

The Mixed-Integer Misclosure Test in Spectral Form

- Understanding its Powerfunction Characteristics -

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Mixed-integer model:

$$\mathcal{H}_0 : \underline{y} \sim \mathcal{N}_m(Aa + Bb, Q_{yy})$$

$$a \in \mathbb{Z}^n, b \in \mathbb{R}^p$$

IAR-Detector:

$$\underline{T} = ||\underline{\hat{e}}||^2_{Q_{yy}} + ||\underline{\check{\epsilon}}||^2_{Q_{\hat{a}\hat{a}}}$$

$$\underline{\check{\epsilon}} = \underline{\hat{a}} - \underline{\check{a}}$$

$$\underline{\check{a}} = \mathcal{I}(\underline{\hat{a}})$$

[PJG Teunissen (2024) J.Geod.]

$$\mathcal{I} : \mathbb{R}^n \mapsto \mathbb{Z}^n$$

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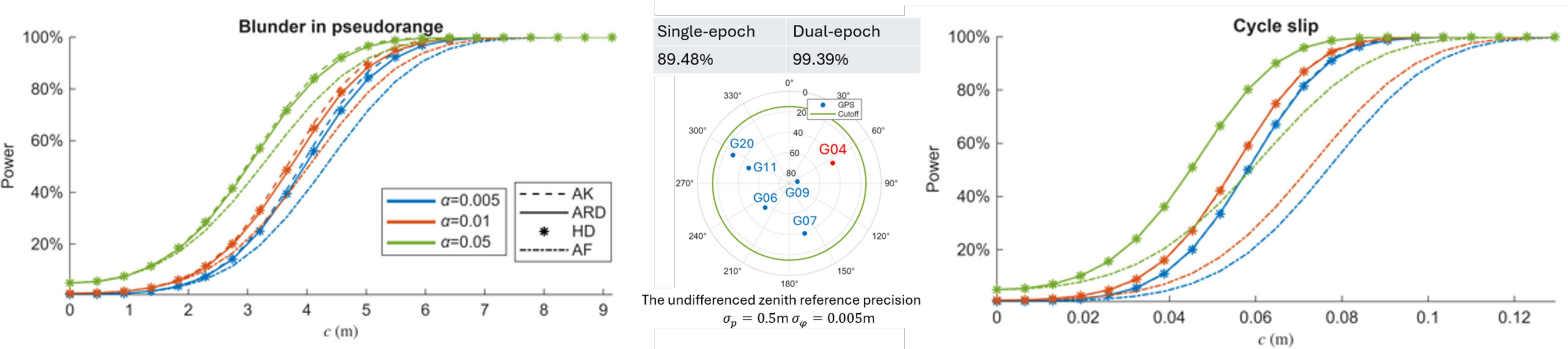
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AR-detector vs AF-detector

for pseudorange-blunder & carrier-phase cycle-slip



$\mathcal{H}_i : \underline{y} \sim \mathcal{N}_m(Aa + Bb + C_i c_i, Q_{yy})$

$c_i \in \mathbb{R}_{\neq 0}^{q_i}$

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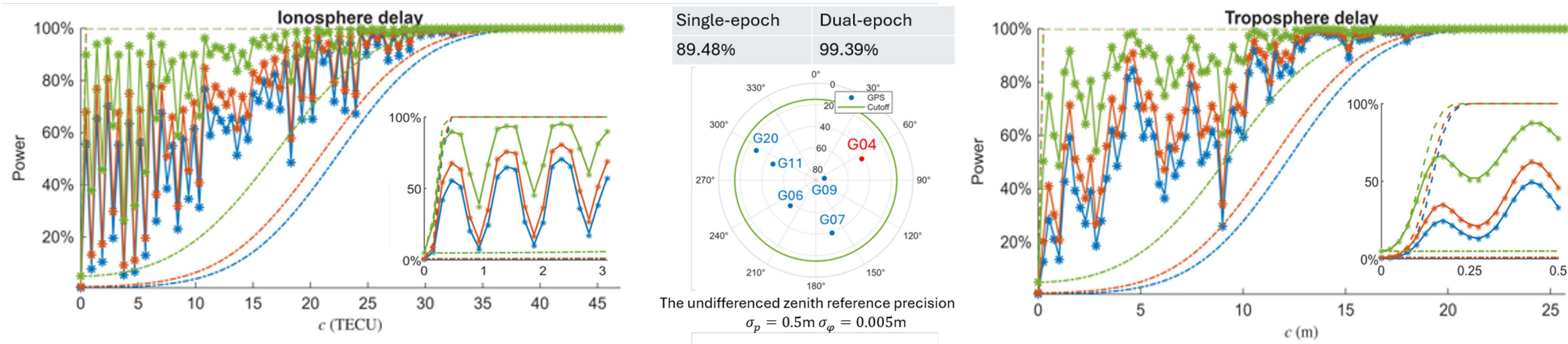
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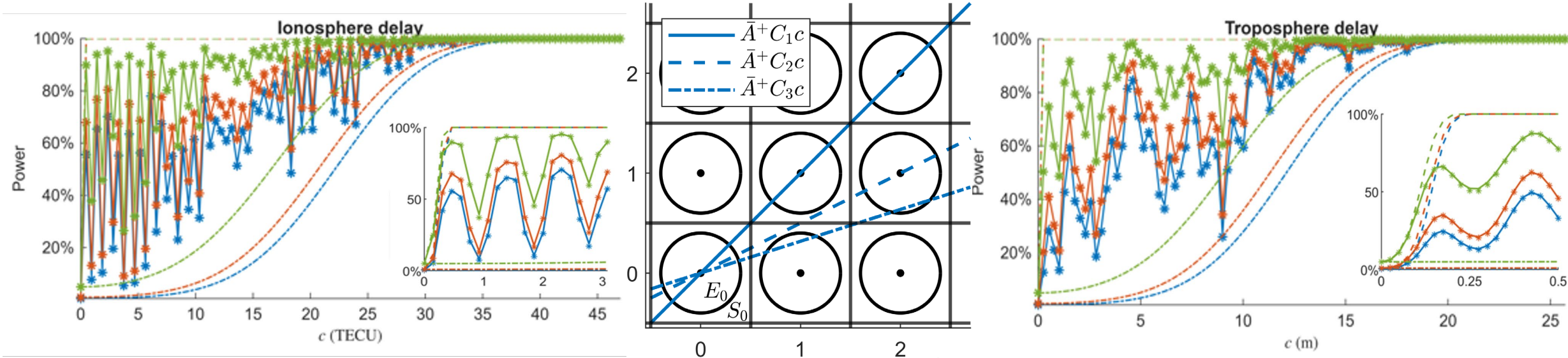
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AR-detector vs AF-detector
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$\underline{\hat{a}} \overset{\mathcal{H}_a}{\sim} \mathcal{N}_n(a + \bar{A}^+Cc, Q_{\hat{a}\hat{a}})$

