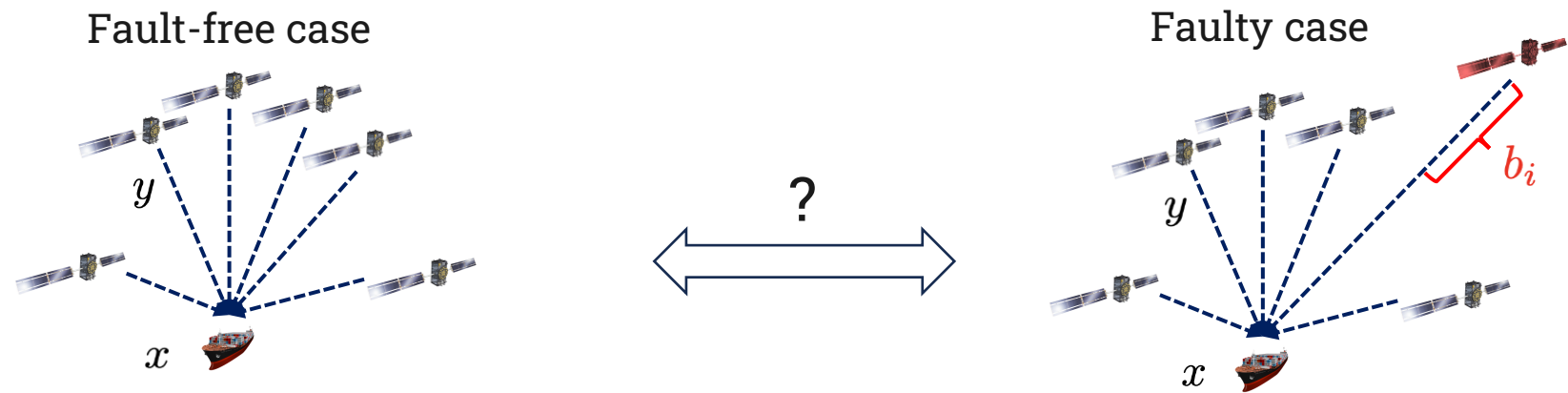


On the choice of penalty functions and their effect on the probability of positioning failure

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XI Hotine-Marussi Symposium, Genova,
Session S2A: Geodetic quality/integrity modelling

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$\mathcal{H}_0 : \hat{x}_0$ - all-in-view solution

estimation

$\mathcal{H}_i : \hat{x}_i$ - excluding biased observations

Tienstra transformation:

$$\begin{bmatrix} \hat{x}_0 \\ \underline{t} \end{bmatrix} = \mathcal{T} \underline{y}$$

pseudorange **observations** in linearized GNSS model

testing

$\underline{t}$ - misclosures ('tegenspraak')

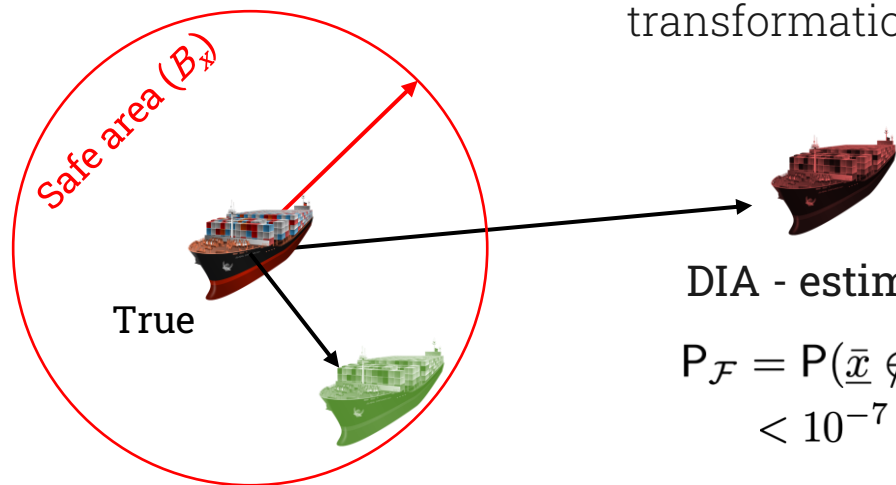
DIA - estimator

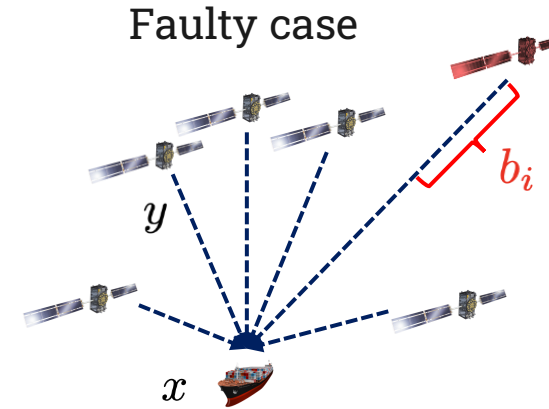
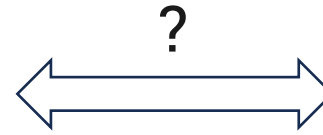
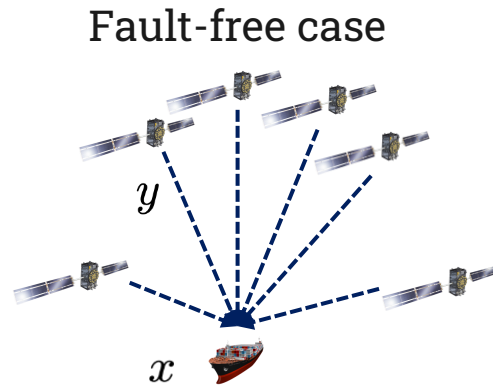
$$P_{\mathcal{F}} = P(\bar{x} \notin \mathcal{B}_x) < 10^{-7} \dots 10^{-5}$$

$$\bar{x} = \sum_{i=0}^k \hat{x}_i p_i(\underline{t})$$

$$p_i(\underline{t}) = \begin{cases} 1 & \text{if select } \mathcal{H}_i \\ 0, & \text{otherwise} \end{cases}$$

indicator function defines testing criteria





Two choices of $p_i(\underline{t})$:

