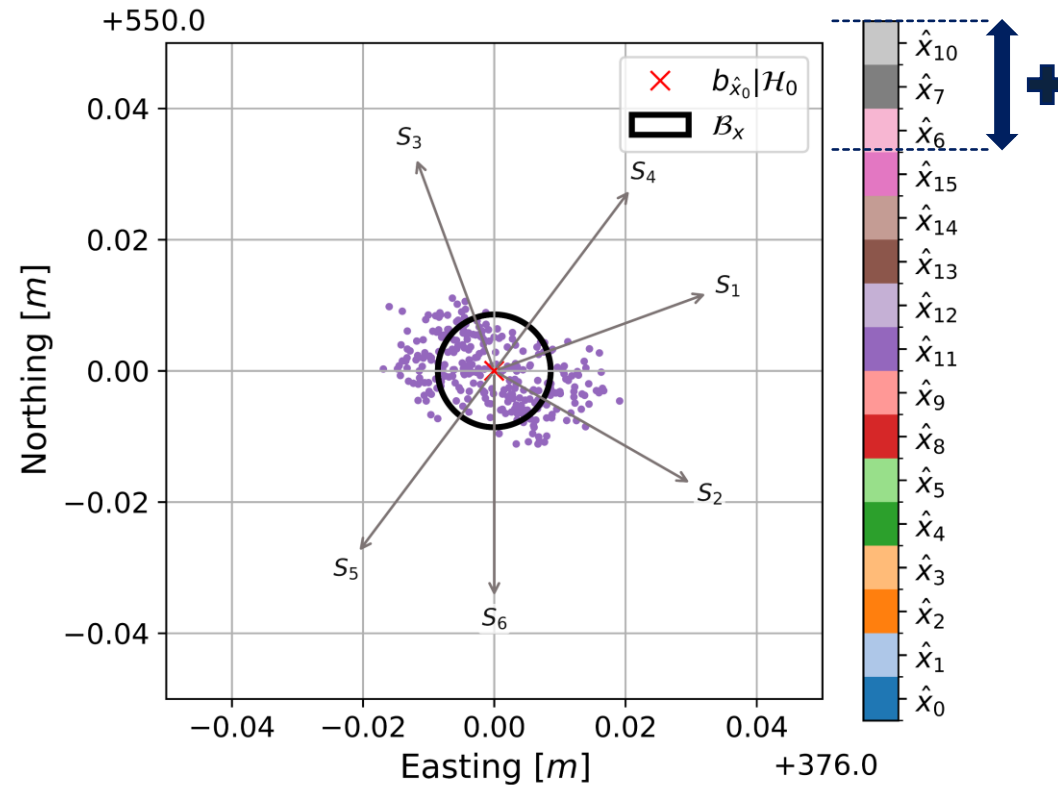
$\underline{t} \in$  (indicated by a red line pointing to  $\mathcal{P}_9$ )

# Adding a Sixth Station

- Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 \mathbf{z}_j \leq 2$$

$$N = 15$$



False alarm due to  $\mathcal{H}_{11}$  ( $S_2, S_4$ )

$$\begin{aligned} &\rightarrow \mathcal{P}_{10} \rightarrow \hat{x}_{10} \\ &\rightarrow \mathcal{P}_7 \rightarrow \hat{x}_7 \\ &\rightarrow \mathcal{P}_6 \rightarrow \hat{x}_6 \\ &\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15} \\ &\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14} \\ &\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13} \\ &\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12} \\ &\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11} \\ &\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9 \\ &\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8 \\ &\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5 \\ &\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4 \\ &\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3 \\ &\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2 \\ &\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1 \\ &\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0 \end{aligned}$$

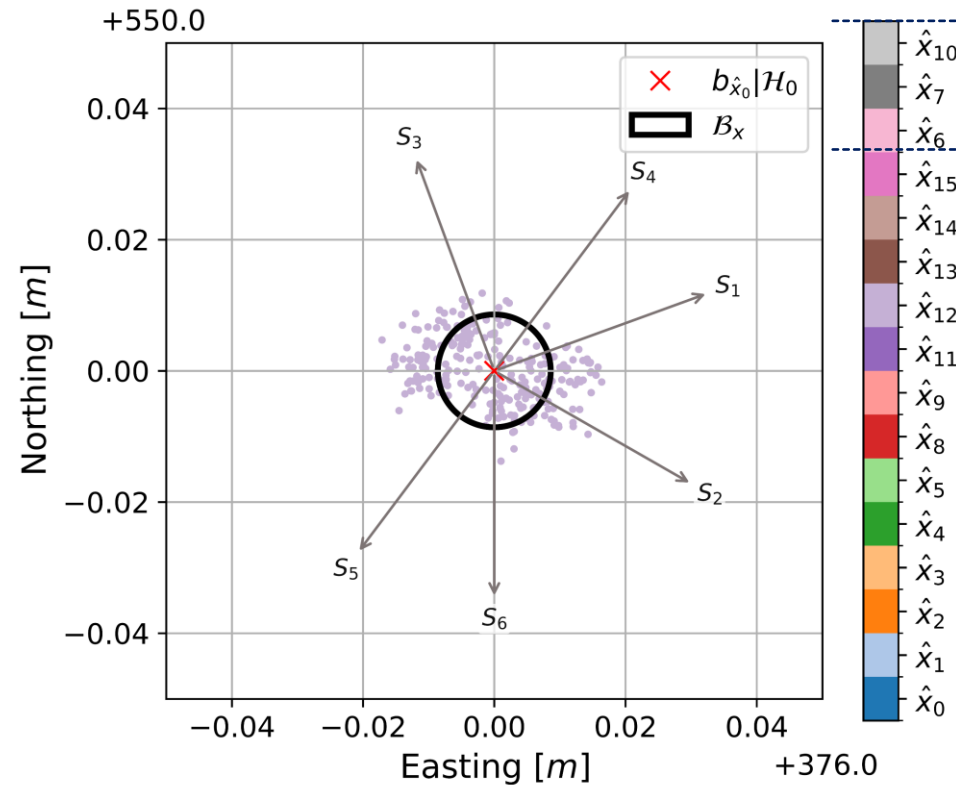
$\underline{t} \in$  (indicated by a purple line pointing to  $\mathcal{P}_{11}$ )

# Adding a Sixth Station

- Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 \mathbf{z}_j \leq 2$$

$$N = 15$$



False alarm due to  $\mathcal{H}_{12}$  ( $S_2, S_5$ )

$$\begin{aligned} &\rightarrow \mathcal{P}_{10} \rightarrow \hat{x}_{10} \\ &\rightarrow \mathcal{P}_7 \rightarrow \hat{x}_7 \\ &\rightarrow \mathcal{P}_6 \rightarrow \hat{x}_6 \\ &\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15} \\ &\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14} \\ &\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13} \\ &\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12} \\ &\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11} \\ &\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9 \\ &\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8 \\ &\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5 \\ &\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4 \\ &\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3 \\ &\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2 \\ &\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1 \\ &\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0 \end{aligned}$$

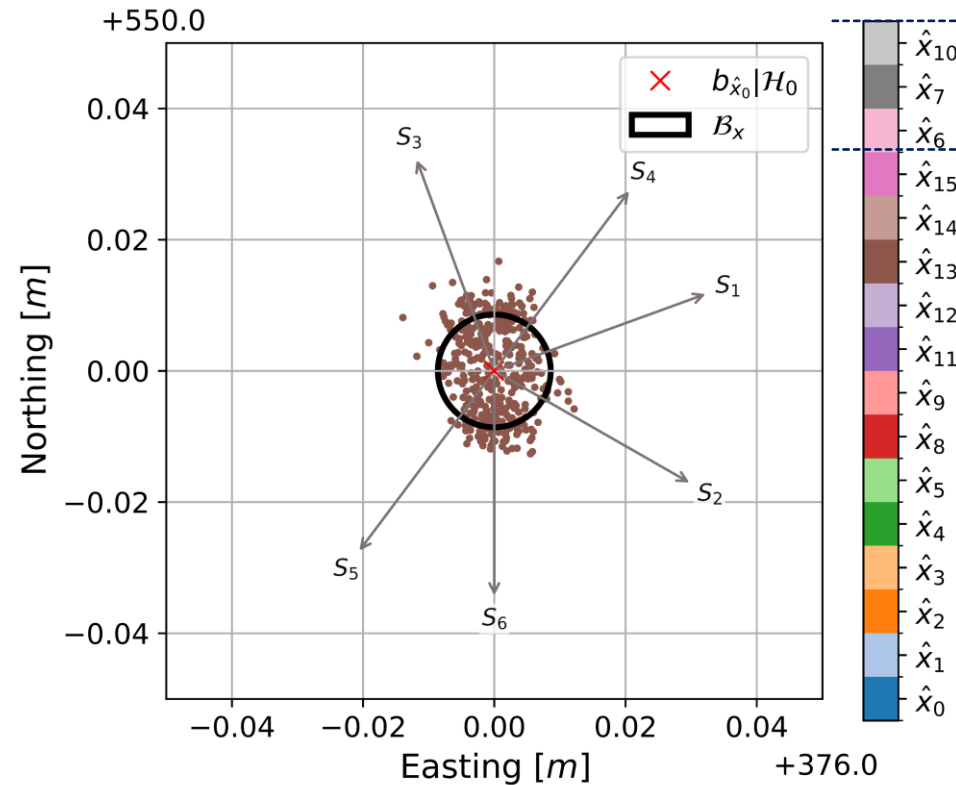
$\underline{t} \in$

# Adding a Sixth Station

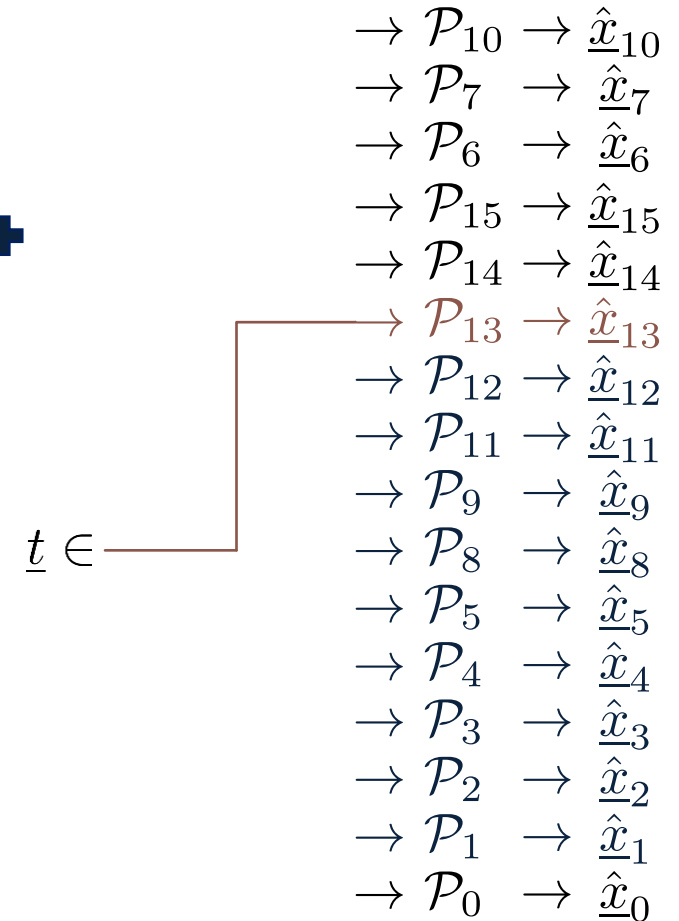
- Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 \mathbf{z}_j \leq 2$$

$$N = 15$$



False alarm due to  $\mathcal{H}_{13}$  ( $S_3, S_4$ )

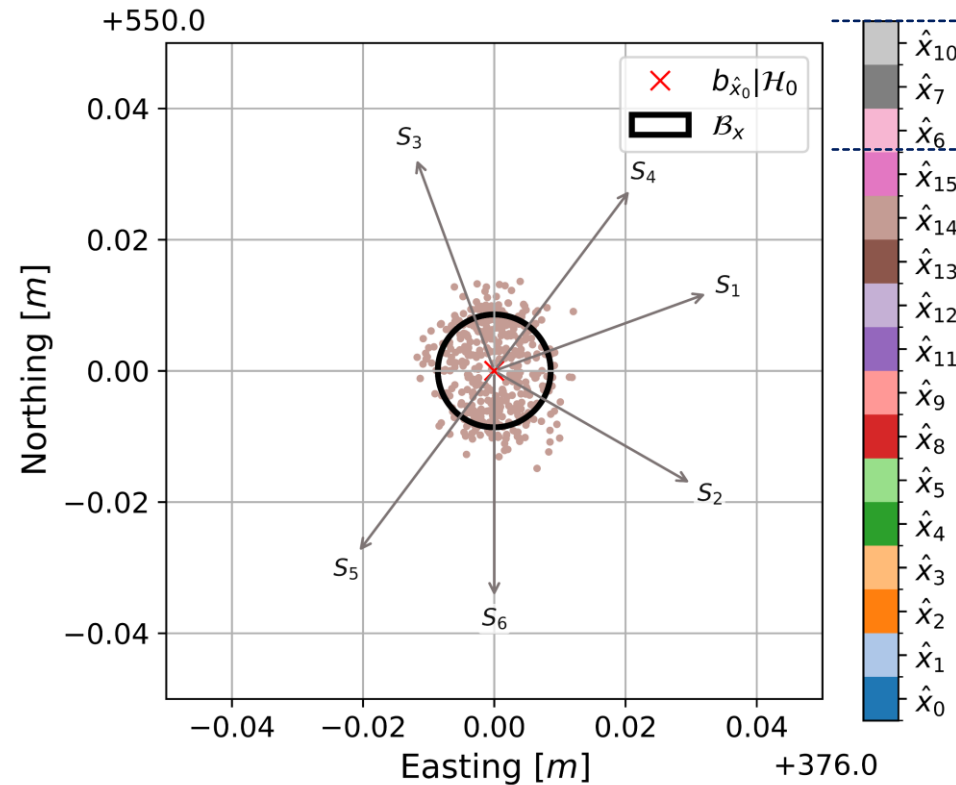


# Adding a Sixth Station

- Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 \mathbf{z}_j \leq 2$$

$$N = 15$$



False alarm due to  $\mathcal{H}_{14}$  ( $S_3, S_5$ )