,

$||\check{\epsilon}(a + \bar{A}^+Cc)||_{Q_{\hat{a}\hat{a}}}^2$

,

Periodic if $\bar{A}^+Cc \in \mathbb{Z}^n$

Power function in Spectral Form

$$P(c) = \mathbb{P}[\underline{T} > \kappa | \mathcal{H}_a(c)]$$

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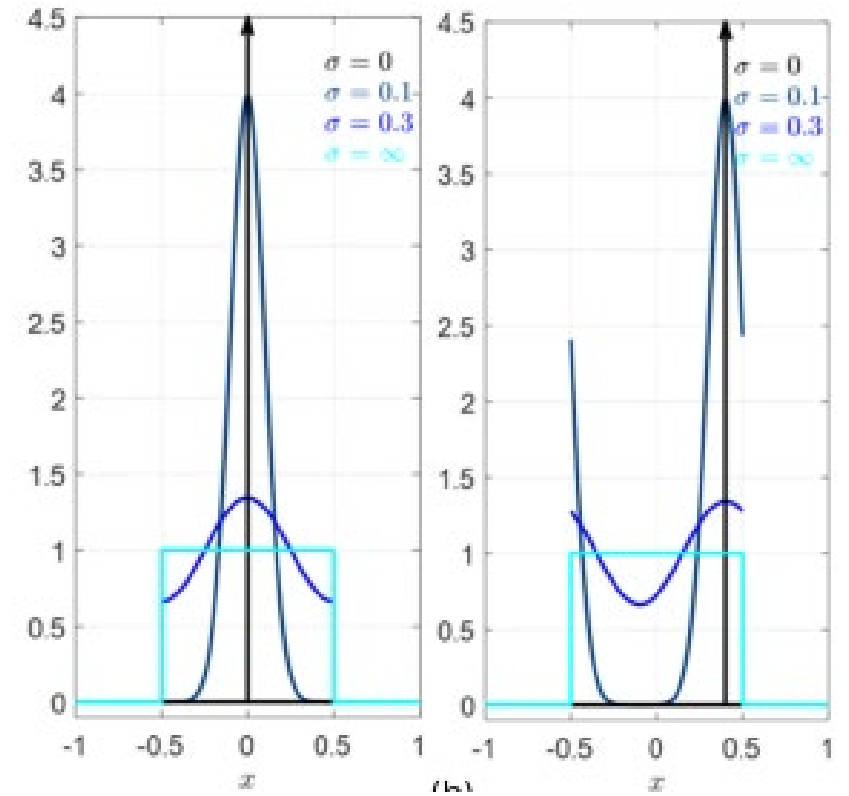
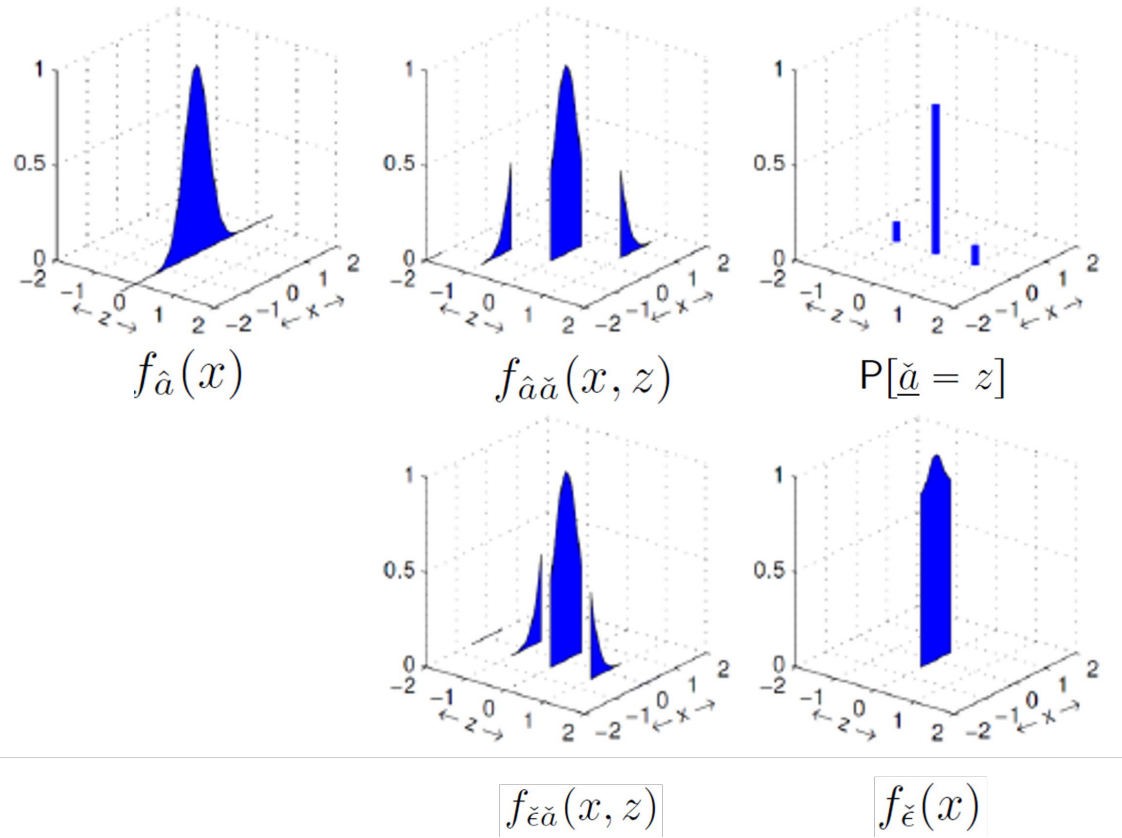
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Convolution – PDF of ambiguity-residual – periodic continuation – Fourier transforms

PDF of ambiguity-residual (scalar case)



$$f_{\check{e}}(x) = \sum_{z \in \mathbb{Z}^n} f_{\hat{a}}(x + z) s_0(x)$$

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$$P(c) = F_0(c) + 2 \sum_{z \in \mathbb{Z}_+^n} e^{-2\pi^2 z^T Q_{\hat{a}\hat{a}} z} F_z(c) \cos(2\pi z^T \alpha(c))$$

damped cosine-series

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$$\sigma_{z^T \hat{a}}^2 = z^T Q_{\hat{a}\hat{a}} z$$

Ambiguity-variance

$$\alpha(c) = \bar{A}^+ C c$$

Ambiguity-bias

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Amplitude
(z-mode size)

$$e^{-2\pi^2 z^T Q_{\hat{a}\hat{a}} z} |F_z(c)| > \delta$$

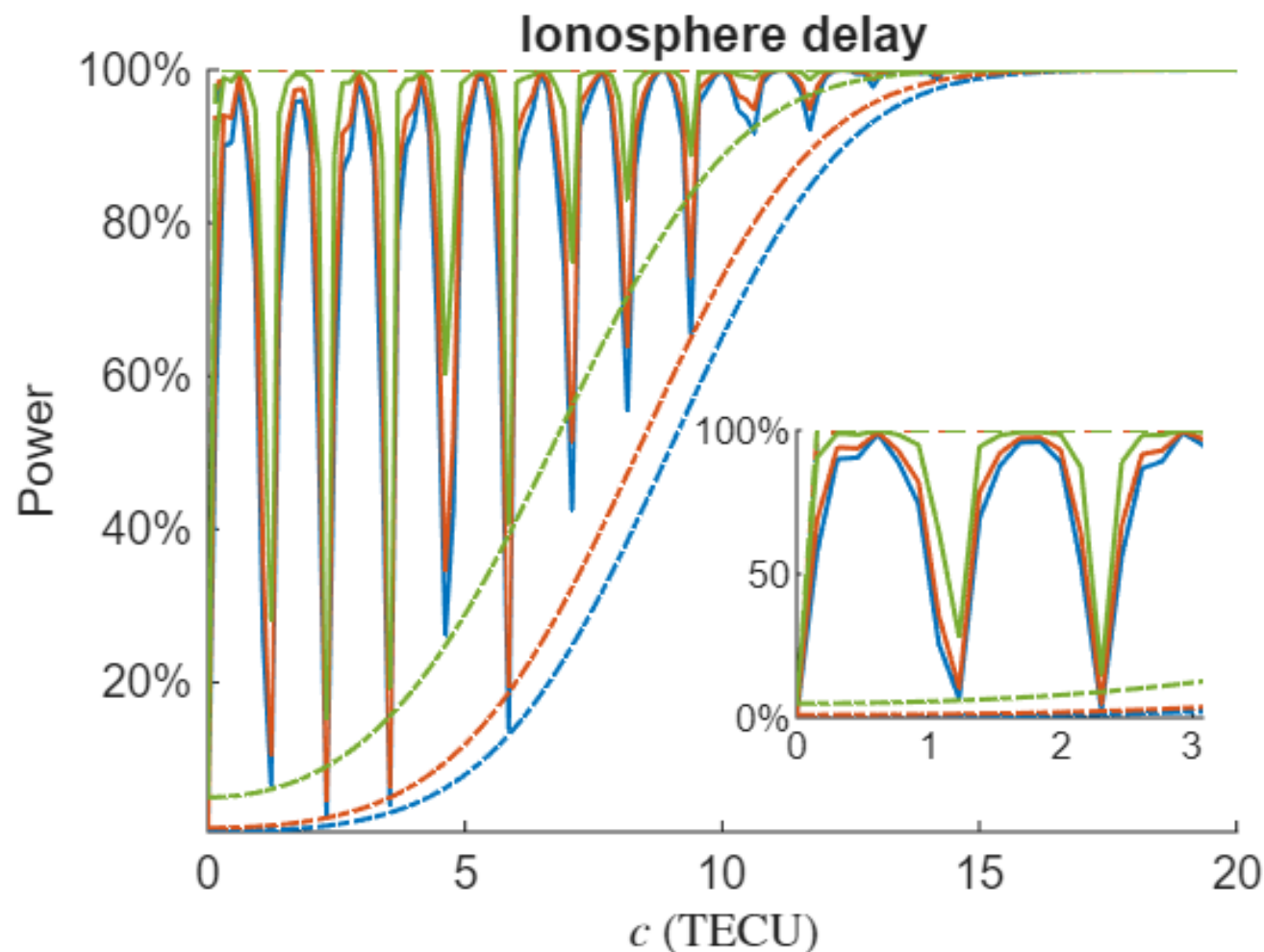
Frequency
(z-mode oscillation-rate)
 $[0, c_{max}]$

$$\left\{ \begin{array}{l} |z^T \alpha(c_{\max})| \ll 1 \\ |z^T \alpha(c_{\max})| \approx 1 \\ |z^T \alpha(c_{\max})| \gg 1 \end{array} \right.$$