Option 1. Maximize probability of correct identification P_{CI} :

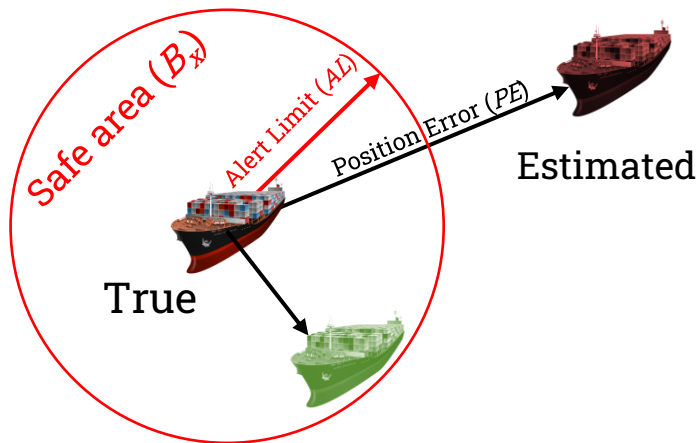
Option 2. Minimize P_F :

select $\mathcal{H}_i$ if $i = \arg \max_{j \in [0, \dots, k]} P^0(\mathcal{H}_j | t)$

select $\mathcal{H}_i$ if $i = \arg \min_{j \in [0, \dots, k]} P^0(\hat{x}_j \notin B_x | t) \implies \min P^0(\bar{x} \notin \mathcal{B}_x)$

functions of unknown approximated $b_j := \hat{b}_j(t)$

non-optimal, but an upper bound



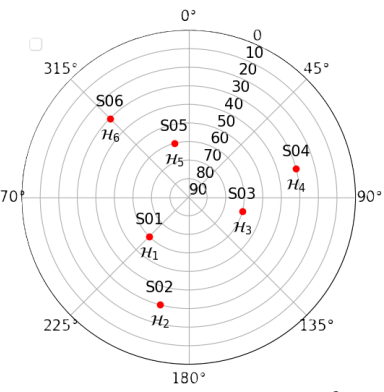
Benefits of Option 2?

Option 1 vs. Option 2, estimated bias

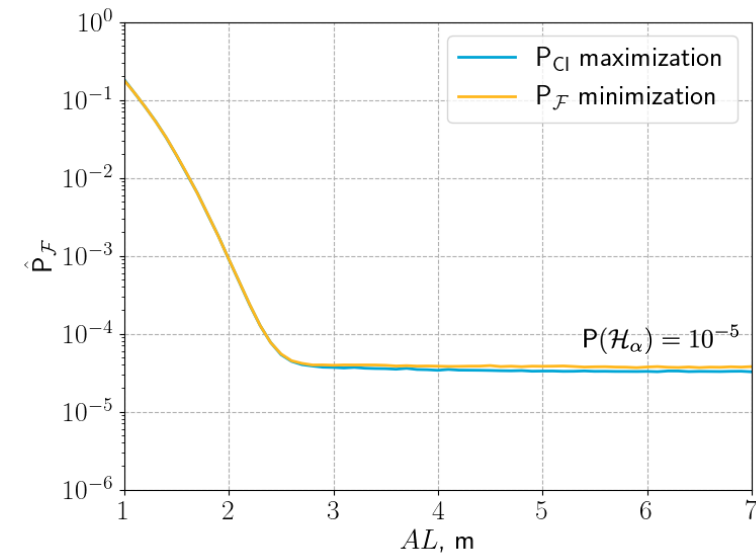
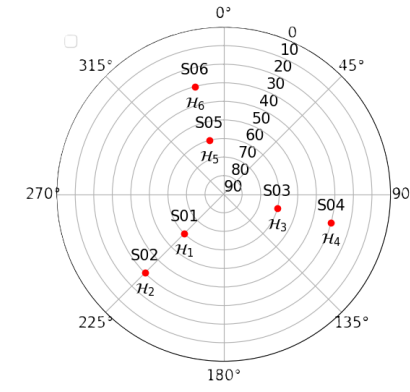
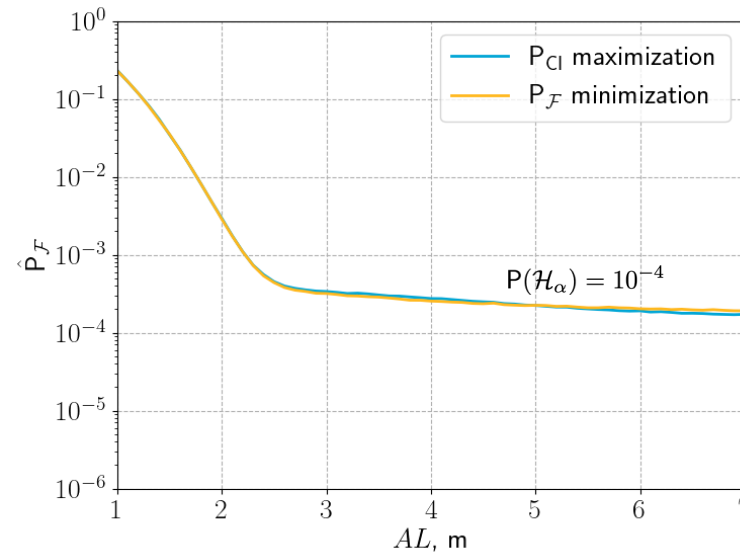
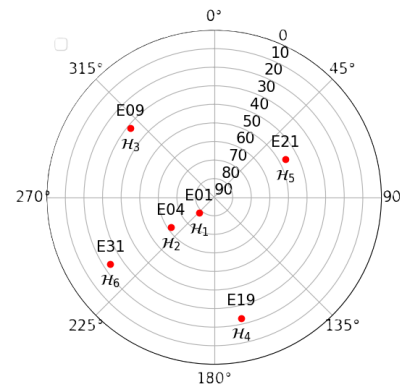
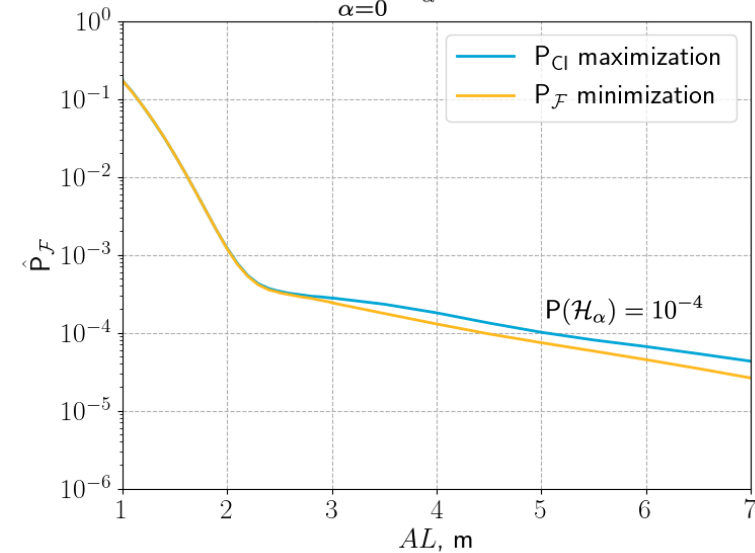
Scenario:
horizontal positioning:

