



GLONASS FDMA PPP-RTK realization using integer-estimability theory

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Motivation

IAR: Integer Ambiguity Resolution | GNSS: Global Navigation Satellite Systems

Three-part structure of PPP-RTK

1 - Network

Process GNSS data from ref. stations with a preferred S-basis



2 - Estimable net. solutions

Sat. orbits, clocks, biases

Rec. clocks, biases

Atm. delays



Certain combinations of biases allow the **recovery of integerness** of user-ambiguities.

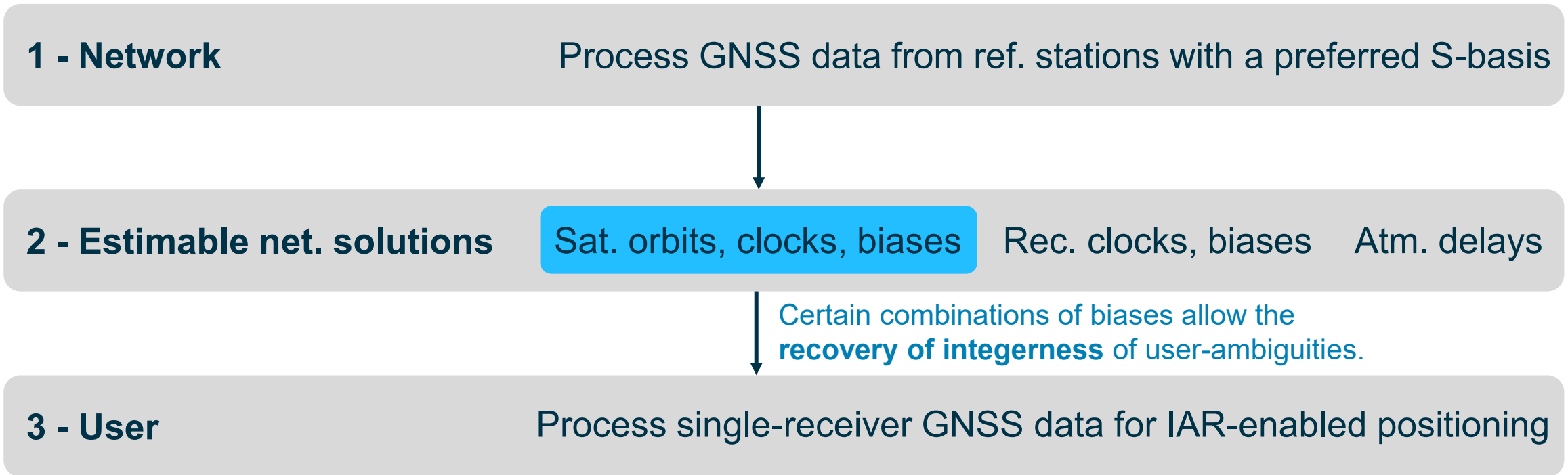
3 - User

Process single-receiver GNSS data for IAR-enabled positioning

Motivation

IAR: Integer Ambiguity Resolution | **GNSS:** Global Navigation Satellite Systems

Three-part structure of PPP-RTK



PPP-RTK is an integer-estimability problem.

Integer-estimability: cornerstone for PPP-RTK

integer-valued UD ambiguities $\swarrow$ $\searrow$ real-valued parameters

$$E(y) = Az + Bb$$

rank-defect mixed-integer model

Integer-estimability: cornerstone for PPP-RTK

integer-valued UD ambiguities $\swarrow$ $\searrow$ real-valued parameters

$$E(y) = Az + Bb \quad \text{rank-defect mixed-integer model}$$



$$E(y) = AZ_1\tilde{z}_1 + B\tilde{b} \quad \text{full-rank integer-estimable model}$$

Integer and estimable ambiguities are not necessarily integer-estimable!