Single-frequency L1, single-epoch GPS short-baseline model;
success-rate under H_0 : 98.7%

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$$\underline{T} = \|\hat{\underline{e}}\|_{Q_{yy}}^2 + \|\check{\underline{e}}\|_{Q_{\hat{a}\hat{a}}}^2$$



$$E \begin{bmatrix} p \\ \phi \end{bmatrix} = \begin{bmatrix} 0 & G \\ \lambda_1 I_m & G \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\hat{a} = \frac{1}{\lambda_1} [\phi - G\hat{b}] \text{ , } \hat{b} = G^+ p$$

$$Q_{\hat{a}\hat{a}} = \frac{1}{\lambda_1^2} [Q_{\phi\phi} + GQ_{\hat{b}\hat{b}}G^T]$$

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$$\text{Iono-bias: } Cc = \begin{bmatrix} +e_1 \\ -e_1 \end{bmatrix} c$$

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$$\text{Z-ambiguity-bias: } Z^T \bar{A}^+ Cc = -\frac{c}{\lambda_1} [Z^T e_1 + \delta]$$

$$Z^T e_1 = \text{integer} \quad \delta = Z^T P_G e_1 = \text{small}$$

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$$f_z = |z^T Z^T \bar{A}^+ Cc| \approx \frac{c}{\lambda_1} |z^T Z^T e_1| = \frac{c}{\lambda_1} |j_z|, \quad j_z \in \mathbb{Z}$$

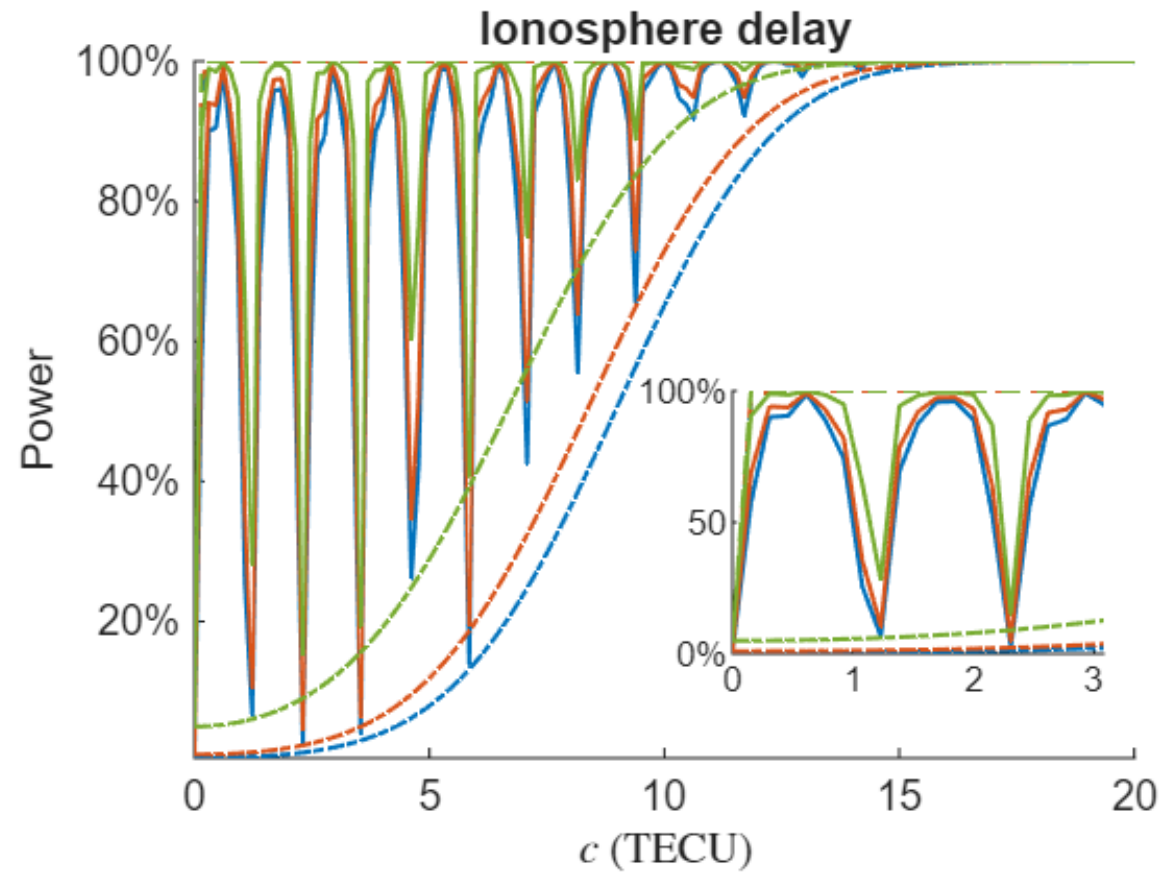
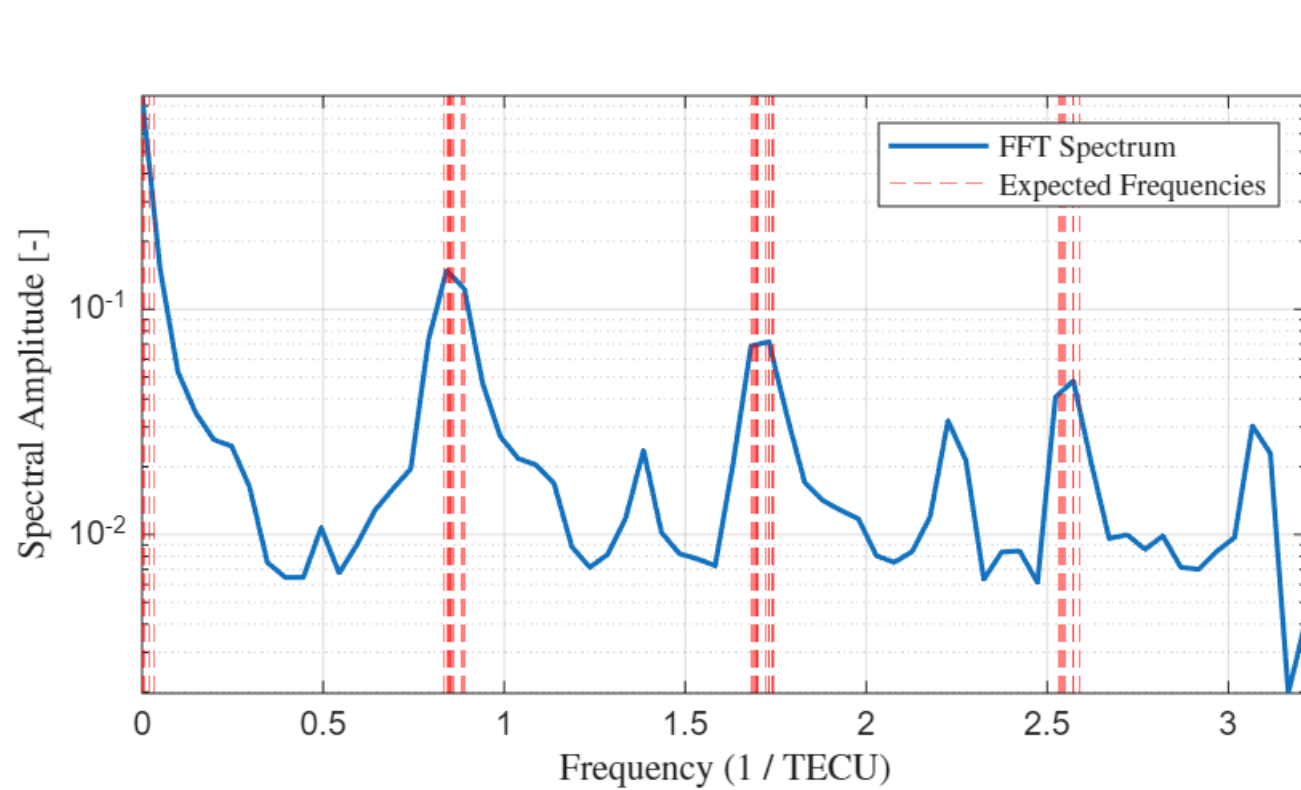
Centre-frequencies

$$\delta_z = \frac{c}{\lambda_1} |z^T \delta|$$

Small groups

$$Z^T e_1 = [-1, 0, -6, 5, 0, -3, -2, -1]^T$$

$$\frac{c}{\lambda_1} = \frac{16.24\text{cm/TECU}}{19.03\text{cm}} = 0.8533 \text{ cycles/TECU}$$