$$\sigma_y = 0.65 \text{ m}$$

$$Q_{yy} = \sigma_y^2 I_6$$



$$\hat{P}_{\mathcal{F}} = \sum_{\alpha=0}^k \max_{b_{\alpha}} \hat{P}(\bar{x} \notin \mathcal{B}_x | \mathcal{H}_{\alpha}) P(\mathcal{H}_{\alpha})$$



Not always beneficial
due to approximation

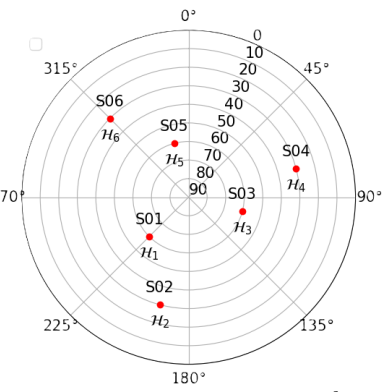
$$r_{j\alpha}(t) := P^0(\hat{x}_j \notin \mathcal{B}_x | \hat{b}_{\alpha}(t), \mathcal{H}_{\alpha}, t)$$

Option 1 vs. Option 2, known bias

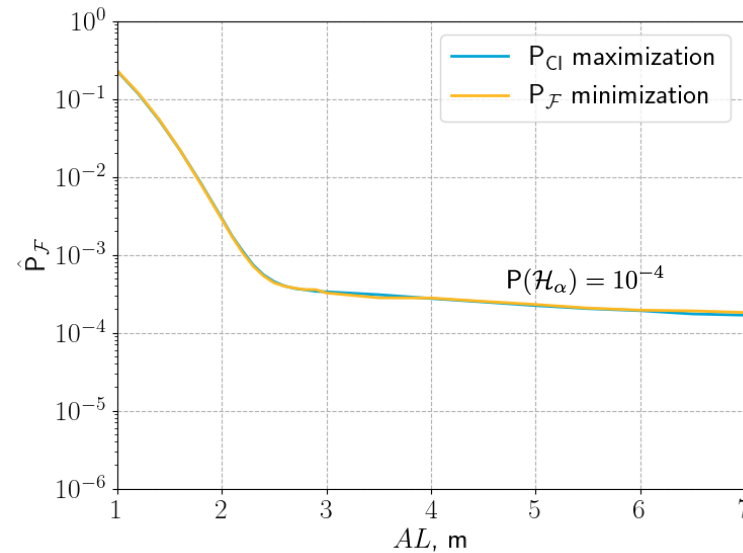
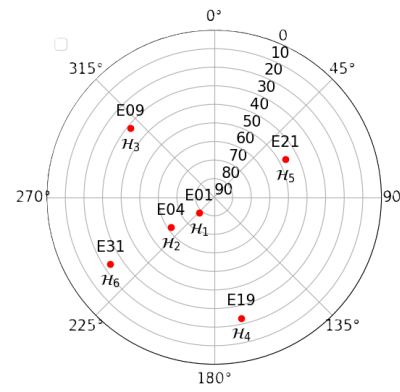
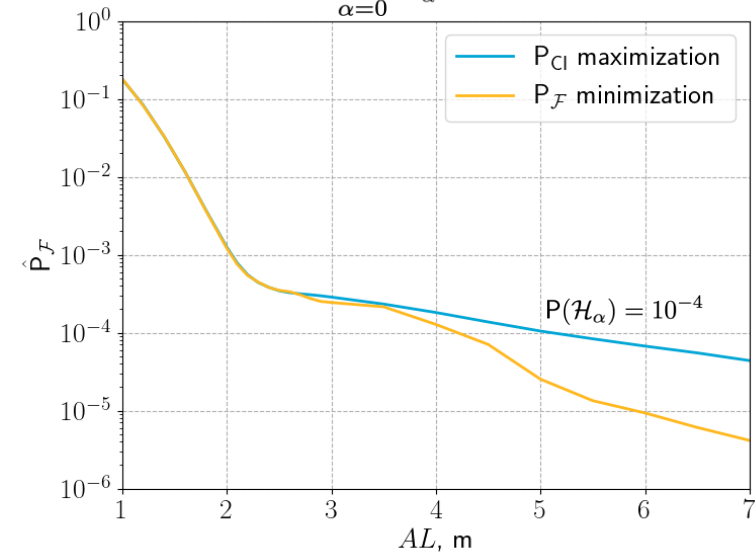
Scenario:
horizontal positioning:

$$\sigma_y = 0.65 \text{ m}$$

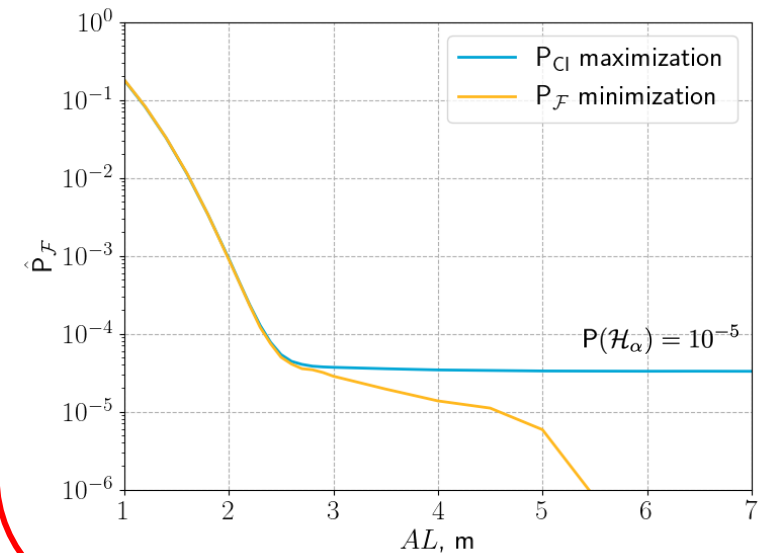
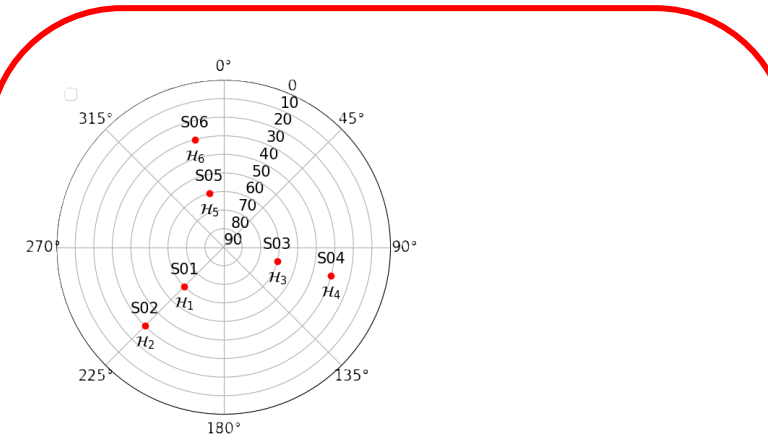
$$Q_{yy} = \sigma_y^2 I_6$$



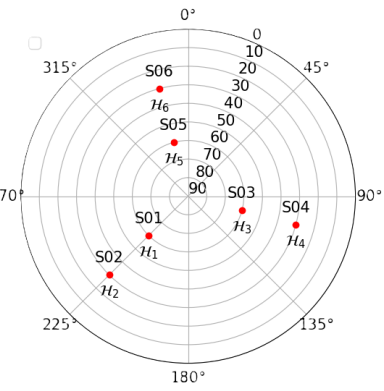
$$\hat{P}_{\mathcal{F}} = \sum_{\alpha=0}^k \max_{b_{\alpha}} \hat{P}(\underline{\hat{x}} \notin \mathcal{B}_x | \mathcal{H}_{\alpha}) P(\mathcal{H}_{\alpha})$$



Reference results
(known bias in penalties)