Without ensuring integer-estimability, one may fix the UD integer ambiguities to non-integer values → the model is wrongly constrained

Teunissen (2019); Teunissen & Khodabandeh (2022)

Integer-estimability: cornerstone for PPP-RTK

integer-valued UD ambiguities $\longleftarrow$ $\longrightarrow$ real-valued parameters

$$E(y) = Az + Bb \quad \text{rank-defect mixed-integer model}$$



$$E(y) = AZ_1\tilde{z}_1 + B\tilde{b} \quad \text{full-rank integer-estimable model}$$

Integer and estimable ambiguities are not necessarily integer-estimable!

function $\tilde{z} = F^T z$ is said to be **integer-estimable** if and only if:

$$\begin{cases} (i) & F = A^T B^\perp X \quad , F \in \mathbb{Z} \text{ and some } X \\ (ii) & F^T Z = [I, 0] \quad , \text{ for some } Z \in \mathbb{Z} \end{cases}$$

Teunissen (2019); Teunissen & Khodabandeh (2022)

Integer-estimability: algorithmic means to obtain it

Our objective is to establish the full-rank integer-estimable parameterization ...

$$Az + Bb = AZ_1 \tilde{z}_1 + B\tilde{b}$$

integer-estimable ambiguities

estimable phase biases

where:

$$\tilde{z}_1 = \tilde{Z}_1^T z$$
$$\tilde{z}_2 = \tilde{Z}_2^T z$$
$$\tilde{b} = b + B^+ AZ_2 \tilde{z}_2$$

... through the construction of an admissible ambiguity transformation $Z = [Z_1, Z_2]$

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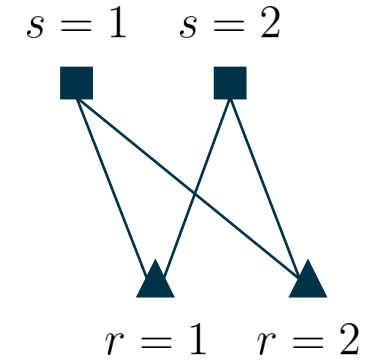
... through the construction of an admissible ambiguity transformation $Z = [Z_1, Z_2]$

Integer-sweeping algorithm (Teunissen & Khodabandeh, 2022)

- Repeated subtraction of integer multiples of one column from another until the row is reduced to its Greatest Common Divisor
- Taking as input an arbitrary integer matrix M , we get: $MZ = [MZ_1, MZ_2] = [L, 0]$
- Inputs: $M = B^T A^{-T}$ or $M = B^{\perp T} A$

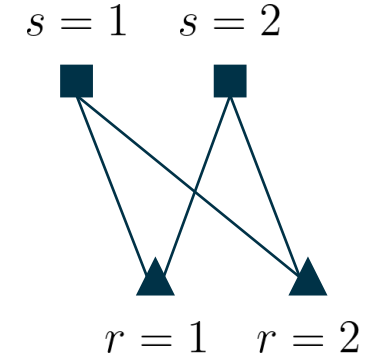
Revisiting CDMA | All-in-view (1/5)

$$E(\phi_{r,j}^s) = \dots + \delta_{r,j} - \delta_{,j}^s + z_{r,j}^s \quad (\text{in cycles})$$



Revisiting CDMA | All-in-view (2/5)

$$E(\phi_{r,j}^s) = \dots + \delta_{r,j} - \delta_{,j}^s + z_{r,j}^s \quad (\text{in cycles})$$

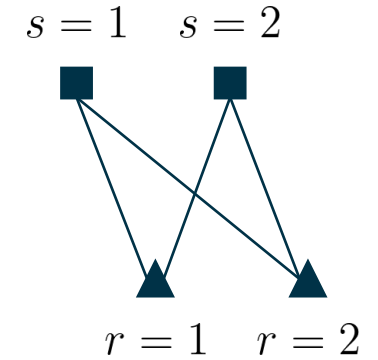


(for 1 frequency)

$$E(\phi) = \dots + I_4 \begin{bmatrix} z_1^1 \\ z_1^2 \\ z_2^1 \\ z_2^2 \end{bmatrix} + \underbrace{\begin{bmatrix} +1 & 0 & -1 & 0 \\ +1 & 0 & 0 & -1 \\ 0 & +1 & -1 & 0 \\ 0 & +1 & 0 & -1 \end{bmatrix}}_{\text{incidence (connectivity) matrix}} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta^1 \\ \delta^2 \end{bmatrix}$$