Centre-frequencies 0, 0.85, 1.70, 2.55, 3.40, ... cycles/TECU
 with smaller 'peaks'
 as the ambiguity-precision gets poorer and damping increases

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Pseudorange-bias: $Cc = \begin{bmatrix} +e_1 \\ 0 \end{bmatrix} c$

Ambiguity-bias: $\bar{A}^+ Cc = -\frac{1}{\lambda_1} P_G e_1 c$

Z-ambiguity-bias: $|Z^T \bar{A}^+ Cc| = |\frac{c}{\lambda_1} Z^T P_G e_1|$
= small

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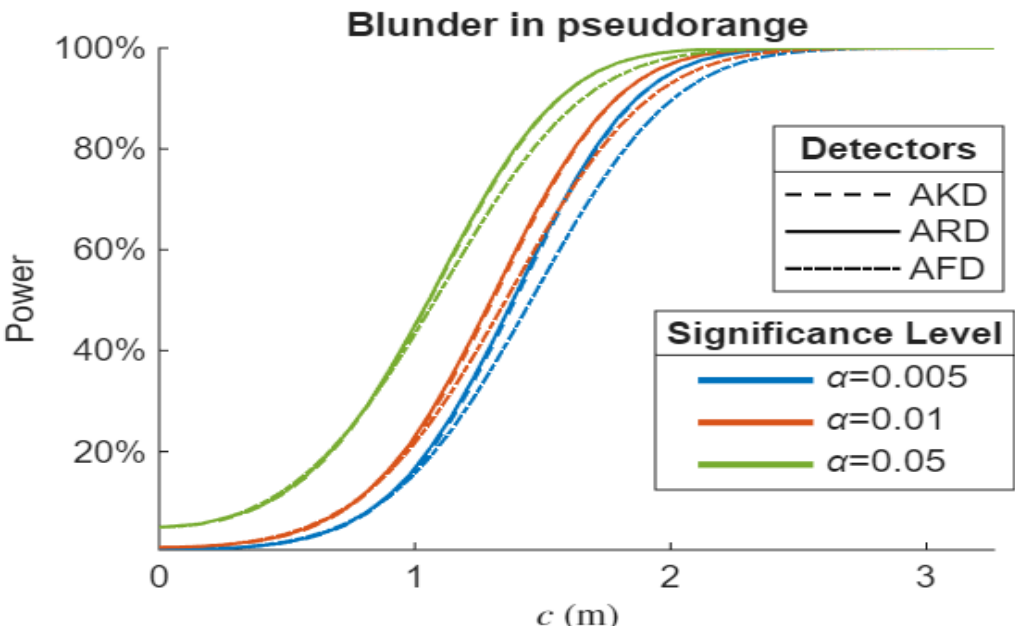
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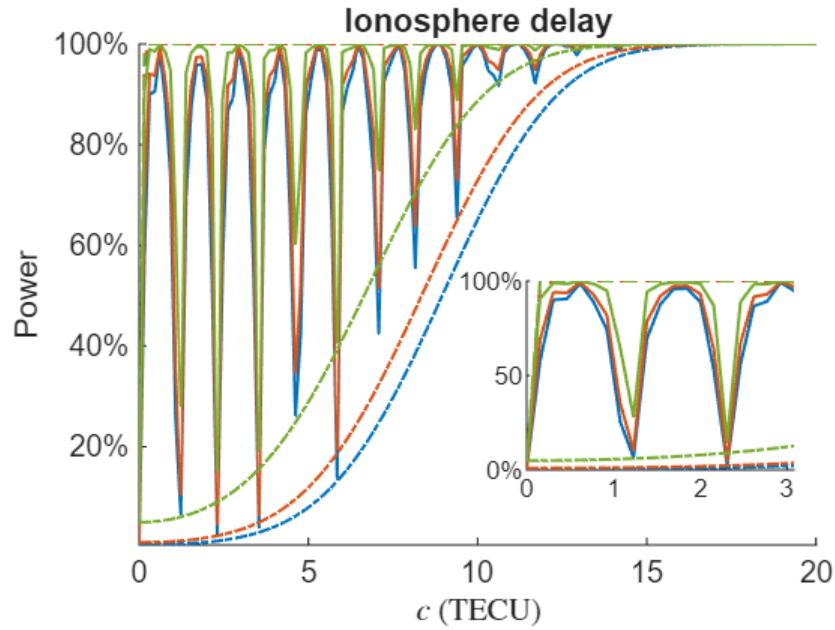
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Conclusions

- AR-detector significantly larger power than AF-detector
- Power improvements are even realized for ambiguity-fixing weak-models
- Power improvements spectacular for ionosphere and troposphere delay detection
- Damped oscillatory character of power function explained by its spectral form:
damped cosine-series
 - Ambiguity-bias in z-domain $\Rightarrow$ frequencies
 - Ambiguity-precision in z-domain $\Rightarrow$ amplitudes
- Power function in spectral form can be used for targeted designing purposes

Figure 6 – fluctuation frequency of ionosphere



With 50 vertical lines (from the top 100 non-zero integer vectors)

High peaks overlap with the expected frequencies

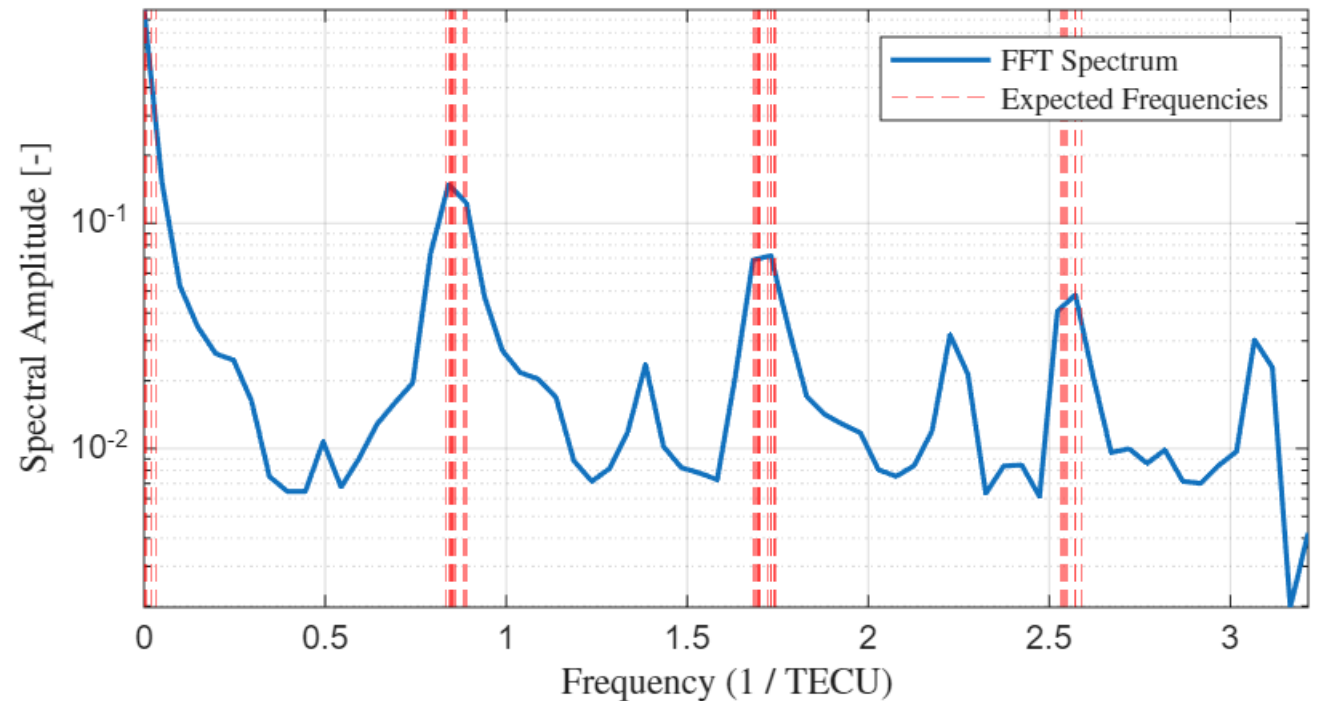
$|z^T Z^T \bar{A}^+ C|$ for the top 30 non-zero integers that give minimum values of $z^T Q_{zz} z$ (Below shows half of them)