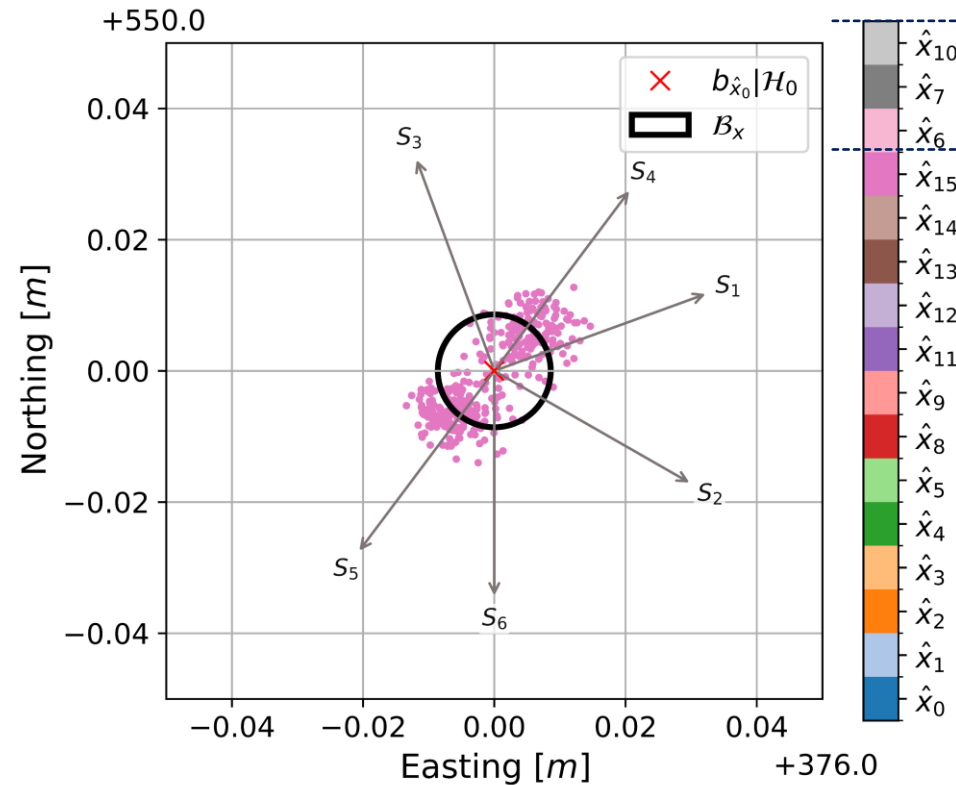
$$\begin{aligned} &\rightarrow \mathcal{P}_{10} \rightarrow \hat{x}_{10} \\ &\rightarrow \mathcal{P}_7 \rightarrow \hat{x}_7 \\ &\rightarrow \mathcal{P}_6 \rightarrow \hat{x}_6 \\ &\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15} \\ &\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14} \\ &\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13} \\ &\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12} \\ &\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11} \\ &\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9 \\ &\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8 \\ &\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5 \\ &\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4 \\ &\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3 \\ &\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2 \\ &\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1 \\ &\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0 \end{aligned}$$

# Adding a Sixth Station

- Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 \mathbf{z}_j \leq 2$$

$$N = 15$$



$$\begin{aligned} &\rightarrow \mathcal{P}_{10} \rightarrow \hat{\underline{x}}_{10} \\ &\rightarrow \mathcal{P}_7 \rightarrow \hat{\underline{x}}_7 \\ &\rightarrow \mathcal{P}_6 \rightarrow \hat{\underline{x}}_6 \\ &\rightarrow \mathcal{P}_{15} \rightarrow \hat{\underline{x}}_{15} \\ &\rightarrow \mathcal{P}_{14} \rightarrow \hat{\underline{x}}_{14} \\ &\rightarrow \mathcal{P}_{13} \rightarrow \hat{\underline{x}}_{13} \\ &\rightarrow \mathcal{P}_{12} \rightarrow \hat{\underline{x}}_{12} \\ &\rightarrow \mathcal{P}_{11} \rightarrow \hat{\underline{x}}_{11} \\ &\rightarrow \mathcal{P}_9 \rightarrow \hat{\underline{x}}_9 \\ &\rightarrow \mathcal{P}_8 \rightarrow \hat{\underline{x}}_8 \\ &\rightarrow \mathcal{P}_5 \rightarrow \hat{\underline{x}}_5 \\ &\rightarrow \mathcal{P}_4 \rightarrow \hat{\underline{x}}_4 \\ &\rightarrow \mathcal{P}_3 \rightarrow \hat{\underline{x}}_3 \\ &\rightarrow \mathcal{P}_2 \rightarrow \hat{\underline{x}}_2 \\ &\rightarrow \mathcal{P}_1 \rightarrow \hat{\underline{x}}_1 \\ &\rightarrow \mathcal{P}_0 \rightarrow \hat{\underline{x}}_0 \end{aligned}$$

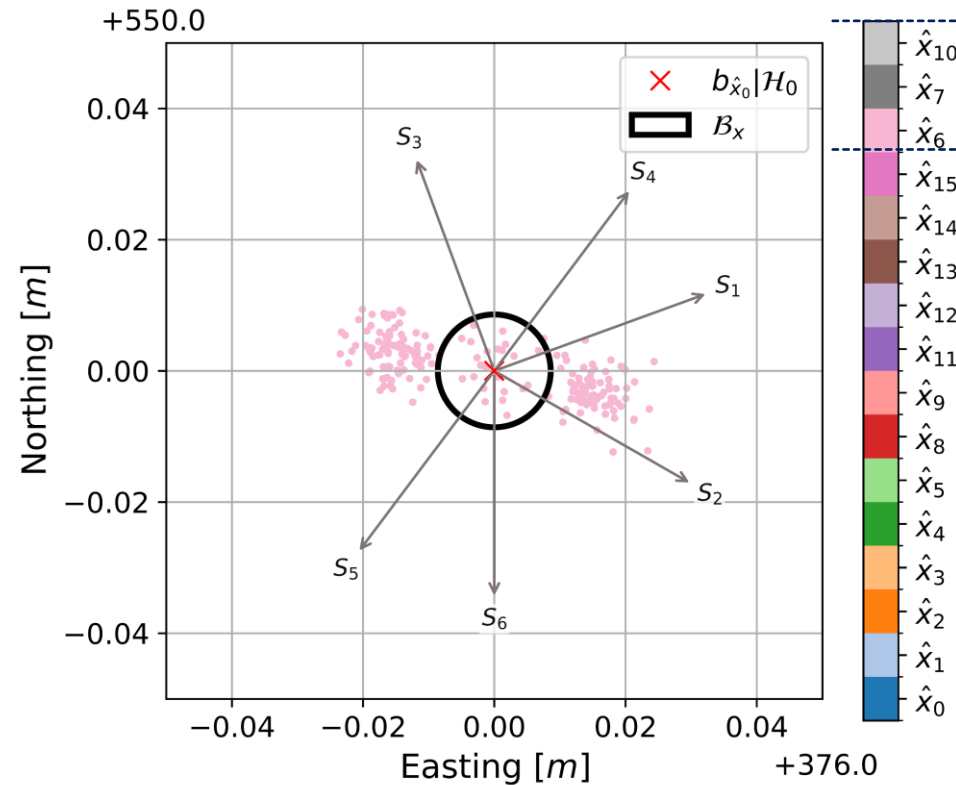
$\underline{t} \in$

# Adding a Sixth Station

- Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 \mathbf{z}_j \leq 2$$

$$N = 15$$



False alarm due to  $\mathcal{H}_6$  ( $S_1, S_2$ )

$\rightarrow \mathcal{P}_{10} \rightarrow \hat{x}_{10}$   
 $\rightarrow \mathcal{P}_7 \rightarrow \hat{x}_7$   
 $\rightarrow \mathcal{P}_6 \rightarrow \hat{x}_6$   
 $\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 $\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 $\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 $\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 $\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 $\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9$   
 $\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8$   
 $\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5$   
 $\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4$   
 $\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3$   
 $\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2$   
 $\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1$   
 $\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0$

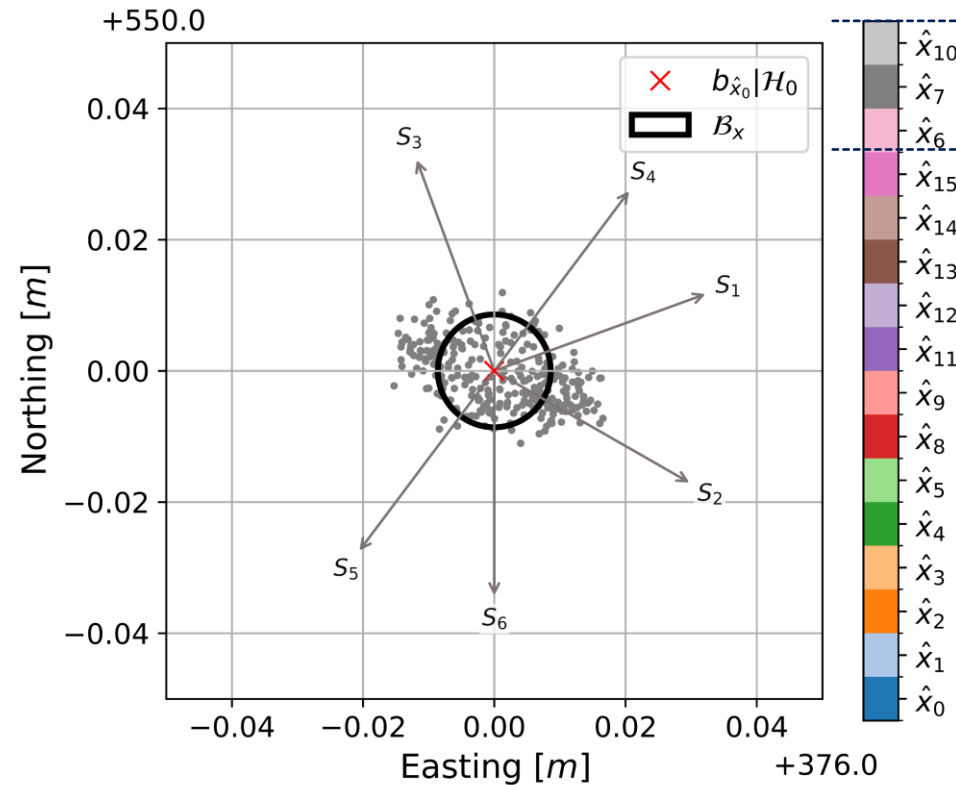
$\underline{t} \in$

# Adding a Sixth Station

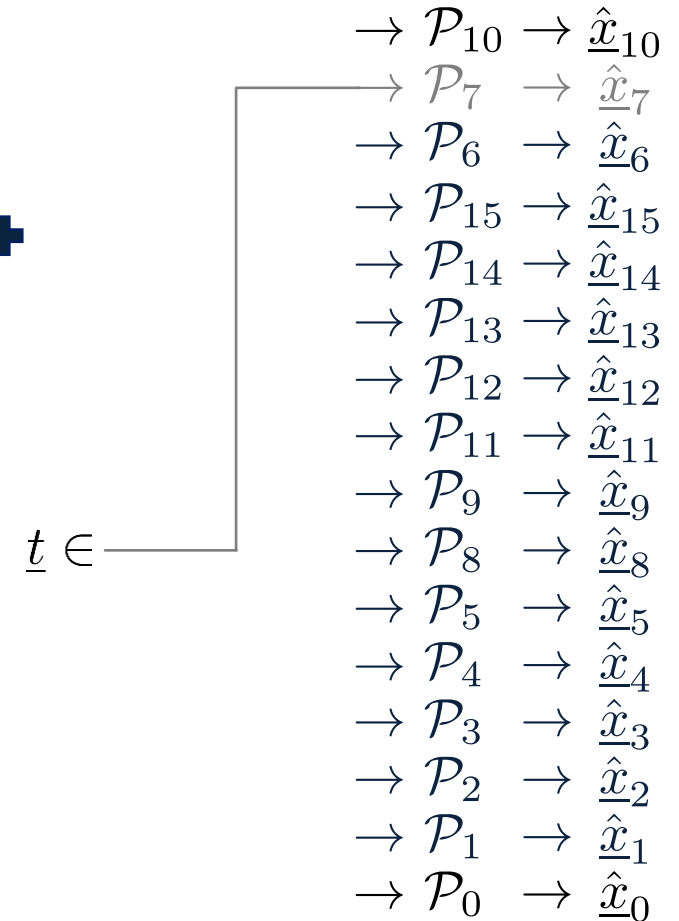
- Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2$$

$$N = 15$$



False alarm due to  $\mathcal{H}_7$  ( $S_1, S_3$ )