Revisiting CDMA | All-in-view (3/5)

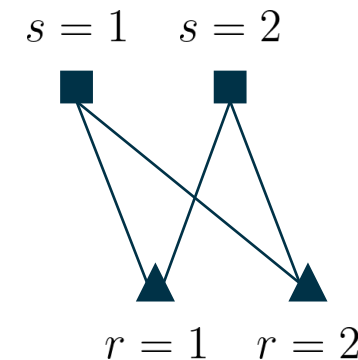
$$E(\phi_{r,j}^s) = \dots + \delta_{r,j} - \delta_{,j}^s + z_{r,j}^s \quad (\text{in cycles})$$



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$$E(\phi) = \dots + I_4 \begin{bmatrix} z_1^1 \\ z_1^2 \\ z_2^1 \\ z_2^2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ +1 & -1 & 0 \\ +1 & 0 & -1 \end{bmatrix}}_{\text{reduced incidence matrix}} \begin{bmatrix} \delta_2 - \delta_1 \\ \delta^1 - \delta_1 \\ \delta^2 - \delta_1 \end{bmatrix}$$

$$E(\phi_{r,j}^s) = \dots + \delta_{r,j} - \delta_{r,j}^s + z_{r,j}^s \quad (\text{in cycles})$$

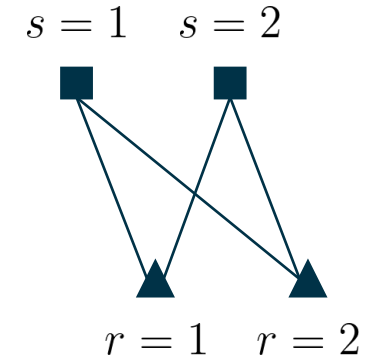


$$(for\ 1\ frequency)\ E(\phi) = \dots + \underbrace{I_4}_A \underbrace{\begin{bmatrix} z_1^1 \\ z_1^2 \\ z_2^1 \\ z_2^2 \end{bmatrix}}_z + \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ +1 & -1 & 0 \\ +1 & 0 & -1 \end{bmatrix}}_B \underbrace{\begin{bmatrix} \delta_2 - \delta_1 \\ \delta^1 - \delta_1 \\ \delta^2 - \delta_1 \end{bmatrix}}_b$$

Revisiting CDMA | All-in-view (5/5)

Integer-sweeping output $\tilde{Z} = \left[\begin{array}{ccc|c} 0 & +1 & 0 & +1 \\ 0 & 0 & +1 & -1 \\ +1 & 0 & 0 & -1 \\ 0 & 0 & 0 & +1 \end{array} \right]$ and $Z = \tilde{Z}^{-T}$

$\underbrace{\quad}_{\tilde{Z}_2} \quad \underbrace{\quad}_{\tilde{Z}_1}$

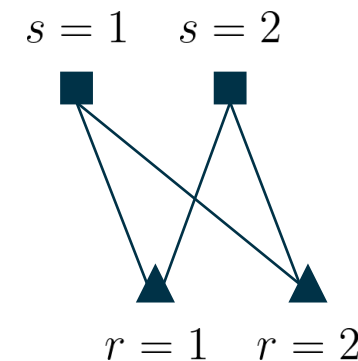


Revisiting CDMA | All-in-view (5/5)

Integer-sweeping output $\tilde{Z} = \left[\begin{array}{ccc|c} 0 & +1 & 0 & +1 \\ 0 & 0 & +1 & -1 \\ +1 & 0 & 0 & -1 \\ 0 & 0 & 0 & +1 \end{array} \right]$ and $Z = \tilde{Z}^{-T}$

$\underbrace{\hspace{10em}}_{\tilde{Z}_2} \quad \underbrace{\hspace{2em}}_{\tilde{Z}_1}$