0.8509	1.7397	4.2369	2.5376	5.0878	1.7018	1.6955
2.5906	3.3885	0.0038	2.5465	2.5413	5.1190	0.8887
0.8325						

$\exp(-2\pi^2 z^T Q z)$

0.8551	0.5719	0.5668	0.5580	0.5358	0.5346	0.5240
0.5095	0.5033	0.5004	0.4911	0.4835	0.4805	0.4694
0.4625						

In this example, the frequency resolution is ~ 0.305 (1/TECU)



Explain fluctuation in power function

1) The vector to compute the frequencies can be written as

$$\begin{aligned} Z^T \bar{A}^+ C &= [-0.8325 \ 0.0076 \ -5.1190 \ 4.2369 \ -0.0038 \ -2.5376 \ -1.7397 \ -0.8509]^T \\ &= [-1, 0, -6, 5, 0, -3, -2, -1]^T * 0.8533 + res \end{aligned}$$

with $res = [0.0208, 0.0076, 0.0008, -0.0296, -0.0038, 0.0223, -0.0031, 0.0024]^T$

Therefore, $Z^T \bar{A}^+ C$ is very close to an integer vector times 0.8533

2) As a result, for any integer vector z , $z^T Z^T \bar{A}^+ C$ will be close to an integer times of 0.8533. This explains why the frequencies are around integer times of 0.8533

3) Due to the res vector, the frequencies within each group are not exactly the same.

4) Where does 0.8533 comes from? $1TECU \sim 0.1624 \text{ m}$

$$\frac{0.1624 \text{ m}/TECU}{0.1903 \text{ m}} = 0.8533 \text{ cycles}/TECU$$

Power-function of test:

$$T = \|t\|_{Q_{tt}}^2 + \|\hat{a} - \check{a}\|_Q^2 > \kappa$$

$$\mathcal{H}_0 : \underline{y} \sim \mathcal{N}_m(Aa + Bb, Q_{yy}) \qquad a \in \mathbb{Z}^n, b \in \mathbb{R}^p \qquad \mathcal{H}_i : \underline{y} \sim \mathcal{N}_m(Aa + Bb + C_i c_i, Q_{yy}) \quad c_i \in \mathbb{R}_{\neq 0}^{q_i}$$

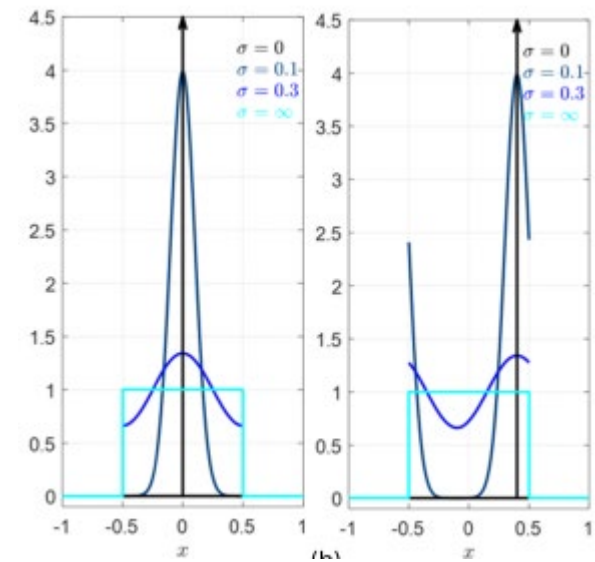
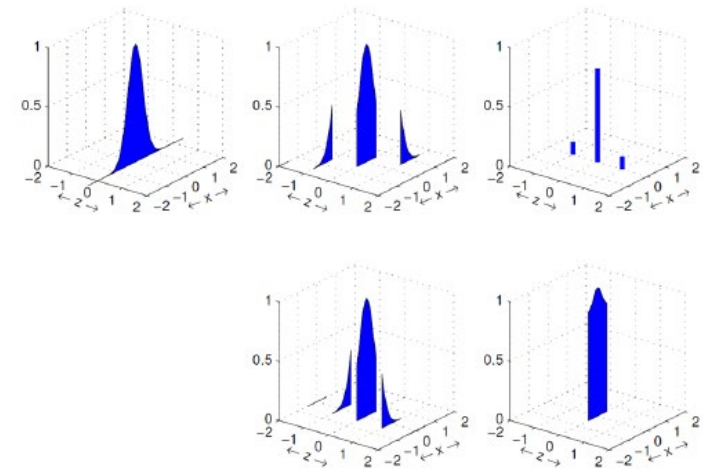
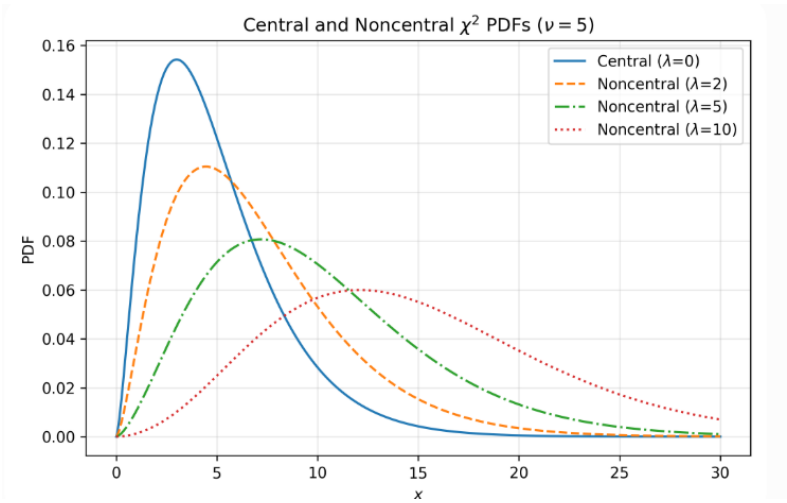
$X \sim \chi_r^2(\lambda), \qquad \lambda = \mu^T Q_{tt}^{-1} \mu = b^T C^T Q_{tt}^{-1} C b.$

t and echeck independent

$f_\epsilon(e; b) = \sum_{z \in \mathbb{Z}^n} \phi_n(e + z - d; Q), \qquad e \in V.$

Testable and ambiguity-influential bias

periodic



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(z-mode size)

$$e^{-2\pi^2 z^T Q_{\hat{a}\hat{a}} z} |F_z(c)| > \delta$$

Frequency

(z-mode oscillation-rate)

$$[0, c_{max}]$$

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$$|F_{z \neq 0}(c)| \rightarrow 0 \text{ if } ||z|| \rightarrow \infty \quad \kappa |\mathcal{H}_a(c)|$$

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$$F_0(c) \rightarrow 1 \text{ if } ||c|| \rightarrow \infty$$

$$P(c) = F_0(c) + 2 \sum_{z \in \mathbb{Z}_+^n} e^{-2\pi^2 z^T Q_{\hat{a}\hat{a}} z} F_z(c) \cos(2\pi z^T \alpha(c))$$

$$\sigma_{z^T \hat{a}}^2 = z^T Q_{\hat{a}\hat{a}} z$$

Ambiguity-variance

$$\alpha(c) = \bar{A}^+ C c$$

Ambiguity-bias

$$\underline{T} = ||\hat{e}||_{Q_{yy}}^2 + ||\check{e}||_{Q_{\hat{a}\hat{a}}}^2$$