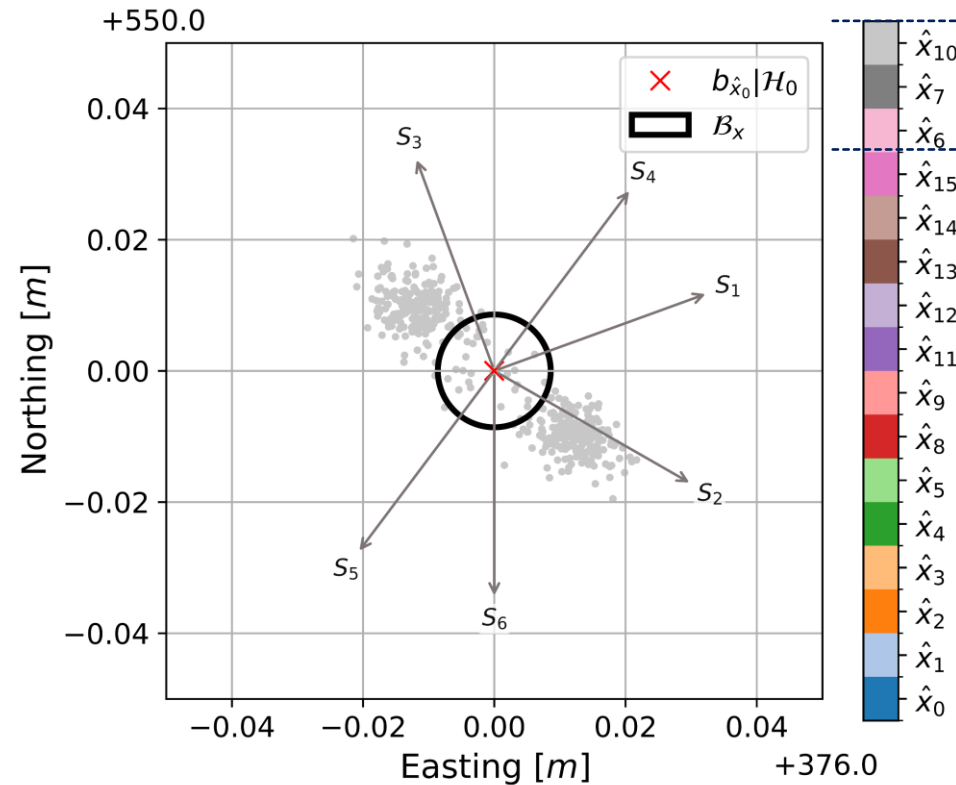


# Adding a Sixth Station

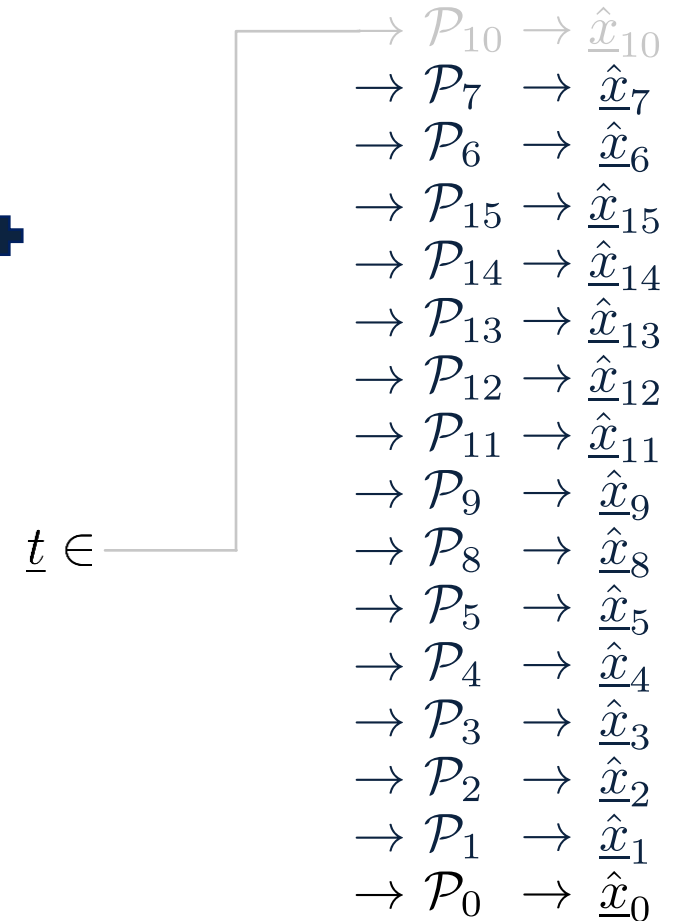
- Under null hypothesis  $\mathcal{H}_0$ :

$$\text{IC: } \sum_{j=1}^5 \mathbf{z}_j \leq 2$$

$$N = 15$$



False alarm due to  $\mathcal{H}_{10}$  ( $S_2, S_3$ )



# Adding a Sixth Station

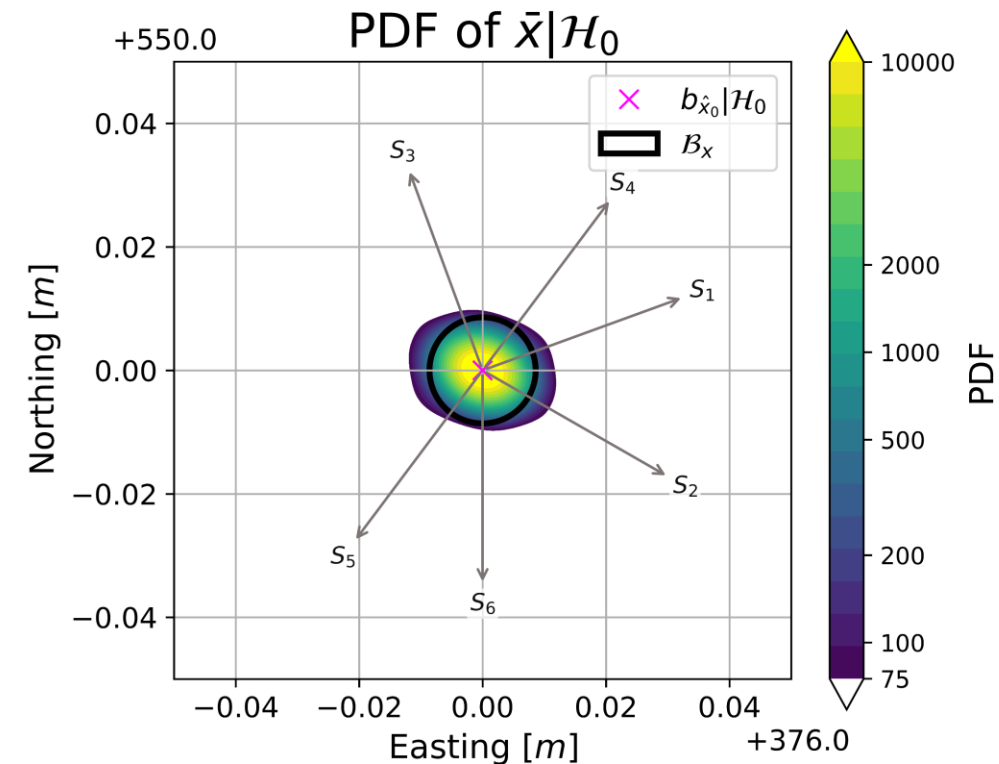
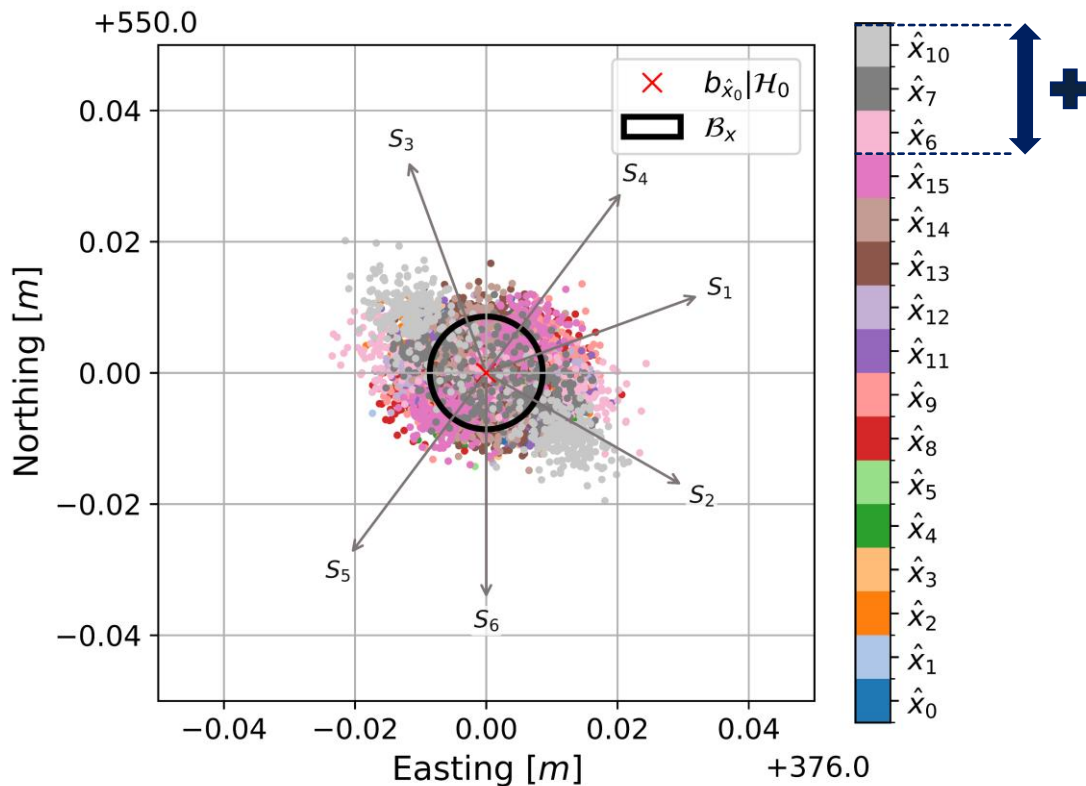
- Under null hypothesis  $\mathcal{H}_0$ :

$$\bar{x} = \sum_{i=0}^N \hat{x}_i \mathbf{1}_{\mathcal{P}_i}(t)$$

$$\text{IC: } \sum_{j=1}^5 \mathbf{z}_j \leq 2$$

$$N = 15$$

$$f_{\bar{x}}(x) = \sum_{i=0}^N \mathbb{E}[f_{\hat{x}_0}(x + A^+ C \bar{Z}(t) \bar{b}(t)) \mathbf{1}_{\mathcal{P}_i}(t)]$$



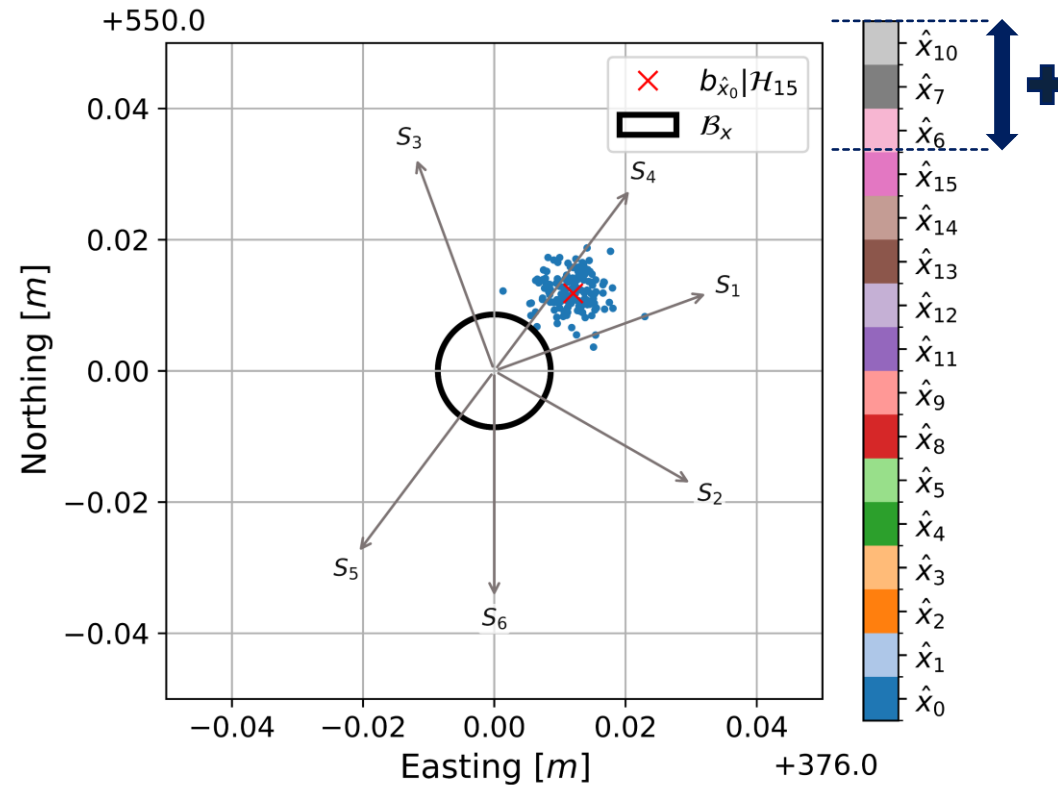
# Adding a Sixth Station

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$

$$\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2$$

$$N = 15$$



$\rightarrow \mathcal{P}_{10} \rightarrow \hat{x}_{10}$   
 $\rightarrow \mathcal{P}_7 \rightarrow \hat{x}_7$   
 $\rightarrow \mathcal{P}_6 \rightarrow \hat{x}_6$   
 $\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 $\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 $\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 $\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 $\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 $\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9$   
 $\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8$   
 $\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5$   
 $\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4$   
 $\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3$   
 $\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2$   
 $\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1$   
 $\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0$