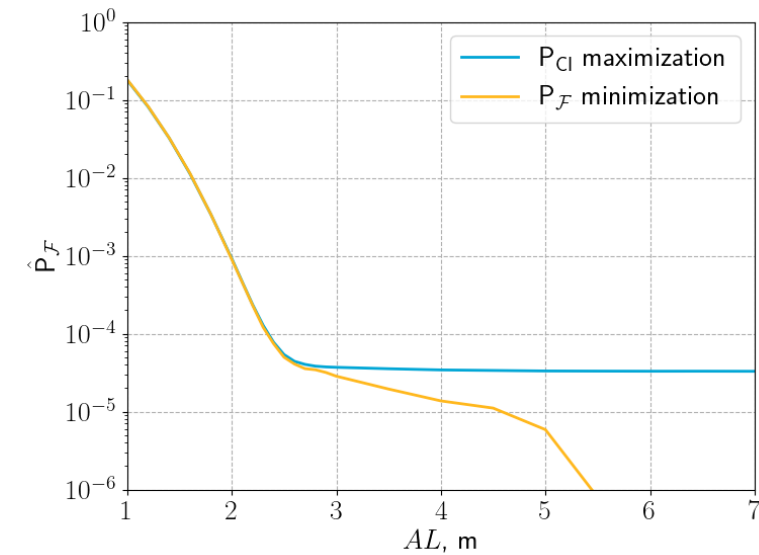
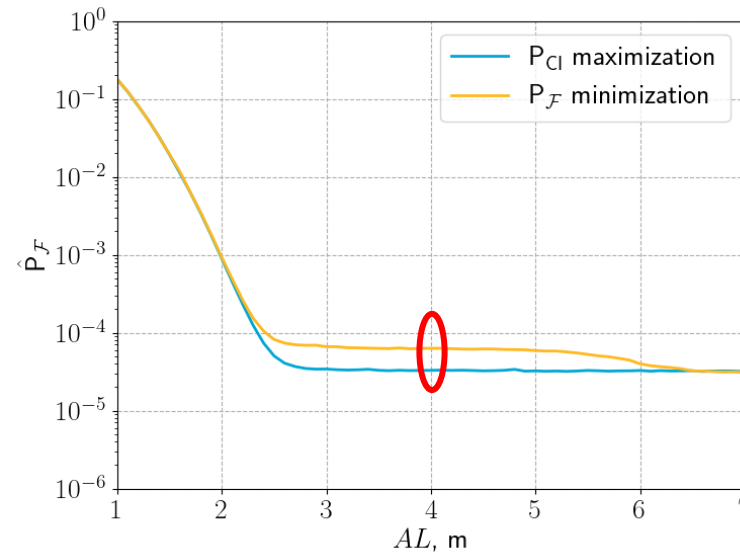
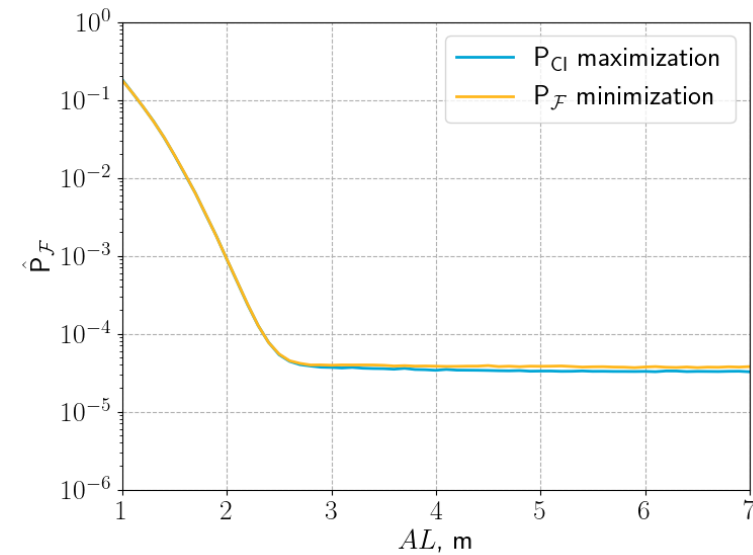
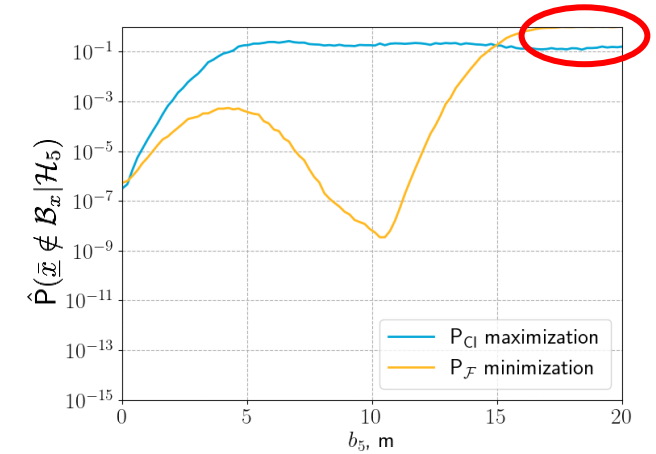


$$r_{j\alpha}(t) := P(\underline{\hat{x}}_j \notin \mathcal{B}_x | \mathbf{b}_{\alpha}, \mathcal{H}_{\alpha}, t)$$



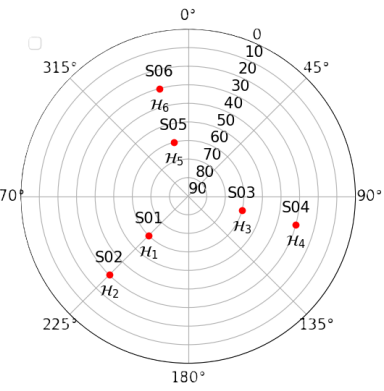
$$r_{j\alpha}(t) := r_{j\alpha}(b_{crit,\alpha}, t)$$

$$b_{crit,\alpha} := \arg \max_{b_\alpha} \hat{P}_{\mathcal{F}(P_{CI\max})} | \mathcal{H}_\alpha, b_\alpha$$