$F_z(c)$ are Fourier-coefficients of $P[\underline{T} > \kappa | \check{e}, c]$

Amplitude
(z-mode size)

$$e^{-2\pi^2 z^T Q_{\hat{a}\hat{a}} z} |F_z(c)| > \delta$$

Frequency
(z-mode oscillation-rate)
 $[0, c_{max}]$

$$\left\{ \begin{array}{l} |z^T \alpha(c_{\max})| \ll 1 \\ |z^T \alpha(c_{\max})| \approx 1 \\ |z^T \alpha(c_{\max})| \gg 1 \end{array} \right.$$

$$\sigma_{z^T \hat{a}}^2 = z^T Q_{\hat{a}\hat{a}} z$$

Ambiguity-variance

$$\alpha(c) = \bar{A}^+ C c$$

Ambiguity-bias

$$|F_{z \neq 0}(c)| \leq F_0(c) \leq 1$$

$$|F_{z \neq 0}(c)| \rightarrow 0 \text{ if } ||z|| \rightarrow \infty$$

$$|F_{z \neq 0}(c)| \rightarrow 0 \text{ if } ||c|| \rightarrow \infty$$

$$F_0(c) \rightarrow 1 \text{ if } ||c|| \rightarrow \infty$$

$$T = \|t\|_{Q_{tt}}^2 + \|\hat{a} - \check{a}\|_Q^2$$

$$f_{\epsilon}(e;b) = \sum_{z \in \mathbb{Z}^n} \phi_n(e+z-d;Q), \qquad e \in V.$$

$$f_T(\tau;b) = \int_V f_X(\tau - \|e\|_Q^2) \, f_{\epsilon}(e;b) \, de,$$

$$f_{\epsilon}(e;b) = \sum_{k \in \mathbb{Z}^n} \exp\big(-2\pi^2 k^T Q k\big) \exp\big(2\pi i k^T (e-d)\big) \,, \qquad e \in V.$$

Thus the probability can be viewed as a Fourier-type series in b :

$$\Pr[T > \kappa; b] = \sum_{m \in \mathbb{Z}^n} a_m(b) \cos(2\pi(m^T A)b - \varphi_m) \,,$$

where the amplitudes are damped by

$$e^{-2\pi^2 m^T Q m}$$

and also vary smoothly through the noncentrality

$$\lambda(b) = b^2 C^T Q_{tt}^{-1} C.$$

So there are two types of b -dependence:

$$\text{smooth trend from } \lambda(b)$$

and

$$\text{oscillation from } \cos(2\pi m^T A b).$$

The ambiguity residual density is

$$f_\epsilon(e; b) = \sum_{z \in \mathbb{Z}^n} \phi_n(e + z - Ab; Q), \quad e \in V.$$

By Poisson summation,

$$f_\epsilon(e; b) = \sum_{m \in \mathbb{Z}^n} e^{-2\pi^2 m^T Q m} e^{2\pi i m^T (\epsilon - Ab)}.$$

Substitution gives

$$P(b) = \sum_{m \in \mathbb{Z}^n} e^{-2\pi^2 m^T Q m} e^{-2\pi i (m^T A)b} \int_V \bar{F}_{\chi_r^2(\lambda(b))} (\kappa - \|e\|_Q^2) e^{2\pi i m^T e} de.$$

Define the coefficient

$$J_m(b; \kappa) = \int_V \bar{F}_{\chi_r^2(\lambda(b))} (\kappa - \|e\|_Q^2) e^{2\pi i m^T e} de.$$

Then

$$P(b) = \sum_{m \in \mathbb{Z}^n} e^{-2\pi^2 m^T Q m} J_m(b; \kappa) e^{-2\pi i (m^T A)b}.$$

for the dominant modes, with

$$J_m(b; \kappa) = \int_V w_b(e; \kappa) \cos(2\pi m^T e) de,$$

where

$$w_b(e; \kappa) = \bar{F}_{\chi_r^2(\lambda(b))} (\kappa - \|e\|_Q^2).$$

$$P(b) = J_0(b) + 2 \sum_{m > 0} e^{-2\pi^2 m^T Q m} J_m(b) \cos(2\pi m^T A b).$$

Interpretation 1: Average power over the pull-in region

$$J_0(b; \kappa) = \int_V \Pr[X > \kappa - \|e\|_Q^2] de.$$

J_0 is the power obtained when the ambiguity residual is assumed **uniformly distributed** over the pull in region.

$$J_0(b; \kappa) = E_U [\bar{F}_{\chi_r^2(\lambda(b))}(\kappa - \|U\|_Q^2)],$$

J_0 = power obtained after replacing f_ϵ by the uniform density on V .

$$\lim_{Q \rightarrow \infty} P(b) = J_0(b).$$

$U \sim \text{Uniform}(V)$.

Thus

J_0 = the asymptotic power when the ambiguity residual becomes uniform over V .

$$P(b) = J_0 + (\text{oscillatory lattice corrections}).$$

Therefore

- J_0 is the power that would be obtained if the integer lattice were "washed out",
- all improvements (or degradations) due to integer ambiguity resolution are contained in

for the dominant modes, with

$$J_m(b; \kappa) = \int_V w_b(e; \kappa) \cos(2\pi m^T e) de,$$

where

$$w_b(e; \kappa) = \bar{F}_{\chi_r^2(\lambda(b))}(\kappa - \|e\|_Q^2).$$

$$0 \leq \bar{F}_{\chi_r^2(\lambda)}(\cdot) \leq 1,$$

we have

$$0 \leq J_0(b; \kappa) \leq 1.$$

Moreover, if

$$C^T Q_{tt}^{-1} C > 0,$$

then

$$\lambda(b) \rightarrow \infty \quad \text{as } |b| \rightarrow \infty,$$

and therefore

$$J_0(b; \kappa) \rightarrow 1 \quad \text{as } |b| \rightarrow \infty.$$

So, qualitatively,

$J_0(b; \kappa)$ is a smooth, even, generally increasing, saturating power curve.

It gives the nonoscillatory envelope/background around which the lattice-induced Fourier corrections oscillate.

Yes. Let the phase be

$$\xi = Ab \pmod{\mathbb{Z}^n},$$

and define the power as a function of this ambiguity-lattice phase:

$$P(\xi) = \int_V w(e) f_\epsilon(e; \xi) de,$$

where

$$w(e) = \bar{F}_{\chi_r^2(\lambda)}(\kappa - \|e\|_Q^2),$$

and

$$f_\epsilon(e; \xi) = \sum_{z \in \mathbb{Z}^n} \phi(e + z - \xi; Q), \quad e \in V.$$

Here V is any unit-volume pull-in region, so $|V| = 1$.

We want to prove that the average of $P(\xi)$ over one pull-in region equals J_0 :

$$\int_V P(\xi) d\xi = J_0 = \int_V w(e) de.$$

Start with

$$\int_V P(\xi) d\xi = \int_V \int_V w(e) \sum_{z \in \mathbb{Z}^n} \phi(e + z - \xi; Q) de d\xi.$$

Interchange the integrations:

$$\int_V P(\xi) d\xi = \int_V w(e) \left[\int_V \sum_{z \in \mathbb{Z}^n} \phi(e + z - \xi; Q) d\xi \right] de.$$

Now consider the bracketed term. For fixed

Now consider the bracketed term. For fixed e ,

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$$\int_V \sum_{z \in \mathbb{Z}^n} \phi(e + z - \xi; Q) d\xi = \sum_{z \in \mathbb{Z}^n} \int_V \phi(e + z - \xi; Q) d\xi.$$

Make the change of variable

$$u = e + z - \xi.$$

As ξ ranges over V , u ranges over

$$e + z - V.$$

Hence

$$\sum_{z \in \mathbb{Z}^n} \int_V \phi(e + z - \xi; Q) d\xi = \sum_{z \in \mathbb{Z}^n} \int_{e+z-V} \phi(u; Q) du.$$

Because V is a fundamental pull-in region of $\mathbb{Z}^n$, the translated regions

Because V is a fundamental pull-in region of $\mathbb{Z}^n$, the translated regions

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$$e + z - V, \quad z \in \mathbb{Z}^n,$$

tile $\mathbb{R}^n$ without overlap except on boundaries of measure zero. Therefore

$$\sum_{z \in \mathbb{Z}^n} \int_{e+z-V} \phi(u; Q) du = \int_{\mathbb{R}^n} \phi(u; Q) du = 1.$$

Thus

$$\int_V \sum_{z \in \mathbb{Z}^n} \phi(e + z - \xi; Q) d\xi = 1.$$

Substituting back,

$$\int_V P(\xi) d\xi = \int_V w(e) de.$$

Since $|V| = 1$, this is also the uniform expectation over the pull-in region:

Since $|V| = 1$, this is also the uniform expectation over the pull-in region:

$$\mathbb{E}_{\Xi \sim \text{Unif}(V)}[P(\Xi)] = \int_V w(e) de = J_0.$$

So J_0 is precisely the pull-in-region average of the power over all relative lattice phases. Equivalently,

$$J_0 = \text{lattice-phase averaged power.}$$

This proof does not require writing the average over the torus explicitly. It uses only the fact that the pull-in region V is a fundamental region whose integer shifts tile $\mathbb{R}^n$.