$\underline{t} \in$

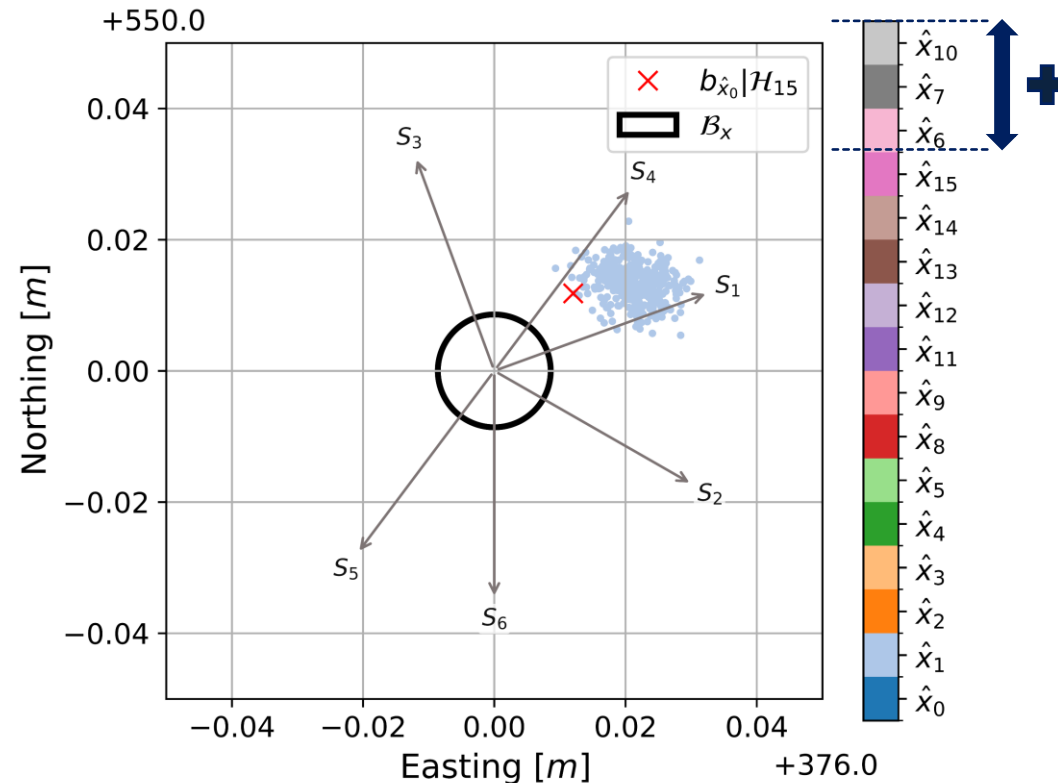
# Adding a Sixth Station

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$

$$\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2$$

$$N = 15$$



$\rightarrow \mathcal{P}_{10} \rightarrow \hat{x}_{10}$   
 $\rightarrow \mathcal{P}_7 \rightarrow \hat{x}_7$   
 $\rightarrow \mathcal{P}_6 \rightarrow \hat{x}_6$   
 $\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 $\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 $\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 $\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 $\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 $\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9$   
 $\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8$   
 $\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5$   
 $\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4$   
 $\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3$   
 $\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2$   
 $\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1$   
 $\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0$

Wrong identification due to  $\mathcal{H}_1 (S_1)$

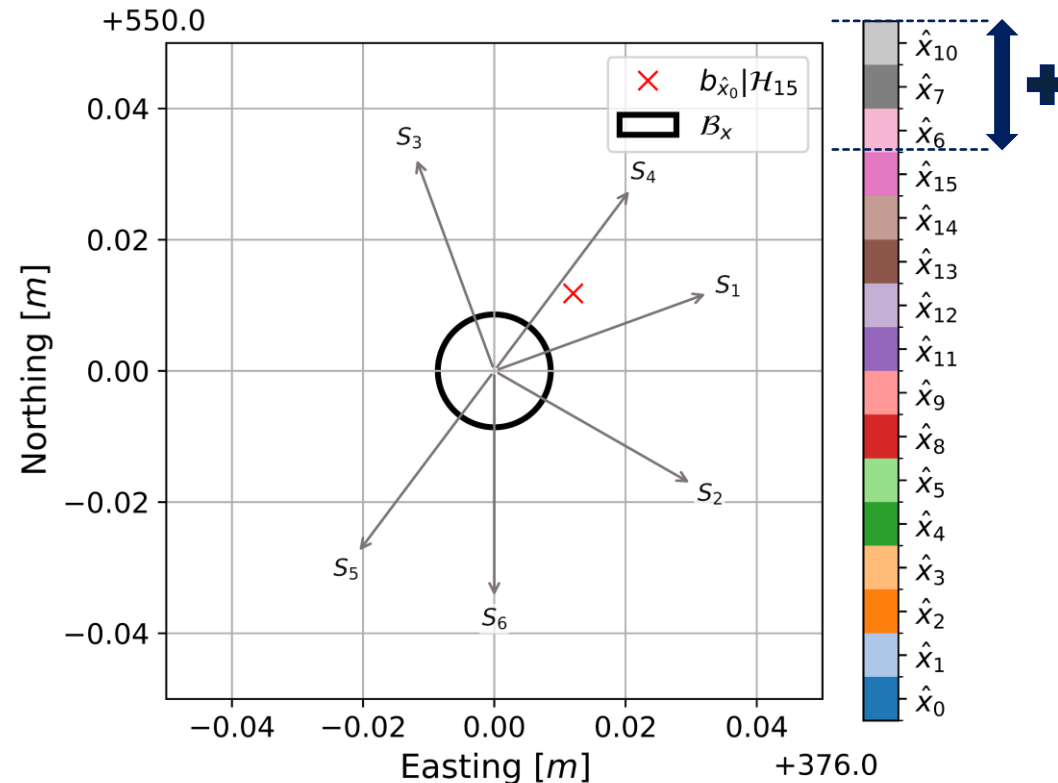
# Adding a Sixth Station

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$

$$\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2$$

$$N = 15$$



$\rightarrow \mathcal{P}_{10} \rightarrow \hat{x}_{10}$   
 $\rightarrow \mathcal{P}_7 \rightarrow \hat{x}_7$   
 $\rightarrow \mathcal{P}_6 \rightarrow \hat{x}_6$   
 $\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 $\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 $\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 $\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 $\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 $\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9$   
 $\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8$   
 $\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5$   
 $\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4$   
 $\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3$   
 $\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2$   
 $\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1$   
 $\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0$

Wrong identification due to  $\mathcal{H}_2 (S_2)$

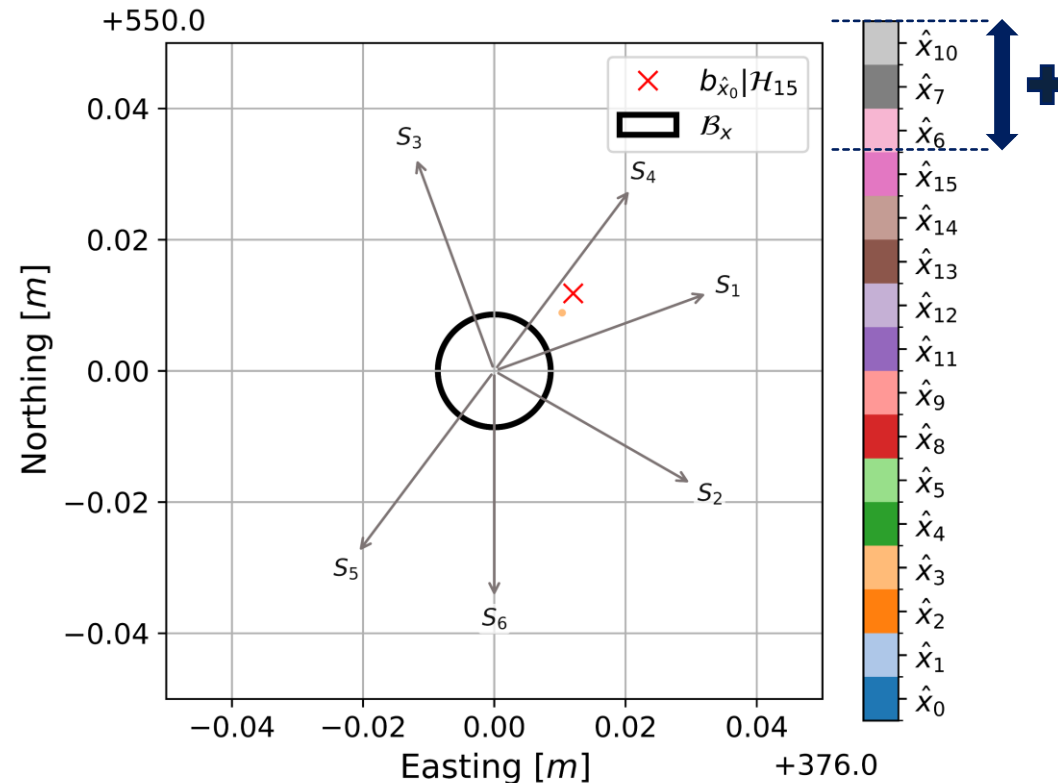
# Adding a Sixth Station

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$

$$\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2$$

$$N = 15$$



$\underline{t} \in$

$\rightarrow \mathcal{P}_{10} \rightarrow \underline{\hat{x}}_{10}$   
 $\rightarrow \mathcal{P}_7 \rightarrow \underline{\hat{x}}_7$   
 $\rightarrow \mathcal{P}_6 \rightarrow \underline{\hat{x}}_6$   
 $\rightarrow \mathcal{P}_{15} \rightarrow \underline{\hat{x}}_{15}$   
 $\rightarrow \mathcal{P}_{14} \rightarrow \underline{\hat{x}}_{14}$   
 $\rightarrow \mathcal{P}_{13} \rightarrow \underline{\hat{x}}_{13}$   
 $\rightarrow \mathcal{P}_{12} \rightarrow \underline{\hat{x}}_{12}$   
 $\rightarrow \mathcal{P}_{11} \rightarrow \underline{\hat{x}}_{11}$   
 $\rightarrow \mathcal{P}_9 \rightarrow \underline{\hat{x}}_9$   
 $\rightarrow \mathcal{P}_8 \rightarrow \underline{\hat{x}}_8$   
 $\rightarrow \mathcal{P}_5 \rightarrow \underline{\hat{x}}_5$   
 $\rightarrow \mathcal{P}_4 \rightarrow \underline{\hat{x}}_4$   
 $\rightarrow \mathcal{P}_3 \rightarrow \underline{\hat{x}}_3$   
 $\rightarrow \mathcal{P}_2 \rightarrow \underline{\hat{x}}_2$   
 $\rightarrow \mathcal{P}_1 \rightarrow \underline{\hat{x}}_1$   
 $\rightarrow \mathcal{P}_0 \rightarrow \underline{\hat{x}}_0$

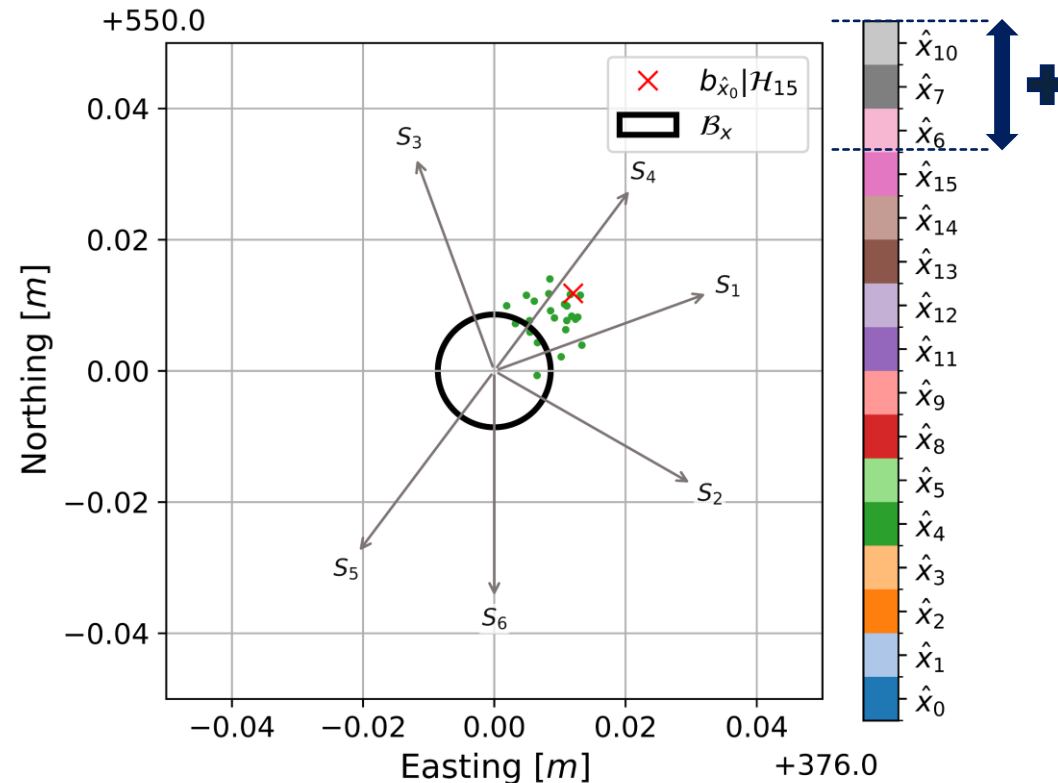
# Adding a Sixth Station

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$

$$\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2$$

$$N = 15$$



$$\begin{aligned} &\rightarrow \mathcal{P}_{10} \rightarrow \hat{x}_{10} \\ &\rightarrow \mathcal{P}_7 \rightarrow \hat{x}_7 \\ &\rightarrow \mathcal{P}_6 \rightarrow \hat{x}_6 \\ &\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15} \\ &\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14} \\ &\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13} \\ &\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12} \\ &\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11} \\ &\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9 \\ &\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8 \\ &\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5 \\ &\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4 \\ &\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3 \\ &\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2 \\ &\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1 \\ &\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0 \end{aligned}$$