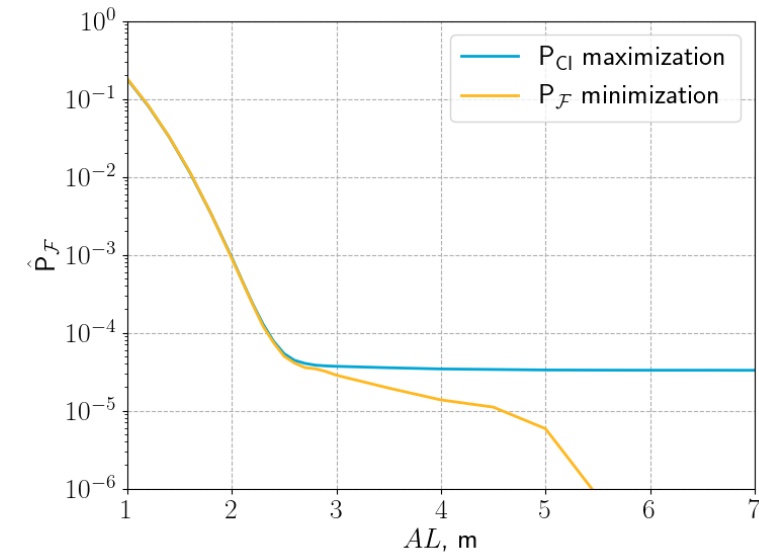
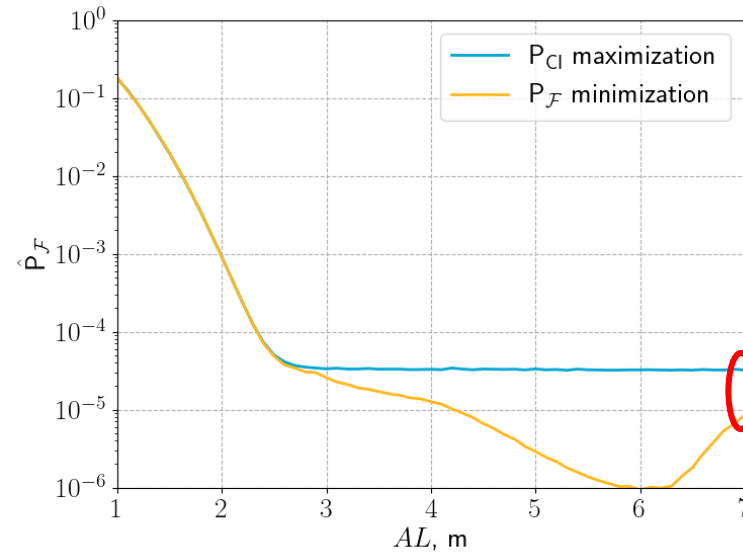
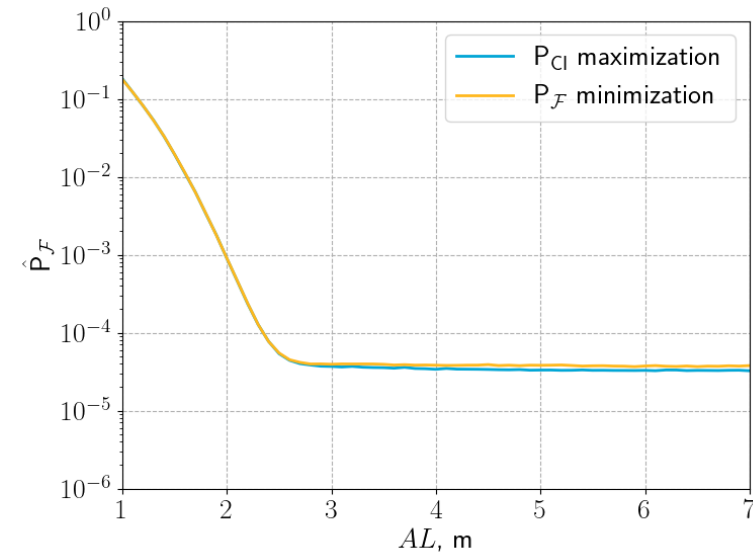
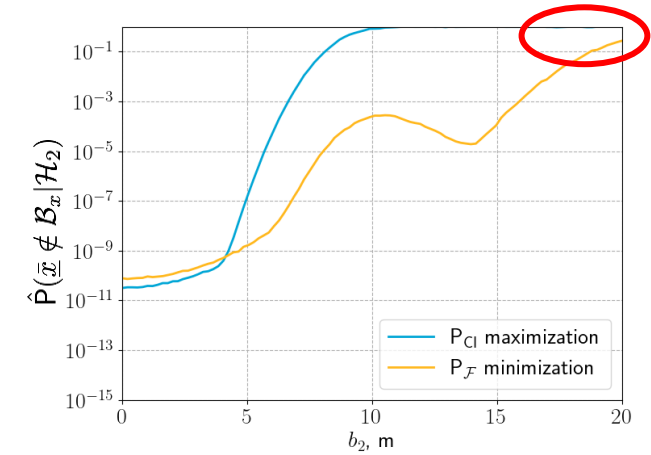
$$r_{j\alpha}(t) := P^0(\hat{x}_j \notin \mathcal{B}_x | \hat{b}_\alpha(t), \mathcal{H}_\alpha, t)$$

$$r_{j\alpha}(t) := P(\hat{x}_j \notin \mathcal{B}_x | b_\alpha, \mathcal{H}_\alpha, t)$$



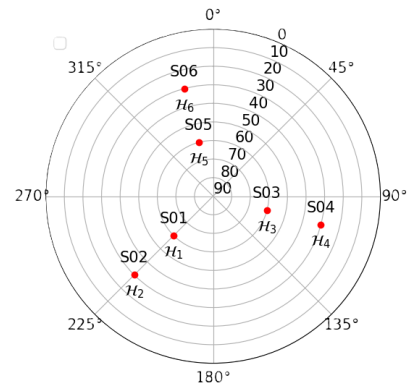
$$r_{j\alpha}(t) := r_{j\alpha}(b_{crit,\alpha}, t)$$

$$b_{crit,\alpha} := MDB_\alpha(10^{-7}, 10^{-7})$$



$$r_{j\alpha}(t) := P^0(\hat{x}_j \notin \mathcal{B}_x | \hat{b}_\alpha(t), \mathcal{H}_\alpha, t)$$