↓



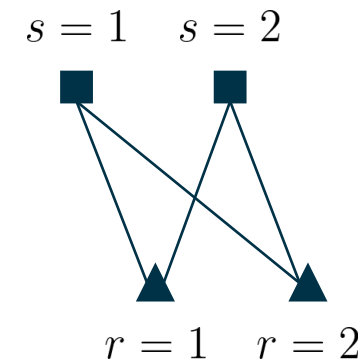
Integer-estimable model: $E(\phi) = \dots + AZ_1\tilde{z}_1 + B\tilde{b}$

$$= \dots + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tilde{z}_1 + \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ +1 & -1 & 0 \\ +1 & 0 & -1 \end{bmatrix} \tilde{b}$$

Revisiting CDMA | All-in-view (5/5)

Integer-sweeping output

$$\tilde{Z} = \left[\begin{array}{ccc|c} 0 & +1 & 0 & +1 \\ 0 & 0 & +1 & -1 \\ +1 & 0 & 0 & -1 \\ 0 & 0 & 0 & +1 \end{array} \right] \quad \text{and} \quad Z = \tilde{Z}^{-T}$$



Equivalent to manually selecting the S-basis as z_1^s and z_2^1

Integer-estimable model: $E(\phi) = \dots + AZ_1\tilde{z}_1 + B\tilde{b}$

$$= \dots + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tilde{z}_1 + \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ +1 & -1 & 0 \\ +1 & 0 & -1 \end{bmatrix} \tilde{b}$$

$$\begin{aligned}\tilde{z}_1 &= \tilde{Z}_1^T z = z_{12}^2 - z_{12}^1 \\ \tilde{b} &= \begin{bmatrix} \delta_2 - \delta_1 \\ \delta^1 - \delta_1 \\ \delta^2 - \delta_1 \end{bmatrix} + \begin{bmatrix} +z_{12}^1 \\ -z_1^1 \\ -z_1^2 \end{bmatrix}\end{aligned}$$

Revisiting CDMA | General case

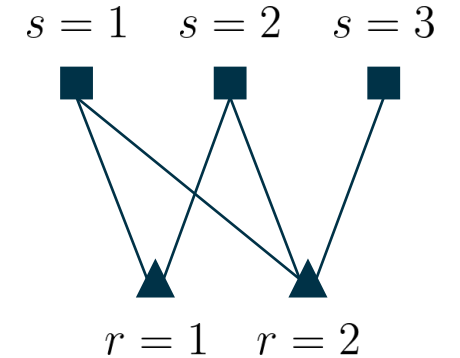
Rank-defect model

$$E(\phi) = \dots + \underbrace{I_5}_A \underbrace{\begin{bmatrix} z_1^1 \\ z_1^2 \\ z_2^1 \\ z_2^2 \\ z_2^3 \end{bmatrix}}_z + \underbrace{\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ +1 & -1 & 0 & 0 \\ +1 & 0 & -1 & 0 \\ +1 & 0 & 0 & -1 \end{bmatrix}}_B \underbrace{\begin{bmatrix} \delta_2 - \delta_1 \\ \delta^1 - \delta_1 \\ \delta^2 - \delta_1 \\ \delta^3 - \delta_1 \end{bmatrix}}_b$$

$$\tilde{Z} = \left[\begin{array}{cccc|c} 0 & +1 & 0 & +1 & +1 \\ 0 & 0 & +1 & 0 & -1 \\ +1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & +1 & 0 \end{array} \right]$$

$$\tilde{z}_1 = \tilde{Z}_1^T z = z_{12}^2 - z_{12}^1$$

$$\tilde{b} = \begin{bmatrix} \delta_2 - \delta_1 \\ \delta^1 - \delta_1 \\ \delta^2 - \delta_1 \\ \delta^3 - \delta_1 \end{bmatrix} + \begin{bmatrix} +z_{12}^1 \\ -z_1^1 \\ -z_1^2 \\ -z_1^1 + z_2^1 - z_2^3 \end{bmatrix}$$



Equivalent to selecting the edges of the spanning tree as S-basis.