Wrong identification due to  $\mathcal{H}_4 (S_4)$

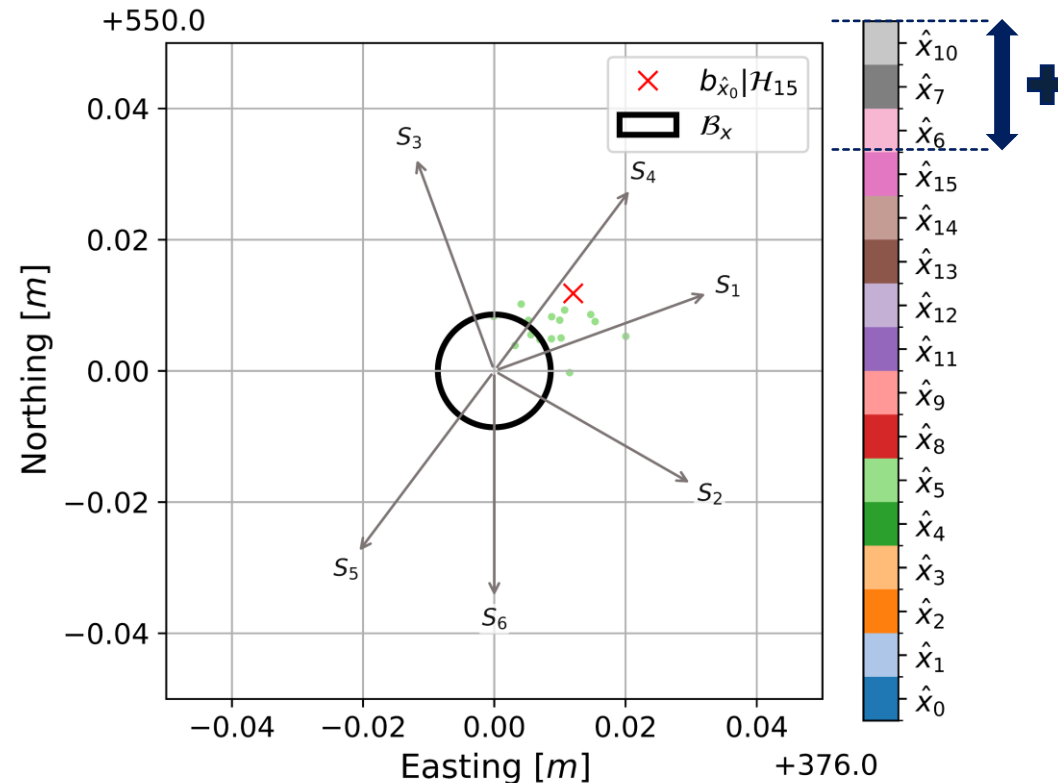
# Adding a Sixth Station

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$

$$\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2$$

$$N = 15$$



$$\begin{aligned} &\rightarrow \mathcal{P}_{10} \rightarrow \hat{x}_{10} \\ &\rightarrow \mathcal{P}_7 \rightarrow \hat{x}_7 \\ &\rightarrow \mathcal{P}_6 \rightarrow \hat{x}_6 \\ &\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15} \\ &\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14} \\ &\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13} \\ &\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12} \\ &\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11} \\ &\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9 \\ &\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8 \\ &\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5 \\ &\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4 \\ &\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3 \\ &\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2 \\ &\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1 \\ &\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0 \end{aligned}$$

Wrong identification due to  $\mathcal{H}_5$  ( $S_5$ )

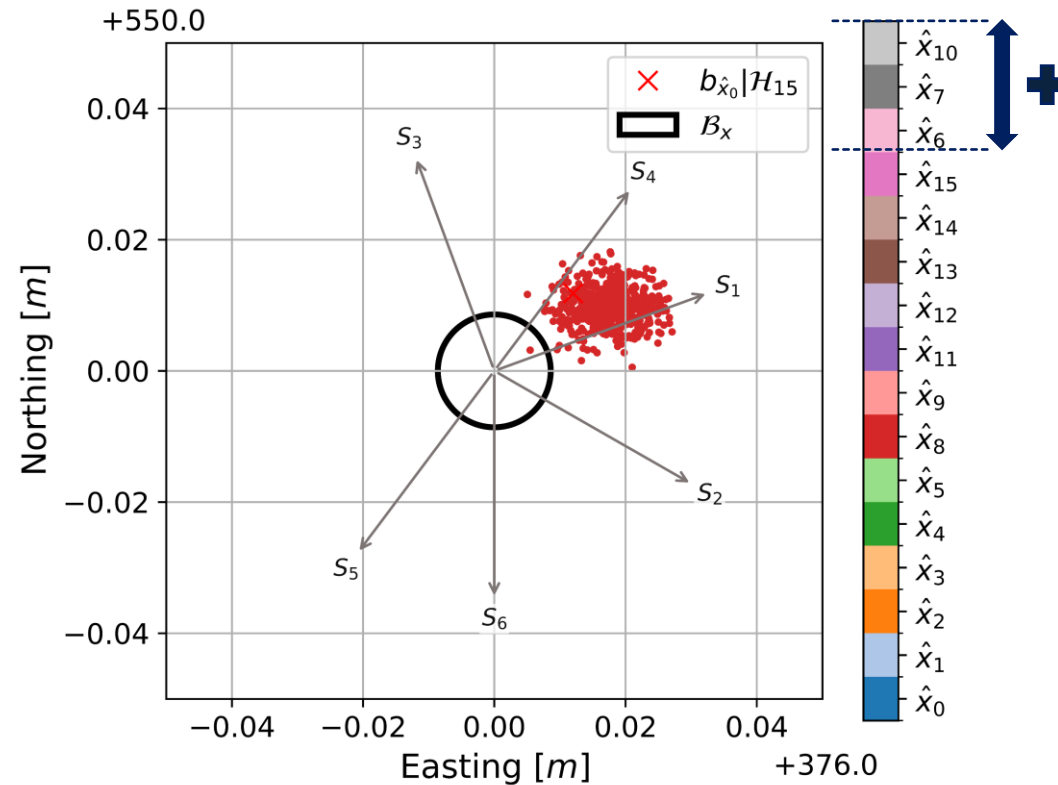
# Adding a Sixth Station

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$

$$\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2$$

$$N = 15$$



$$\begin{aligned} &\rightarrow \mathcal{P}_{10} \rightarrow \hat{x}_{10} \\ &\rightarrow \mathcal{P}_7 \rightarrow \hat{x}_7 \\ &\rightarrow \mathcal{P}_6 \rightarrow \hat{x}_6 \\ &\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15} \\ &\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14} \\ &\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13} \\ &\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12} \\ &\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11} \\ &\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9 \\ &\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8 \\ &\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5 \\ &\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4 \\ &\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3 \\ &\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2 \\ &\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1 \\ &\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0 \end{aligned}$$

Wrong identification due to  $\mathcal{H}_8 (S_1, S_4)$

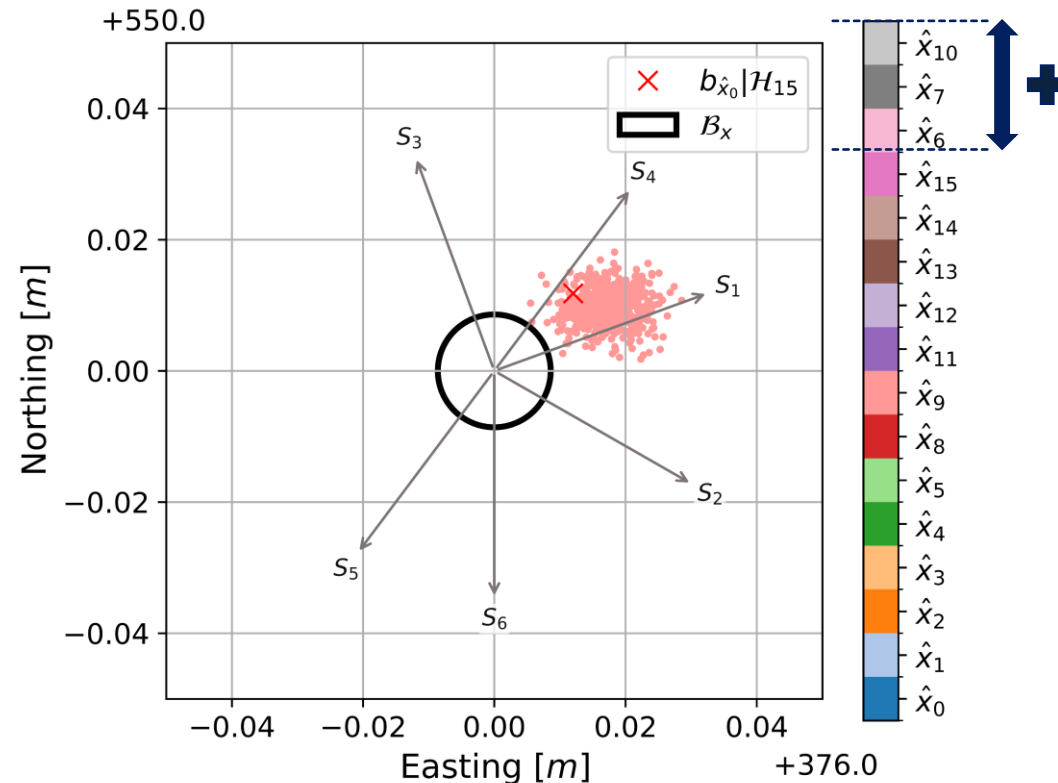
# Adding a Sixth Station

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$

$$\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2$$

$$N = 15$$



$\rightarrow \mathcal{P}_{10} \rightarrow \hat{x}_{10}$   
 $\rightarrow \mathcal{P}_7 \rightarrow \hat{x}_7$   
 $\rightarrow \mathcal{P}_6 \rightarrow \hat{x}_6$   
 $\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15}$   
 $\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14}$   
 $\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13}$   
 $\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12}$   
 $\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11}$   
 $\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9$   
 $\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8$   
 $\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5$   
 $\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4$   
 $\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3$   
 $\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2$   
 $\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1$   
 $\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0$

$\underline{t} \in$

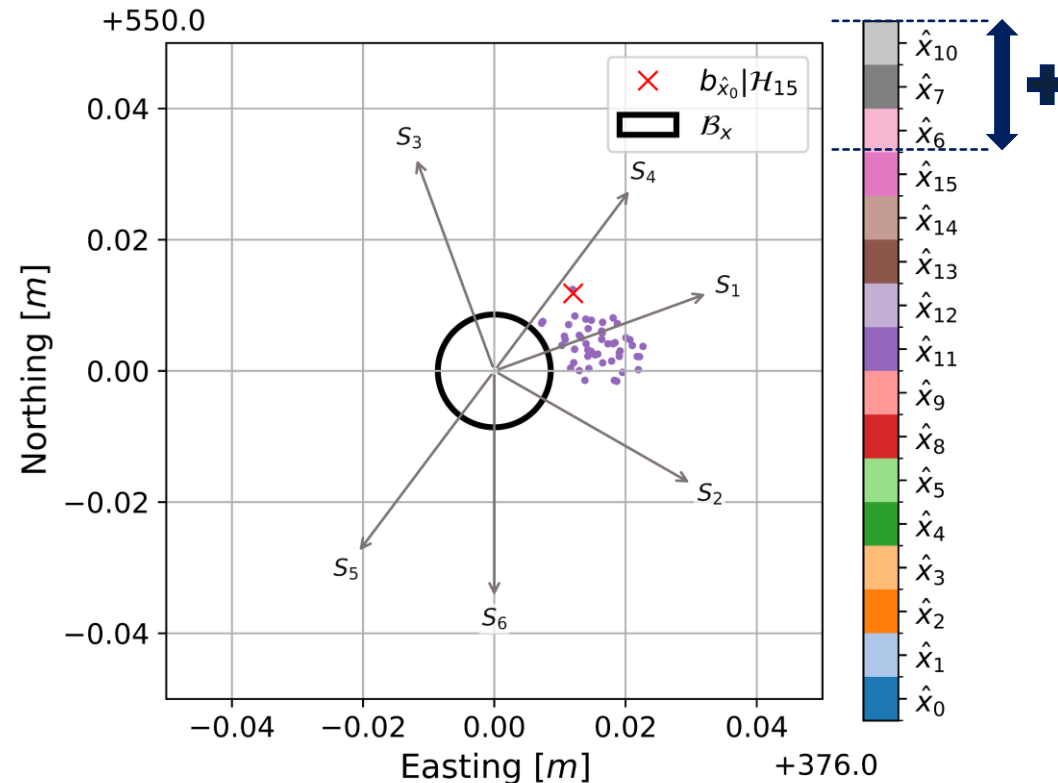
# Adding a Sixth Station

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$

$$\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2$$

$$N = 15$$



$$\begin{aligned} &\rightarrow \mathcal{P}_{10} \rightarrow \hat{x}_{10} \\ &\rightarrow \mathcal{P}_7 \rightarrow \hat{x}_7 \\ &\rightarrow \mathcal{P}_6 \rightarrow \hat{x}_6 \\ &\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15} \\ &\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14} \\ &\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13} \\ &\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12} \\ &\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11} \\ &\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9 \\ &\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8 \\ &\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5 \\ &\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4 \\ &\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3 \\ &\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2 \\ &\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1 \\ &\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0 \end{aligned}$$