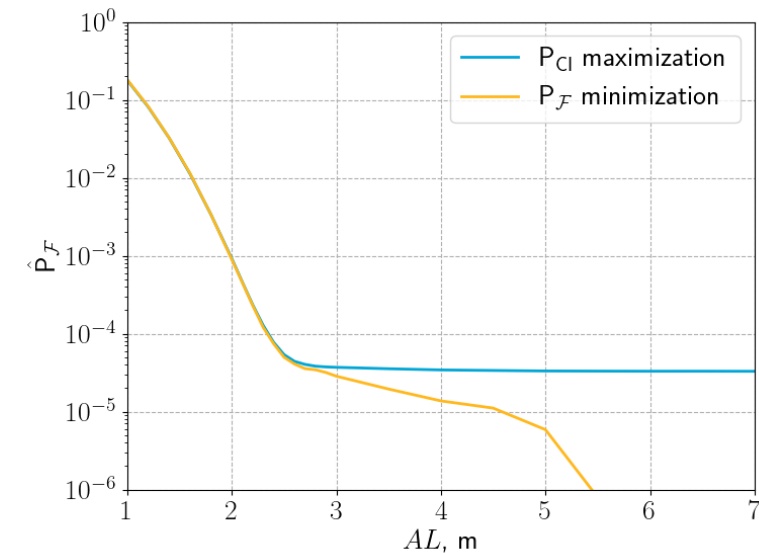
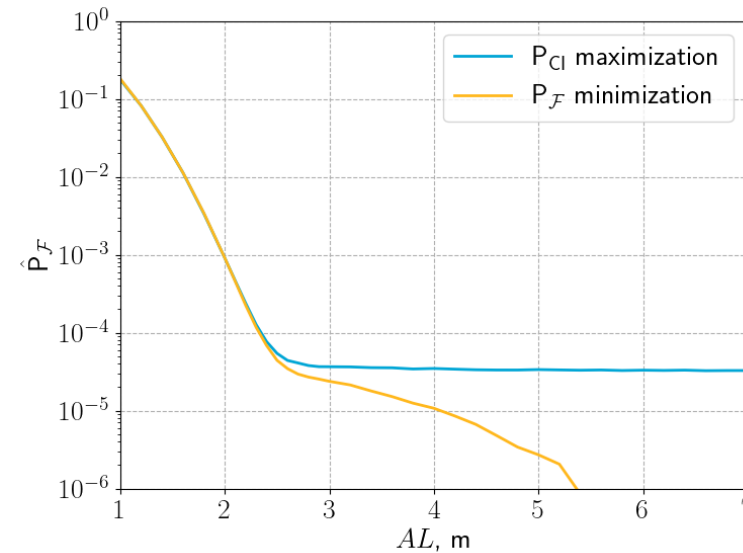
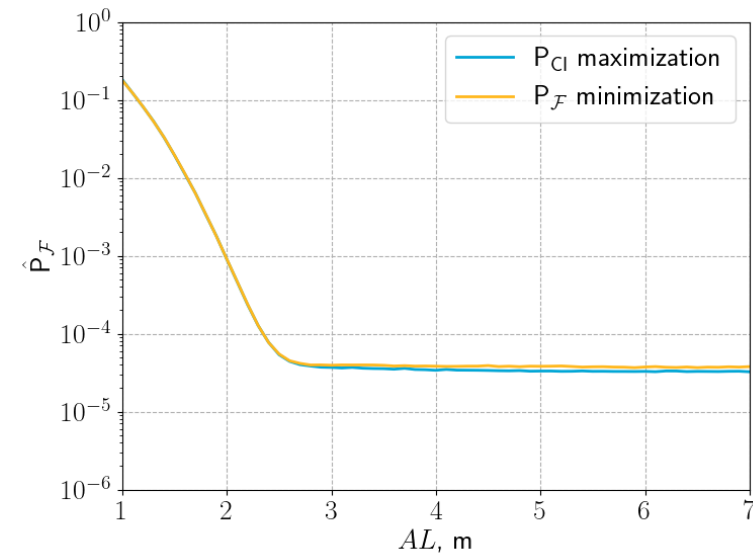
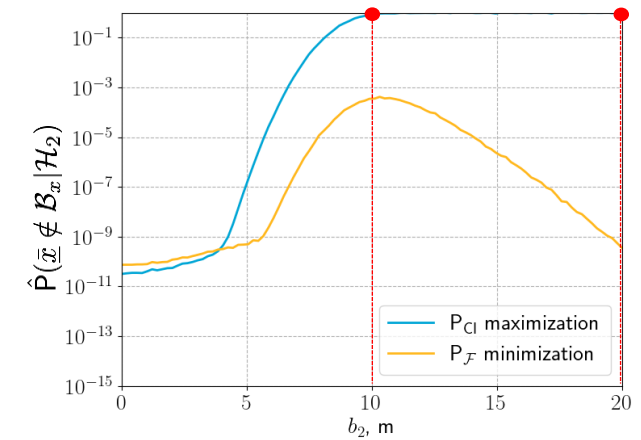
$$r_{j\alpha}(t) := P(\hat{x}_j \notin \mathcal{B}_x | b_\alpha, \mathcal{H}_\alpha, t)$$



$$r_{j\alpha}(t) := \frac{1}{5} \sum_{k=0}^4 r_{j\alpha}(b_{crit,\alpha_k}, t)$$

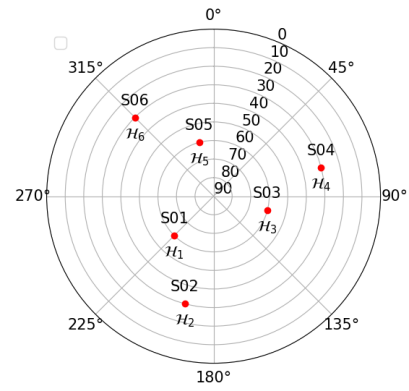
$$b_{crit,\alpha_k} = \arg \max_{b_\alpha} \hat{P}_{\mathcal{F}(P_{Clmax})} | \mathcal{H}_\alpha + k\Delta$$

$$\Delta = 10 \text{ m}$$



$$r_{j\alpha}(t) := P^0(\hat{x}_j \notin \mathcal{B}_x | \hat{b}_\alpha(t), \mathcal{H}_\alpha, t)$$