Psychas et al. (2025) – IAG SA 2025

FDMA: basis of GLONASS

$$E(\phi_{r,j}^s) = \dots + \delta_{r,j} - \delta_{,j}^s + \lambda_j^s z_{r,j}^s \quad (\text{only ambiguities in cycles})$$

➤ **FDMA:** Frequency Division Multiple Access

➤ The broadcast signals are transmitted on 14 different adjacent frequencies:

$$f_j^s = f_j + \kappa^s \Delta f_j, \quad j = 1, 2, \quad s = 1, \dots, m \quad \kappa^s \in [-7, +6]$$

$$\lambda_j^s = 2848 \lambda_j / r^s$$

$$r^s = 2848 + \kappa^s = f_j^s / \Delta f_j$$

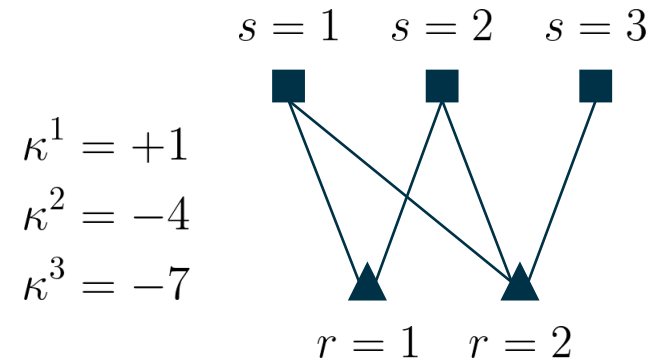
FDMA | General case (1/2)

$$E(\phi_{r,j}^s) = \dots + \delta_{r,j} - \delta_{,j}^s + \lambda_j^s z_{r,j}^s \quad (\text{only ambiguities in cycles})$$

$$f_j^s = r^s \Delta f_j \quad \text{and} \quad \bar{\delta} = \delta(\Delta f_j / c)$$

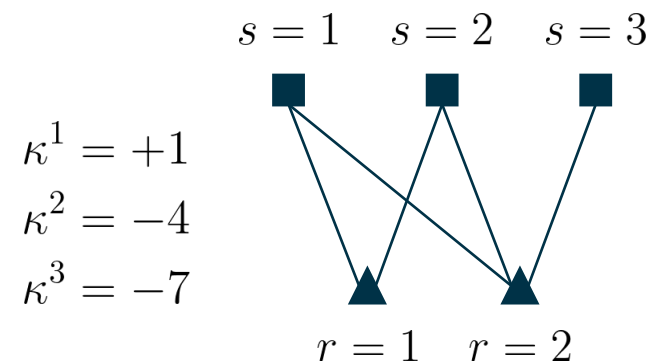
$$E(\bar{\phi}_{r,j}^s) = \dots + r^s (\bar{\delta}_{r,j} - \bar{\delta}_{,j}^s) + z_{r,j}^s \quad (\text{all in cycles})$$

$$E(\phi) = \dots + I_5 \begin{bmatrix} z_1^1 \\ z_1^2 \\ z_2^1 \\ z_2^2 \\ z_2^3 \end{bmatrix} + \begin{bmatrix} 2849 & 0 & 0 & 0 & 0 \\ 0 & 2844 & 0 & 0 & 0 \\ 0 & 0 & 2849 & 0 & 0 \\ 0 & 0 & 0 & 2844 & 0 \\ 0 & 0 & 0 & 0 & 2841 \end{bmatrix} \begin{bmatrix} +1 & 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & -1 & 0 \\ 0 & +1 & -1 & 0 & 0 \\ 0 & +1 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta^1 \\ \delta^2 \\ \delta^3 \end{bmatrix}$$



FDMA | General case (2/2)

$$E(\phi) = \dots + \underbrace{I_5}_A \underbrace{\begin{bmatrix} z_1^1 \\ z_1^2 \\ z_2^1 \\ z_2^2 \\ z_2^3 \end{bmatrix}}_z + \underbrace{\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 2849 & -1 & 0 & 0 \\ 2844 & 0 & -1 & 0 \\ 2841 & 0 & 0 & -1 \end{bmatrix}}_B \underbrace{\begin{bmatrix} \delta_2 - \delta_1 \\ 2849(\delta^1 - \delta_1) \\ 2844(\delta^2 - \delta_1) \\ 2841(\delta^3 - \delta_1) \end{bmatrix}}_b$$