$\underline{t} \in$

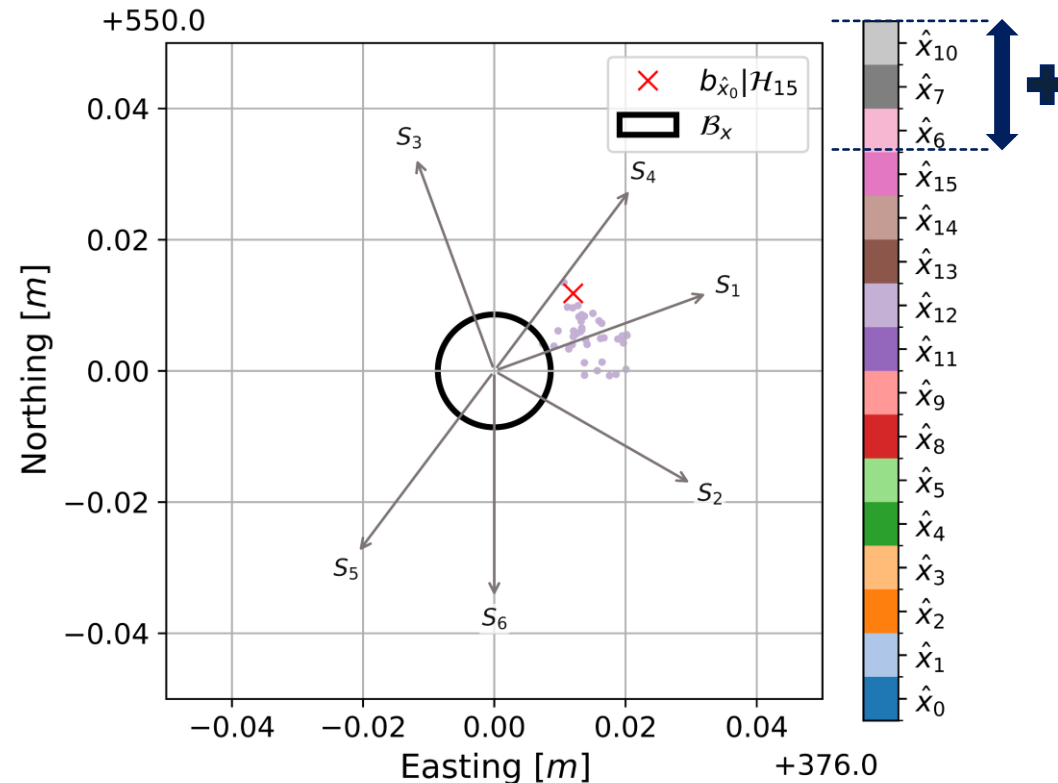
# Adding a Sixth Station

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$

$$\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2$$

$$N = 15$$



$\underline{t} \in$

|                                |                                        |
|--------------------------------|----------------------------------------|
| $\rightarrow \mathcal{P}_{10}$ | $\rightarrow \underline{\hat{x}}_{10}$ |
| $\rightarrow \mathcal{P}_7$    | $\rightarrow \underline{\hat{x}}_7$    |
| $\rightarrow \mathcal{P}_6$    | $\rightarrow \underline{\hat{x}}_6$    |
| $\rightarrow \mathcal{P}_{15}$ | $\rightarrow \underline{\hat{x}}_{15}$ |
| $\rightarrow \mathcal{P}_{14}$ | $\rightarrow \underline{\hat{x}}_{14}$ |
| $\rightarrow \mathcal{P}_{13}$ | $\rightarrow \underline{\hat{x}}_{13}$ |
| $\rightarrow \mathcal{P}_{12}$ | $\rightarrow \underline{\hat{x}}_{12}$ |
| $\rightarrow \mathcal{P}_{11}$ | $\rightarrow \underline{\hat{x}}_{11}$ |
| $\rightarrow \mathcal{P}_9$    | $\rightarrow \underline{\hat{x}}_9$    |
| $\rightarrow \mathcal{P}_8$    | $\rightarrow \underline{\hat{x}}_8$    |
| $\rightarrow \mathcal{P}_5$    | $\rightarrow \underline{\hat{x}}_5$    |
| $\rightarrow \mathcal{P}_4$    | $\rightarrow \underline{\hat{x}}_4$    |
| $\rightarrow \mathcal{P}_3$    | $\rightarrow \underline{\hat{x}}_3$    |
| $\rightarrow \mathcal{P}_2$    | $\rightarrow \underline{\hat{x}}_2$    |
| $\rightarrow \mathcal{P}_1$    | $\rightarrow \underline{\hat{x}}_1$    |
| $\rightarrow \mathcal{P}_0$    | $\rightarrow \underline{\hat{x}}_0$    |

Wrong identification due to  $\mathcal{H}_{12} (S_2, S_5)$

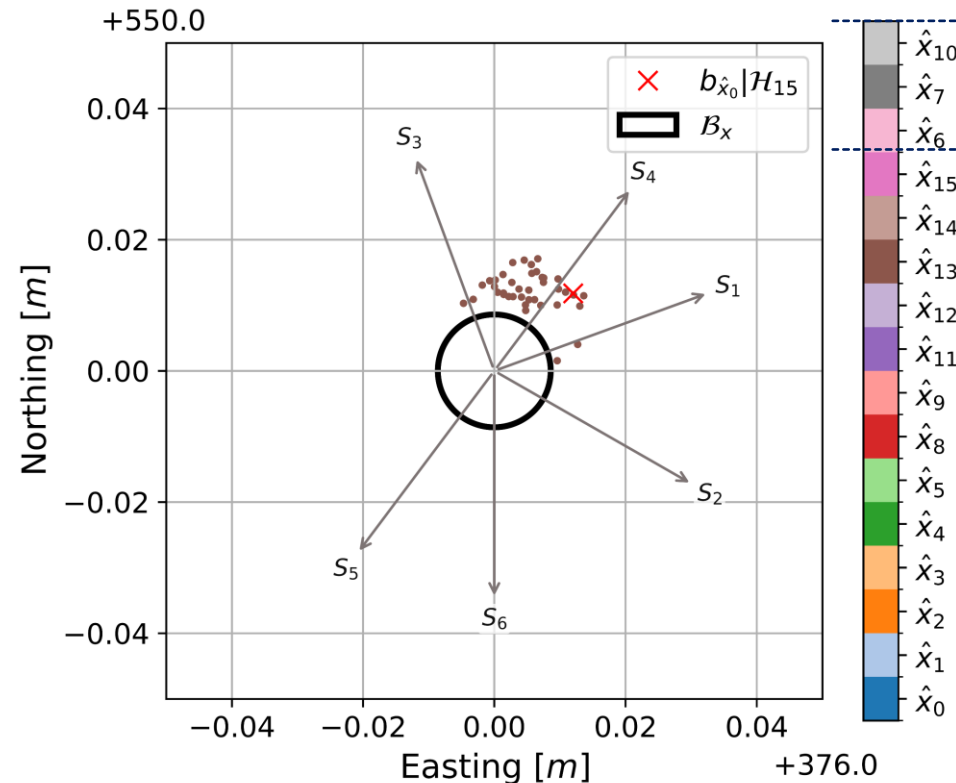
# Adding a Sixth Station

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$

$$\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2$$

$$N = 15$$



$\underline{t} \in$

$\rightarrow \mathcal{P}_{10} \rightarrow \hat{\underline{x}}_{10}$   
 $\rightarrow \mathcal{P}_7 \rightarrow \hat{\underline{x}}_7$   
 $\rightarrow \mathcal{P}_6 \rightarrow \hat{\underline{x}}_6$   
 $\rightarrow \mathcal{P}_{15} \rightarrow \hat{\underline{x}}_{15}$   
 $\rightarrow \mathcal{P}_{14} \rightarrow \hat{\underline{x}}_{14}$   
 $\rightarrow \mathcal{P}_{13} \rightarrow \hat{\underline{x}}_{13}$   
 $\rightarrow \mathcal{P}_{12} \rightarrow \hat{\underline{x}}_{12}$   
 $\rightarrow \mathcal{P}_{11} \rightarrow \hat{\underline{x}}_{11}$   
 $\rightarrow \mathcal{P}_9 \rightarrow \hat{\underline{x}}_9$   
 $\rightarrow \mathcal{P}_8 \rightarrow \hat{\underline{x}}_8$   
 $\rightarrow \mathcal{P}_5 \rightarrow \hat{\underline{x}}_5$   
 $\rightarrow \mathcal{P}_4 \rightarrow \hat{\underline{x}}_4$   
 $\rightarrow \mathcal{P}_3 \rightarrow \hat{\underline{x}}_3$   
 $\rightarrow \mathcal{P}_2 \rightarrow \hat{\underline{x}}_2$   
 $\rightarrow \mathcal{P}_1 \rightarrow \hat{\underline{x}}_1$   
 $\rightarrow \mathcal{P}_0 \rightarrow \hat{\underline{x}}_0$

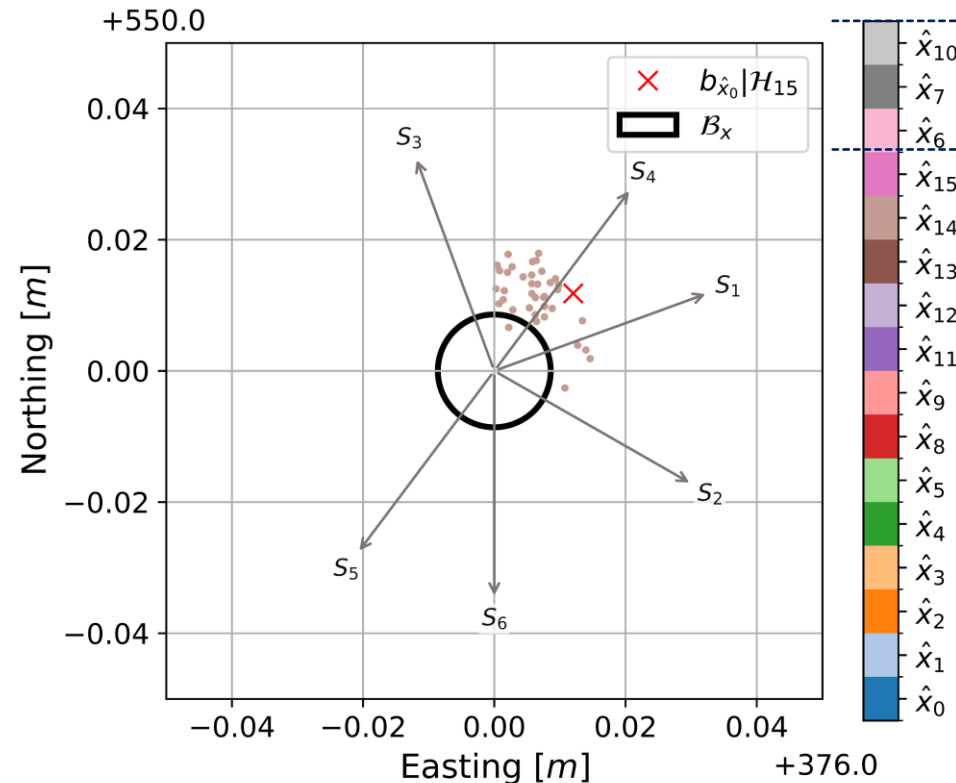
# Adding a Sixth Station

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$

$$\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2$$

$$N = 15$$



$\underline{t} \in$

$\rightarrow \mathcal{P}_{10} \rightarrow \underline{\hat{x}}_{10}$   
 $\rightarrow \mathcal{P}_7 \rightarrow \underline{\hat{x}}_7$   
 $\rightarrow \mathcal{P}_6 \rightarrow \underline{\hat{x}}_6$   
 $\rightarrow \mathcal{P}_{15} \rightarrow \underline{\hat{x}}_{15}$   
 $\rightarrow \mathcal{P}_{14} \rightarrow \underline{\hat{x}}_{14}$   
 $\rightarrow \mathcal{P}_{13} \rightarrow \underline{\hat{x}}_{13}$   
 $\rightarrow \mathcal{P}_{12} \rightarrow \underline{\hat{x}}_{12}$   
 $\rightarrow \mathcal{P}_{11} \rightarrow \underline{\hat{x}}_{11}$   
 $\rightarrow \mathcal{P}_9 \rightarrow \underline{\hat{x}}_9$   
 $\rightarrow \mathcal{P}_8 \rightarrow \underline{\hat{x}}_8$   
 $\rightarrow \mathcal{P}_5 \rightarrow \underline{\hat{x}}_5$   
 $\rightarrow \mathcal{P}_4 \rightarrow \underline{\hat{x}}_4$   
 $\rightarrow \mathcal{P}_3 \rightarrow \underline{\hat{x}}_3$   
 $\rightarrow \mathcal{P}_2 \rightarrow \underline{\hat{x}}_2$   
 $\rightarrow \mathcal{P}_1 \rightarrow \underline{\hat{x}}_1$   
 $\rightarrow \mathcal{P}_0 \rightarrow \underline{\hat{x}}_0$