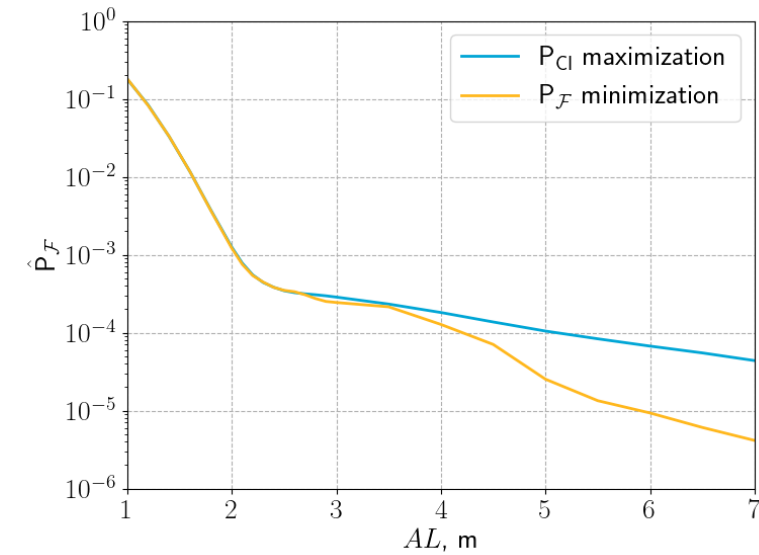
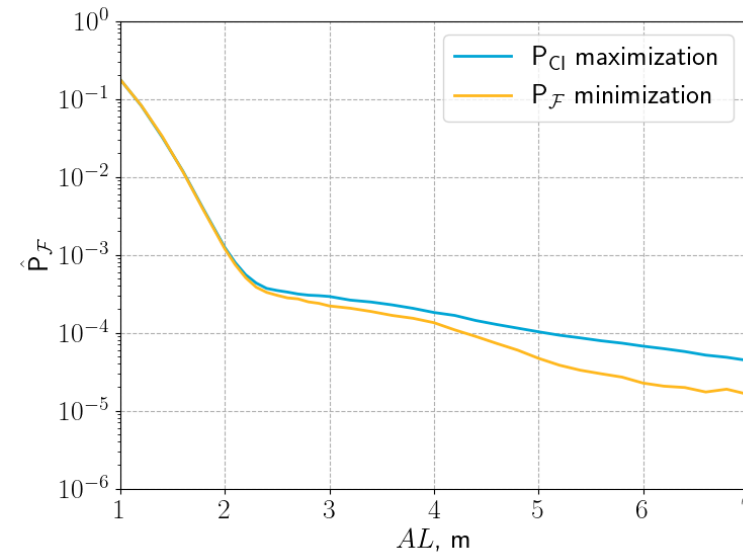
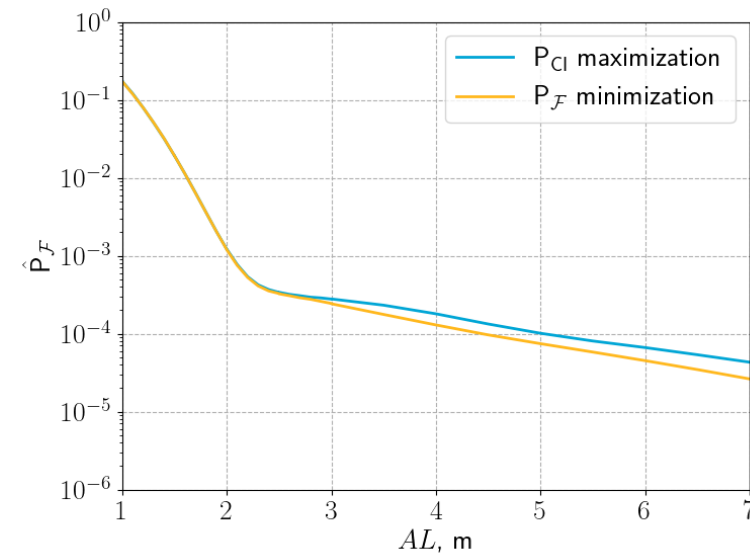
$$r_{j\alpha}(t) := P(\hat{x}_j \notin \mathcal{B}_x | b_\alpha, \mathcal{H}_\alpha, t)$$



$$r_{j\alpha}(t) := \frac{1}{5} \sum_{k=0}^4 r_{j\alpha}(b_{crit,\alpha_k}, t)$$

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$$\Delta = 10 \text{ m}$$



$$r_{j\alpha}(t) := P^0(\hat{x}_j \notin \mathcal{B}_x | \hat{b}_\alpha(t), \mathcal{H}_\alpha, t)$$

$$r_{j\alpha}(t) := P(\hat{x}_j \notin \mathcal{B}_x | \mathbf{b}_\alpha, \mathcal{H}_\alpha, t)$$

- Penalized testing
 - considers consequences of the decision
 - can decrease probability of positioning failure
- Improvement is limited by unknown bias estimation in penalty functions and depends on constellation geometry
- Several alternative penalty functions have been investigated
- Linear combination of penalties for the critical biases performs better than a single penalty for the estimated bias

Thank you for your attention

Serge Kaplev
s.kaplev@tudelft.nl

More math:

DIA- estimator:
(Teunissen, 2018)



Penalized testing:
(Teunissen, 2024)