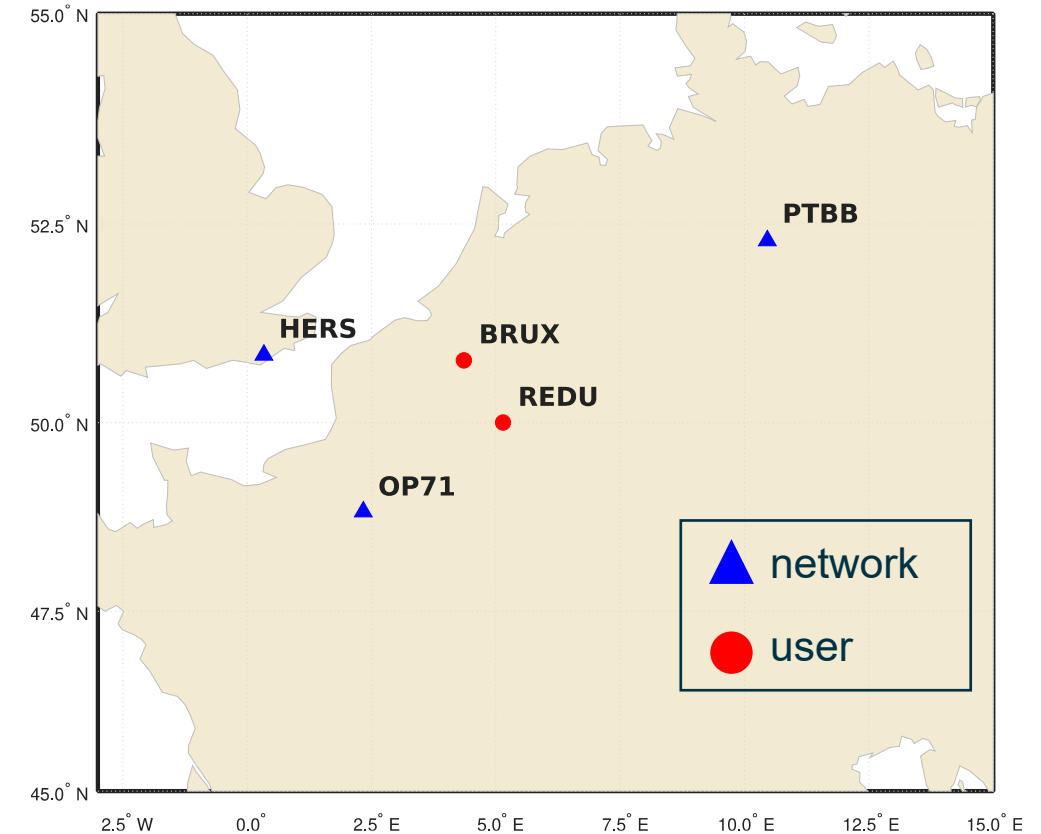
$$\tilde{Z} = \left[\begin{array}{cccc|c} 0 & +1 & 0 & 1137 & -2844 \\ 0 & 0 & +1 & -1138 & +2849 \\ -1 & 0 & 0 & -1137 & +2844 \\ +3 & 0 & 0 & +1138 & -2849 \\ -2 & 0 & 0 & +1 & 0 \end{array} \right]$$

$$\tilde{z}_1 = \tilde{Z}_1^T z = +2844z_{12}^1 - 2849z_{12}^2 \quad \text{no classical DD ambiguities!}$$

$$\tilde{b} = \begin{bmatrix} \delta_2 - \delta_1 \\ \delta^1 - \delta_1 \\ \delta^2 - \delta_1 \\ \delta^3 - \delta_1 \end{bmatrix} + \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2849} & 0 & 0 \\ 0 & 0 & -\frac{1}{2844} & 0 \\ 0 & 0 & 0 & -\frac{1}{2841} \end{bmatrix} \begin{bmatrix} +2279z_{12}^2 - 2275z_{12}^1 \\ z_1^1 \\ z_1^2 \\ z_2^3 - 1137z_{12}^1 + 1138z_{12}^2 \end{bmatrix}$$