Wrong identification due to  $\mathcal{H}_{14} (S_3, S_5)$

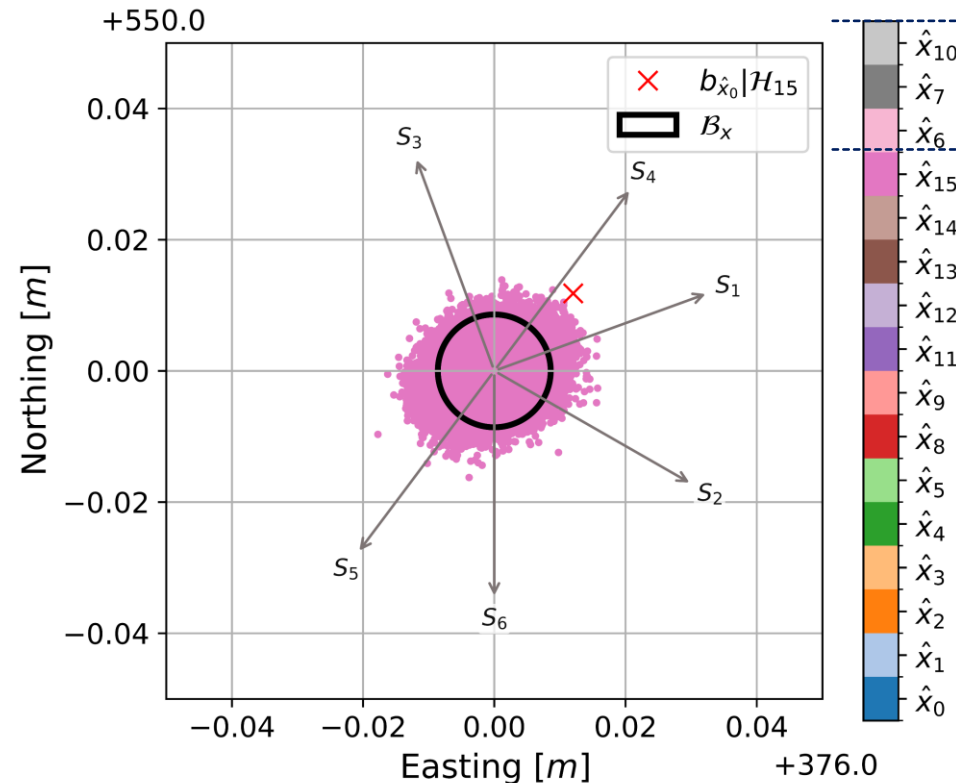
# Adding a Sixth Station

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$

$$\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2$$

$$N = 15$$



Correct identification due to  $\mathcal{H}_{15}$  ( $S_4, S_5$ )

$$\begin{aligned} &\rightarrow \mathcal{P}_{10} \rightarrow \hat{x}_{10} \\ &\rightarrow \mathcal{P}_7 \rightarrow \hat{x}_7 \\ &\rightarrow \mathcal{P}_6 \rightarrow \hat{x}_6 \\ &\rightarrow \mathcal{P}_{15} \rightarrow \hat{x}_{15} \\ &\rightarrow \mathcal{P}_{14} \rightarrow \hat{x}_{14} \\ &\rightarrow \mathcal{P}_{13} \rightarrow \hat{x}_{13} \\ &\rightarrow \mathcal{P}_{12} \rightarrow \hat{x}_{12} \\ &\rightarrow \mathcal{P}_{11} \rightarrow \hat{x}_{11} \\ &\rightarrow \mathcal{P}_9 \rightarrow \hat{x}_9 \\ &\rightarrow \mathcal{P}_8 \rightarrow \hat{x}_8 \\ &\rightarrow \mathcal{P}_5 \rightarrow \hat{x}_5 \\ &\rightarrow \mathcal{P}_4 \rightarrow \hat{x}_4 \\ &\rightarrow \mathcal{P}_3 \rightarrow \hat{x}_3 \\ &\rightarrow \mathcal{P}_2 \rightarrow \hat{x}_2 \\ &\rightarrow \mathcal{P}_1 \rightarrow \hat{x}_1 \\ &\rightarrow \mathcal{P}_0 \rightarrow \hat{x}_0 \end{aligned}$$

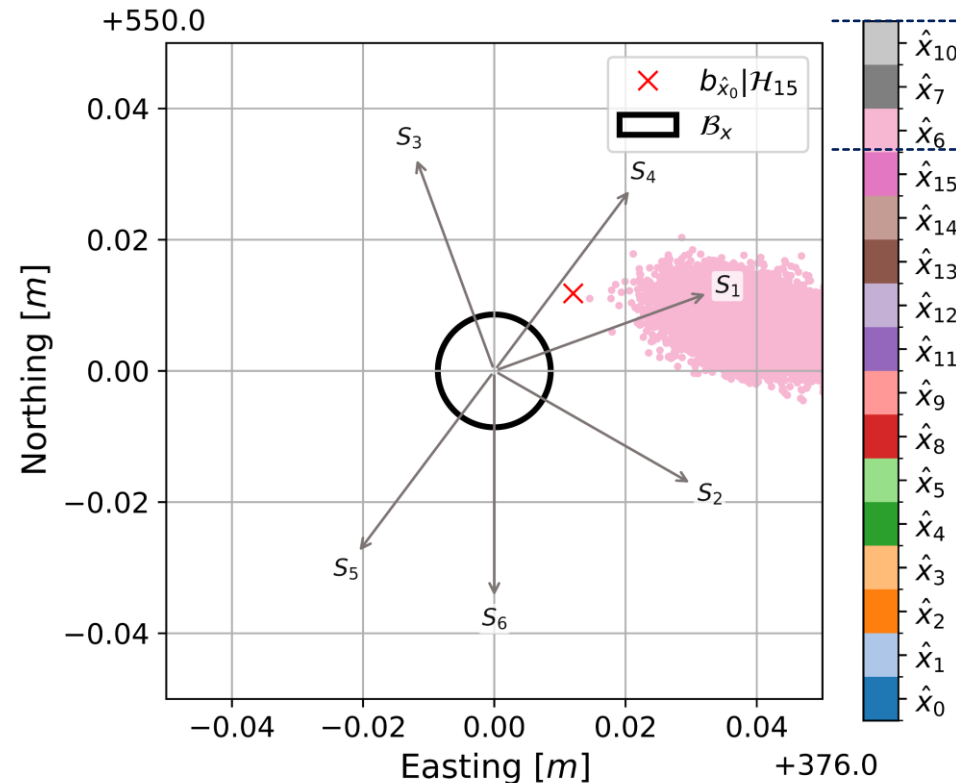
# Adding a Sixth Station

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$

$$\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2$$

$$N = 15$$



$\underline{t} \in$

$\rightarrow \mathcal{P}_{10} \rightarrow \underline{\hat{x}}_{10}$   
 $\rightarrow \mathcal{P}_7 \rightarrow \underline{\hat{x}}_7$   
 $\rightarrow \mathcal{P}_6 \rightarrow \underline{\hat{x}}_6$   
 $\rightarrow \mathcal{P}_{15} \rightarrow \underline{\hat{x}}_{15}$   
 $\rightarrow \mathcal{P}_{14} \rightarrow \underline{\hat{x}}_{14}$   
 $\rightarrow \mathcal{P}_{13} \rightarrow \underline{\hat{x}}_{13}$   
 $\rightarrow \mathcal{P}_{12} \rightarrow \underline{\hat{x}}_{12}$   
 $\rightarrow \mathcal{P}_{11} \rightarrow \underline{\hat{x}}_{11}$   
 $\rightarrow \mathcal{P}_9 \rightarrow \underline{\hat{x}}_9$   
 $\rightarrow \mathcal{P}_8 \rightarrow \underline{\hat{x}}_8$   
 $\rightarrow \mathcal{P}_5 \rightarrow \underline{\hat{x}}_5$   
 $\rightarrow \mathcal{P}_4 \rightarrow \underline{\hat{x}}_4$   
 $\rightarrow \mathcal{P}_3 \rightarrow \underline{\hat{x}}_3$   
 $\rightarrow \mathcal{P}_2 \rightarrow \underline{\hat{x}}_2$   
 $\rightarrow \mathcal{P}_1 \rightarrow \underline{\hat{x}}_1$   
 $\rightarrow \mathcal{P}_0 \rightarrow \underline{\hat{x}}_0$

Wrong identification due to  $\mathcal{H}_6 (S_1, S_2)$

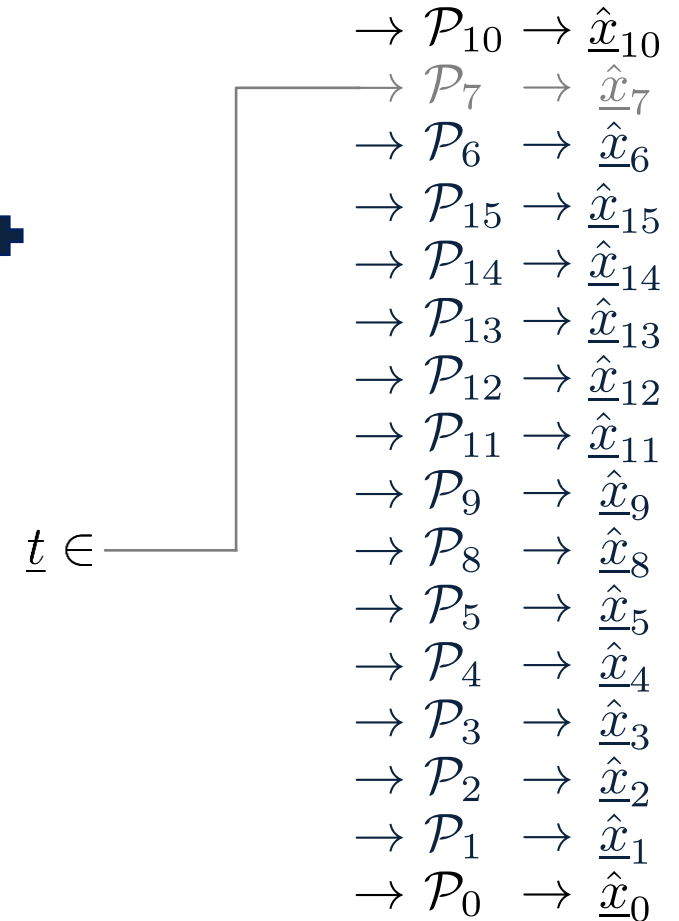
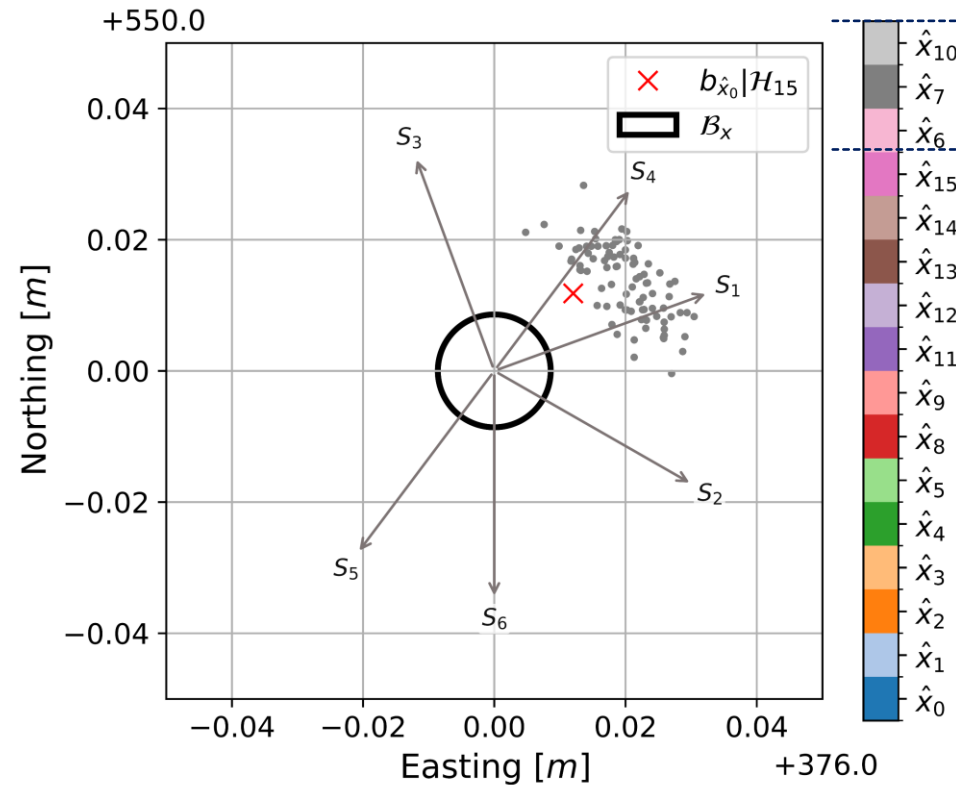
# Adding a Sixth Station