10. Overall characterization

For fixed κ ,

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$$J_m(b; \kappa) \text{ is real and even in both } m \text{ and } b$$

under central symmetry. Furthermore,

$$|J_m| \leq J_0 \leq 1, \quad J_m \rightarrow 0 \text{ as } \|m\| \rightarrow \infty,$$

and for $\gamma > 0$,

$$J_0(b) \rightarrow 1, \quad J_m(b) \rightarrow 0 \ (m \neq 0) \quad \text{as } |b| \rightarrow \infty.$$

So a typical nonzero $J_m(b)$ is a smooth even function that starts at $J_m(0)$, may increase, decrease, or change sign at intermediate b , but ultimately returns to zero as $|b|$ becomes large. The complete contribution to power is then determined by three separate factors:

$$e^{-2\pi^2 m^T Q m}$$

Gaussian lattice damping

$$J_{\gamma \downarrow \kappa}$$

threshold-weight Fourier coefficient

$$\cos(2\pi m^T A b)$$

lattice-phase oscillation

.

For proof, see PDF *

Yes. The cleanest view is that, for each fixed b and κ ,

$$J_m(b; \kappa) = \int_V w_b(e; \kappa) e^{2\pi i m^T e} de,$$

with

$$w_b(e; \kappa) = \overline{F}_{\chi_r^2(\lambda(b))}(\kappa - \|e\|_Q^2), \quad \lambda(b) = \gamma b^2, \quad \gamma = C^T Q_{tt}^{-1} C.$$

Thus $\{J_m(b; \kappa)\}_{m \in \mathbb{Z}^n}$ are the Fourier coefficients, over the pull-in region V , of the bounded function $w_b(\cdot; \kappa)$.

Therefore

$$|J_m(b; \kappa)| \leq J_0(b; \kappa) \leq 1.$$

This implies that the complete m -mode in the power function satisfies

$$\left| 2e^{-2\pi^2 m^T Q m} J_m(b; \kappa) \cos(2\pi m^T A b) \right| \leq 2e^{-2\pi^2 m^T Q m} J_0(b; \kappa).$$

Hence the Gaussian multiplier $e^{-2\pi^2 m^T Q m}$ gives a rigorous envelope, irrespective of the detailed behavior of J_m .

$$P(b) = J_0(b) + 2 \sum_{m>0} e^{-2\pi^2 m^T Q m} J_m(b) \cos(2\pi m^T A b).$$

I now think that your questions over the last hour have actually revealed a much cleaner separation:

- Q determines **how large** each Fourier mode is.
- A determines **how fast** each Fourier mode oscillates.

These are completely different effects and should probably not be merged into one criterion.

In fact, I would now formulate the analysis in two stages:

1. **Amplitude screening:** determine the finite set of relevant modes by

$$e^{-2\pi^2 m^T Q m} |J_m| > \varepsilon.$$

2. **Frequency analysis:** for those retained modes only, investigate whether

$$|m^T A| b_{\max}$$

is small, of order one, or large.

In hindsight, I think there are really two independent questions.

Question 1

Which Fourier modes matter?

Answer:

Those satisfying

$$e^{-2\pi^2 m^T Q m} |J_m(b)|$$

not small.

This depends only on Q .



Question 2

Among those important modes, which actually oscillate over the interval?

Answer:

Those for which

$$|m^T A| b_{\max}$$

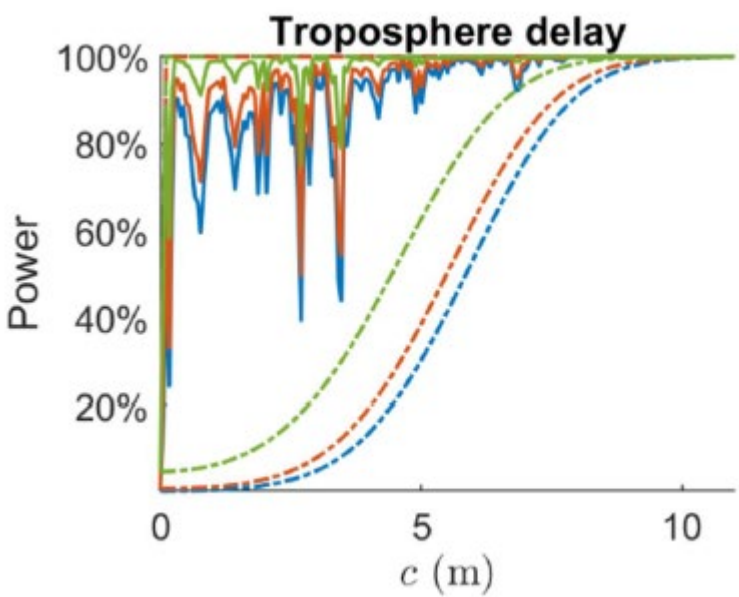
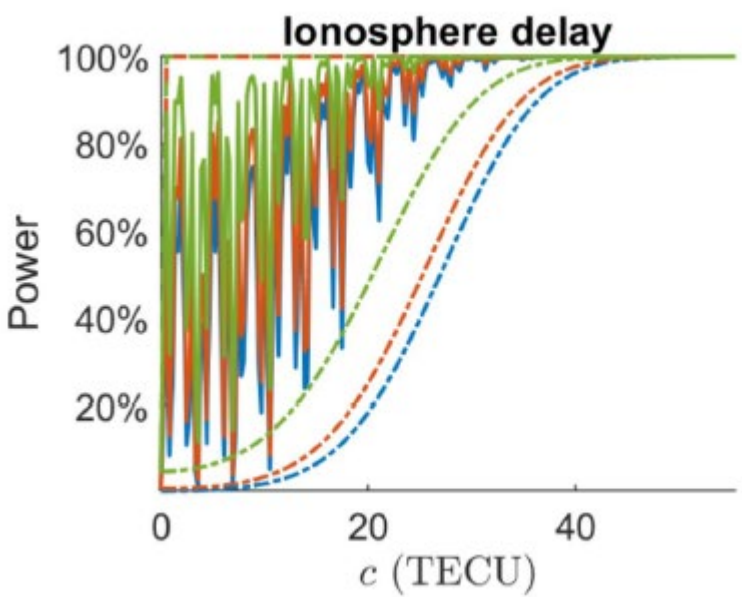
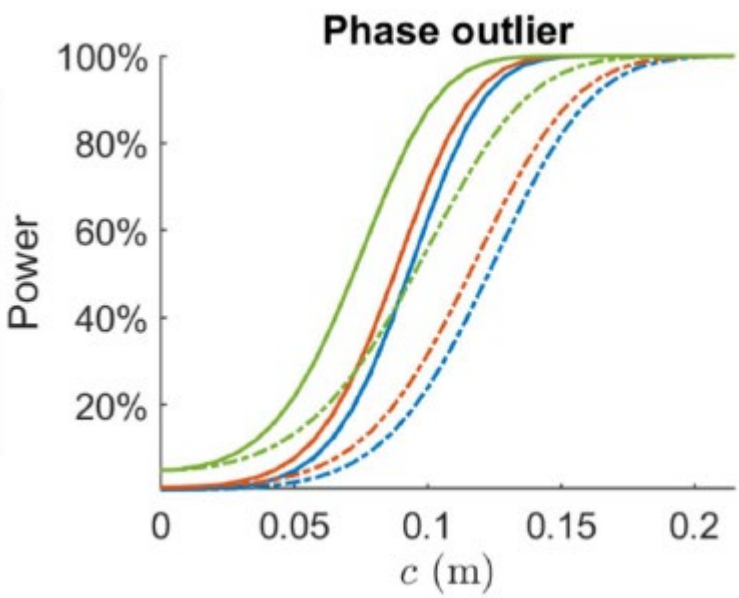
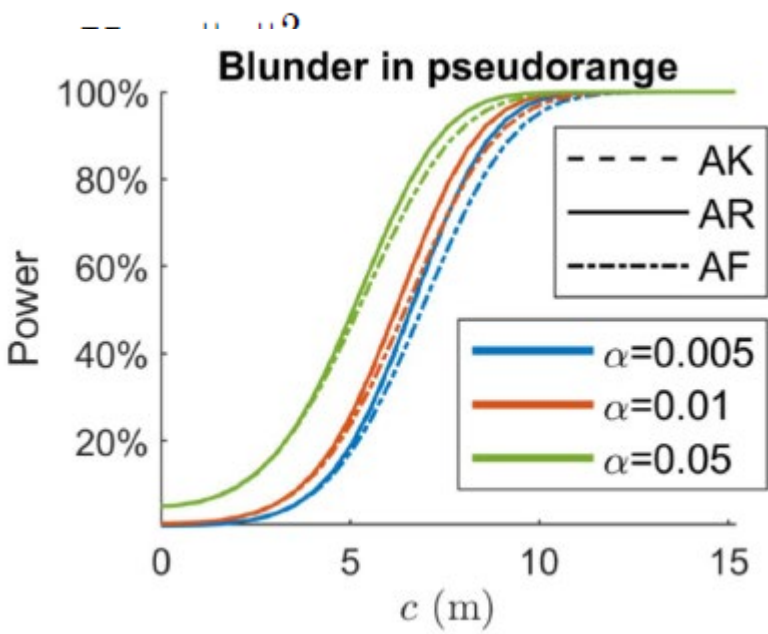
is of order one or larger.