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$

$$\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2$$

$$N = 15$$



Wrong identification due to  $\mathcal{H}_7 (S_1, S_3)$

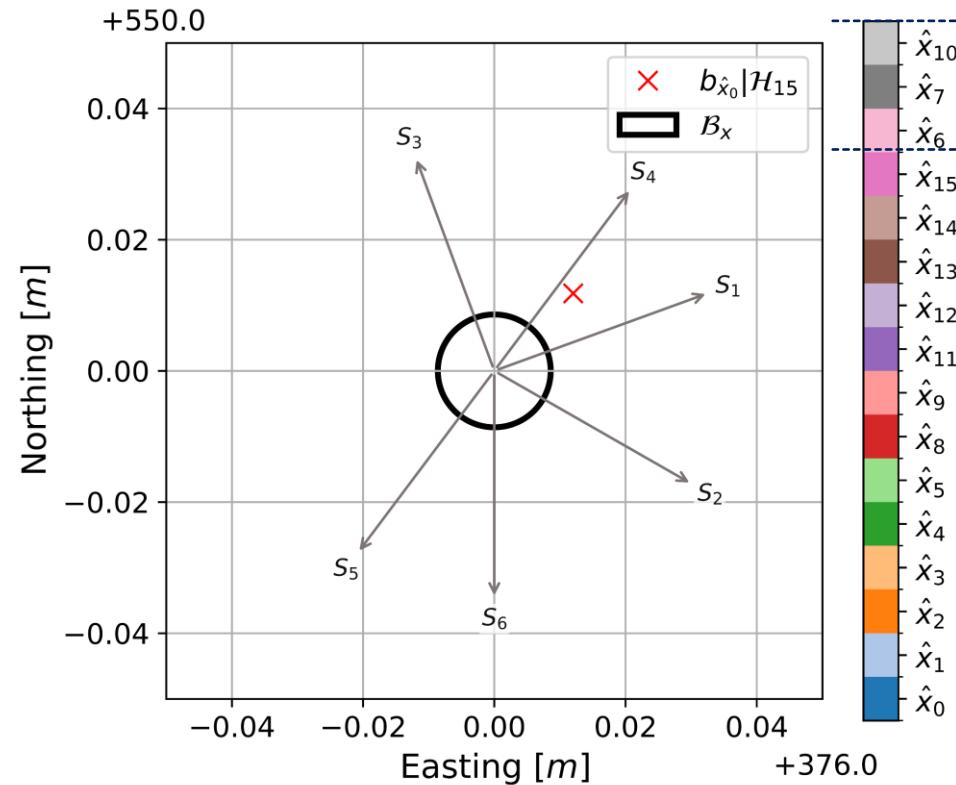
# Adding a Sixth Station

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$

$$\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$$

$$\text{IC: } \sum_{j=1}^5 z_j \leq 2$$

$$N = 15$$



$\underline{t} \in$

$\rightarrow \mathcal{P}_{10} \rightarrow \hat{\underline{x}}_{10}$   
 $\rightarrow \mathcal{P}_7 \rightarrow \hat{\underline{x}}_7$   
 $\rightarrow \mathcal{P}_6 \rightarrow \hat{\underline{x}}_6$   
 $\rightarrow \mathcal{P}_{15} \rightarrow \hat{\underline{x}}_{15}$   
 $\rightarrow \mathcal{P}_{14} \rightarrow \hat{\underline{x}}_{14}$   
 $\rightarrow \mathcal{P}_{13} \rightarrow \hat{\underline{x}}_{13}$   
 $\rightarrow \mathcal{P}_{12} \rightarrow \hat{\underline{x}}_{12}$   
 $\rightarrow \mathcal{P}_{11} \rightarrow \hat{\underline{x}}_{11}$   
 $\rightarrow \mathcal{P}_9 \rightarrow \hat{\underline{x}}_9$   
 $\rightarrow \mathcal{P}_8 \rightarrow \hat{\underline{x}}_8$   
 $\rightarrow \mathcal{P}_5 \rightarrow \hat{\underline{x}}_5$   
 $\rightarrow \mathcal{P}_4 \rightarrow \hat{\underline{x}}_4$   
 $\rightarrow \mathcal{P}_3 \rightarrow \hat{\underline{x}}_3$   
 $\rightarrow \mathcal{P}_2 \rightarrow \hat{\underline{x}}_2$   
 $\rightarrow \mathcal{P}_1 \rightarrow \hat{\underline{x}}_1$   
 $\rightarrow \mathcal{P}_0 \rightarrow \hat{\underline{x}}_0$

# Adding a Sixth Station

- Under alternative hypothesis  $\mathcal{H}_{15}$ : outliers of  $-30$  mm on  $S_4$  and  $30$  mm on  $S_5$

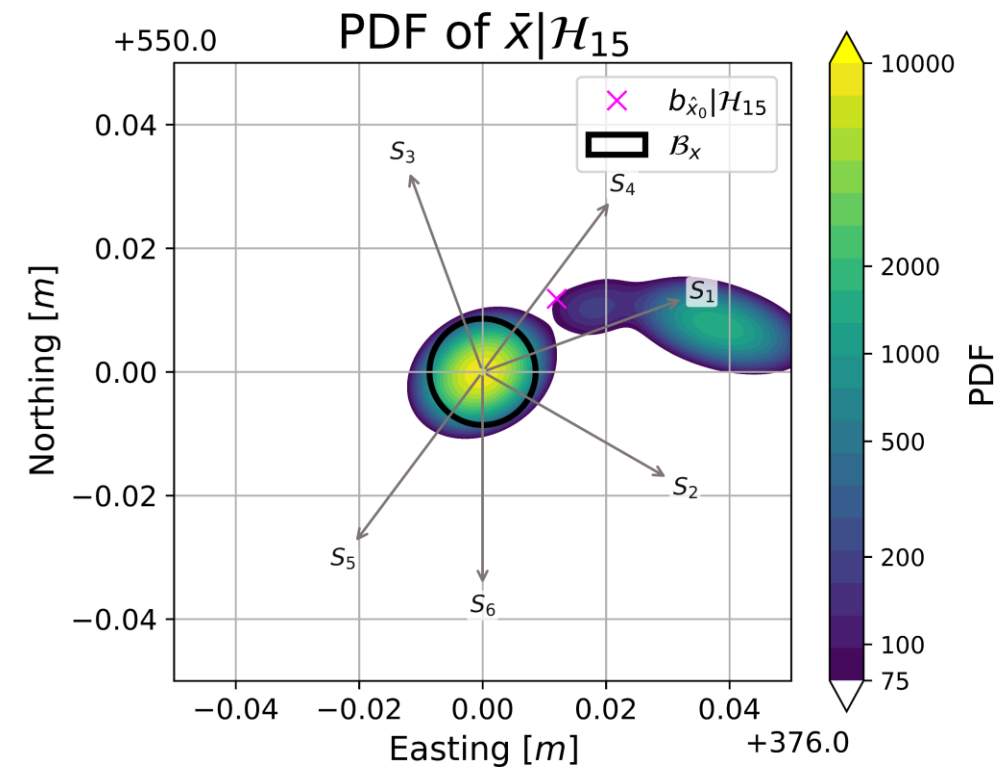
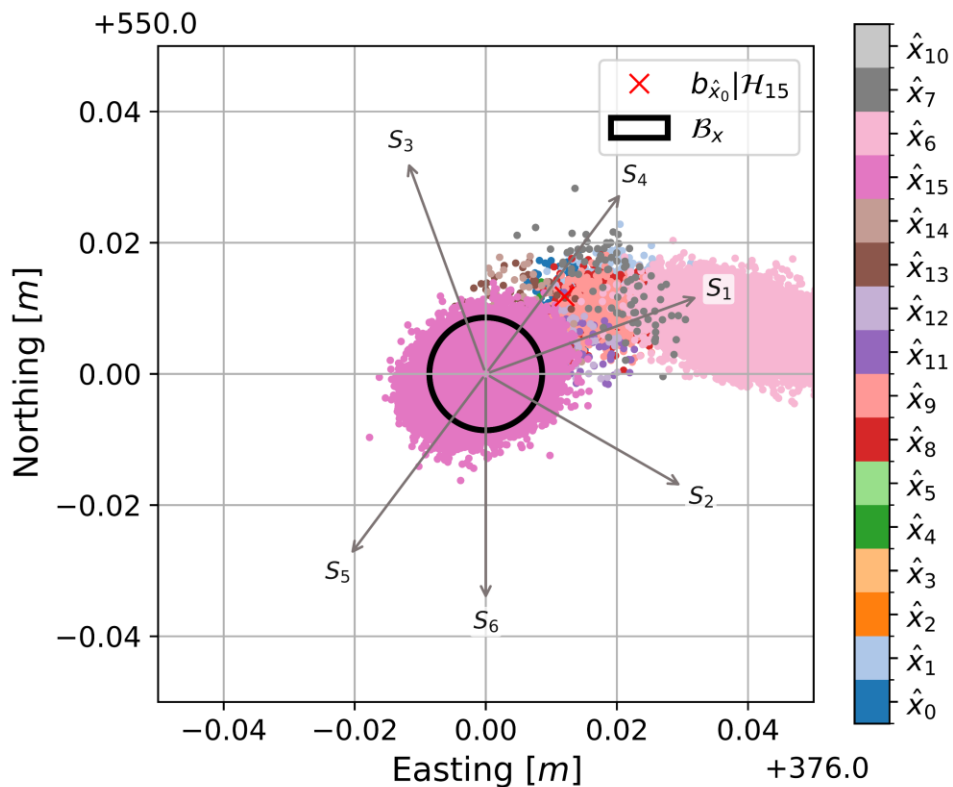
$$\mathcal{H}_{15} : z = [0, 0, 0, 1, 1]$$

$$\bar{x} = \sum_{i=0}^N \hat{x}_i \mathbf{1}_{\mathcal{P}_i}(t)$$

$$\text{IC: } \sum_{j=1}^5 \mathbf{z}_j \leq 2$$

$$N = 15$$

$$f_{\bar{x}}(x) = \sum_{i=0}^N \mathbb{E}[f_{\hat{x}_0}(x + A^+ C \bar{Z}(t) \bar{b}(t)) \mathbf{1}_{\mathcal{P}_i}(t)]$$



$$P(\bar{x} \notin \mathcal{B}_x | \mathcal{H}_{15}) = 24.5\%$$

Extra range improves estimation & identification;  
 $\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_{10}$  restored

# Adding a Sixth Station

$$\text{IC: } \sum_{j=1}^5 \mathbf{z}_j \leq \mathbf{2}$$

$$N = 15$$

► Under  $\mathcal{H}_0$ :

|                                                             | $\mathcal{P}_0$ | $\mathcal{P}_1$ | $\mathcal{P}_2$ | $\mathcal{P}_3$ | $\mathcal{P}_4$ | $\mathcal{P}_5$ | $\mathcal{P}_6$ | $\mathcal{P}_7$ | $\mathcal{P}_8$ | $\mathcal{P}_9$ | $\mathcal{P}_{10}$ | $\mathcal{P}_{11}$ | $\mathcal{P}_{12}$ | $\mathcal{P}_{13}$ | $\mathcal{P}_{14}$ | $\mathcal{P}_{15}$ |
|-------------------------------------------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|--------------------|--------------------|--------------------|--------------------|--------------------|--------------------|
| $P(\underline{t} \in \mathcal{P}_i \mid \mathcal{H}_0)$ [%] | 94.98           | 0.25            | 0.28            | 0.25            | 0.35            | 0.33            | 0.22            | 0.34            | 0.33            | 0.28            | 0.50               | 0.32               | 0.27               | 0.46               | 0.46               | 0.38               |

► Under  $\mathcal{H}_{15}$ :

|                                                                | $\mathcal{P}_0$ | $\mathcal{P}_1$ | $\mathcal{P}_2$ | $\mathcal{P}_3$ | $\mathcal{P}_4$ | $\mathcal{P}_5$ | $\mathcal{P}_6$ | $\mathcal{P}_7$ | $\mathcal{P}_8$ | $\mathcal{P}_9$ | $\mathcal{P}_{10}$ | $\mathcal{P}_{11}$ | $\mathcal{P}_{12}$ | $\mathcal{P}_{13}$ | $\mathcal{P}_{14}$ | $\mathcal{P}_{15}$ |
|----------------------------------------------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|--------------------|--------------------|--------------------|--------------------|--------------------|--------------------|
| $P(\underline{t} \in \mathcal{P}_i \mid \mathcal{H}_{15})$ [%] | 0.16            | 0.40            | 0.00            | 0.00            | 0.03            | 0.02            | 17.46           | 0.10            | 0.58            | 0.55            | 0.00               | 0.05               | 0.04               | 0.04               | 0.04               | 80.54              |