This depends only on A .

$$f_T(t) = \int$$

$$f_T(t) = \sum_{k \in \mathbb{Z}^n} e^{-}$$

T



$$i), \quad e \in V,$$

$$\overline{ik^Te}, \quad e \in V.$$

$$\mathcal{H}_i : \underline{y} \sim \mathcal{N}_m(Aa + Bb + C_i c_i, Q_{yy}) \quad c_i \in \mathbb{R}_{\neq 0}^{q_i}$$

$$T = \|t\|_{Q_{tt}}^2 + \|\hat{a} - \check{a}\|_Q^2 = X + \|\epsilon\|_Q^2.$$

Include here the nice
2D-periodic figures of its contourlines

$$X = \|t\|_{Q_{tt}}^2$$

$$X \sim \chi_r^2(\lambda), \qquad \lambda = \mu^T Q_{tt}^{-1} \mu = b^T C^T Q_{tt}^{-1} C b.$$

$$f_T(\tau;b)=\int_V f_X(\tau-\|e\|_Q^2)\,f_\epsilon(e;b)\,de,$$

$$f_\epsilon(e;b)=\sum_{z\in\mathbb{Z}^n}\phi_n(e+z-d;Q),\qquad e\in V.$$

$$f_\epsilon(e;b)=\sum_{k\in\mathbb{Z}^n}\exp(-2\pi^2k^TQk)\exp(2\pi ik^T(e-d))\,,\qquad e\in V.$$

$$f_T(\tau;b)=\sum_{k\in\mathbb{Z}^n}e^{-2\pi^2k^TQk}e^{-2\pi ik^TAb}\int_{V_\tau}f_{\chi_r^2(\lambda)}(\tau-\|e\|_Q^2)\,e^{2\pi ik^Te}\,de,$$

$$V_\tau=\{e\in V:\|e\|_Q^2<\tau\}$$

$$f_T(\tau;b)=\sum_{k\in\mathbb{Z}^n}e^{-2\pi^2k^TQk}\int_{V_\tau}f_{\chi_r^2(\lambda)}(\tau-\|e\|_Q^2)\cos(2\pi k^T(e-Ab))\,de.$$

Power function:

$$\Pr[T > \kappa] = \int_V \bar{F}_{\chi_r^2(\lambda)}(\kappa - \|e\|_Q^2) f_\epsilon(e; b) de$$

$$\bar{F}_{\chi_r^2(\lambda)}(x) = \Pr[\chi_r^2(\lambda) > x],$$

$$\Pr[T > \kappa] = \sum_{m \in \mathbb{Z}^n} e^{-2\pi^2 m^T Q m} \int_V \bar{F}_{\chi_r^2(\lambda)}(\kappa - \|e\|_Q^2) \cos(2\pi m^T (e - Ab)) de.$$

Thus the probability can be viewed as a Fourier-type series in b :

$$\Pr[T > \kappa; b] = \sum_{m \in \mathbb{Z}^n} a_m(b) \cos(2\pi(m^T A)b - \varphi_m),$$

where the amplitudes are damped by

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and also vary smoothly through the noncentrality

$$\lambda(b) = b^2 C^T Q_{tt}^{-1} C.$$

So there are two types of b -dependence:

$$\text{smooth trend from } \lambda(b)$$

and

$$\text{oscillation from } \cos(2\pi m^T Ab).$$

$$\mathcal{H}_0: \underline{y} \sim \mathcal{N}_m(Aa+Bb, Q_{yy}) \qquad a \in \mathbb{Z}^n, b \in \mathbb{R}^p$$

$$\mathsf{T}_{\{D\}} = ||\underline{t}_0||_{Q_{t_0t_0}}^2 + ||\underline{\check{\epsilon}}_0||_{Q_{\hat{a}_0\hat{a}_0}}^2$$

$$\underline{\check{\epsilon}}_0 = \hat{\underline{\epsilon}}_0(\hat{\underline{a}}_0) = \hat{\underline{a}}_0 - \underline{\check{a}}_0$$

$$\mathcal{H}_0: \underline{y} \sim \mathcal{N}_m(Aa+Bb,Q_{yy}) \qquad a \in \mathbb{Z}^n, b \in \mathbb{R}^p$$

$$\mathcal{H}_i: \underline{y} \sim \mathcal{N}_m(Aa+Bb+C_i c_i,Q_{yy}) \; c_i \in \mathbb{R}_{\neq 0}^{q_i}$$

$$\underline{T}(\check{\underline{a}}_0,\check{\underline{a}})=||\hat{\underline{c}}||^2_{\underline{Q}_{\check{c}\check{c}}}+||\check{\underline{\epsilon}}_0||^2_{\underline{Q}_{\hat{a}_0\hat{a}_0}}-||\check{\underline{\epsilon}}_a||^2_{\underline{Q}_{\hat{a}\hat{a}}}$$

$$\check{\underline{\epsilon}}_a = \hat{\underline{\epsilon}}_a(\check{\underline{a}}) = \hat{\underline{a}} - \check{\underline{a}}$$

Every nonzero integer vector can be written unique



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$$m = k m_0,$$

where

- $k \geq 1$,
- the components of m_0 have greatest common divisor equal to one (a *primitive* lattice vector).

Then

$$P(c) = J_0(c) + 2 \sum_{m_0 \text{ primitive}} \sum_{k=1}^{\infty} e^{-2\pi^2 k^2 m_0^T Q m_0} J_{k m_0}(c) \cos(2\pi k m_0^T A c).$$

This reorganizes the power function into **independent harmonic families**, one for each primitive lattice direction. Each family has:

- a **fundamental frequency** $m_0^T A$,
- a hierarchy of harmonics $2m_0^T A, 3m_0^T A, \dots$ ↓
- amplitudes that decay essentially like

Chengyu-figures

- Power pseudorange (AK-AR-AF); Skyplot; Power cycle-slip (AK-AR-AF)
- Power iono (AK-AR-AF); Skyplot; Power tropo (AK-AR-AF)
- PeriodicFunction as function of bias $\|\hat{a} - a_{\text{check}}\|_{\mathbf{Q}}^2$
- Periodic function of echeck PDF (for different sigma)
- $z^T \mathbf{Q} z$ and $z^T \mathbf{A} \bar{a}^{\text{check}} = C$ for iono and pseudorange
- Chengyu this is my Power-formula:

$$P(c) = \sum_{z \in \mathbb{Z}^n} e^{-2\pi^2 z^T \mathbf{Q} z} F_z(c; \kappa) \cos(2\pi z^T \bar{\mathbf{A}}^+ C c)$$

$$F_z(c; \kappa) = \int_{S_0} \bar{F}_{\chi^2(r, \lambda(c))}(\kappa - \|e\|_{\mathbf{Q}}^2) \cos(2\pi z^T e) de$$

Title: The Mixed-Integer Misclosure Test in Spectral Form

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Department of Geoscience and Remote Sensing, The
Netherlands

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$$F_z(c; \kappa) = \int_{S_0} \bar{F}_{\chi^2(r, \lambda(c))} (\kappa - \|e\|_{Q_{\hat{a}\hat{a}}}^2) \cos(2\pi z^T e) de$$

Some examples of performance of detector for model-misspecifications

- Showing the improved effect of AR-detection vs AF-detection
- Show oscillations of ionopower and tropo-power
 - Oscillations can be explained in spatial domain with train of gaussian pulses
 - But we would like to predict it and design for it => Fourier-approach