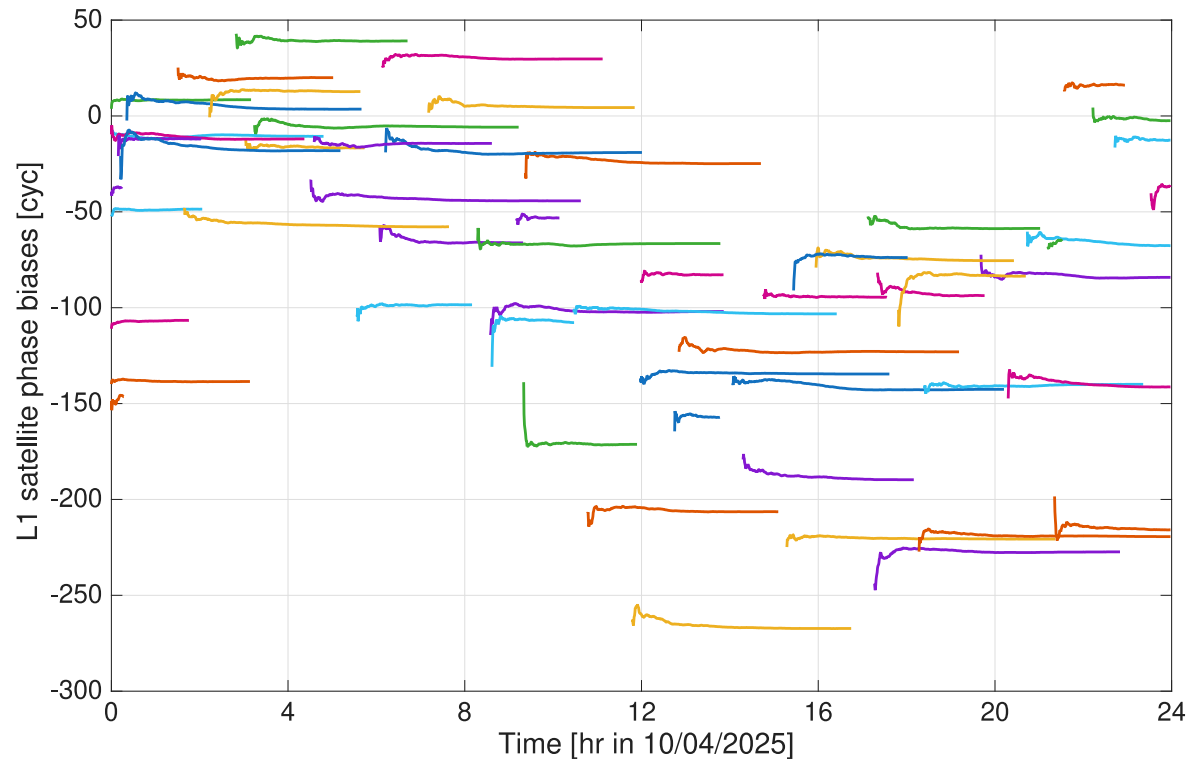
Real-world data processing | Strategy

- Homogeneous network (free of code IFBs)
- Receivers: SEPT POLARX5TR 5.5.0
- GPS L1/L2 and GLONASS L1/L2 data with 30s period
- **Software:** GAIUS (Psychas, 2022)
 - **Integrated network processing**
 - UDUC measurement model
 - S-system theory
 - Generalized Kalman filter
 - Sat. biases: time-constant
 - Sat. clocks: unlinked-in-time
 - **User processing**
 - Kinematic processing mode
 - Ionosphere-float model
 - LAMBDA-based ILS (PAR) w/o ambiguity validation

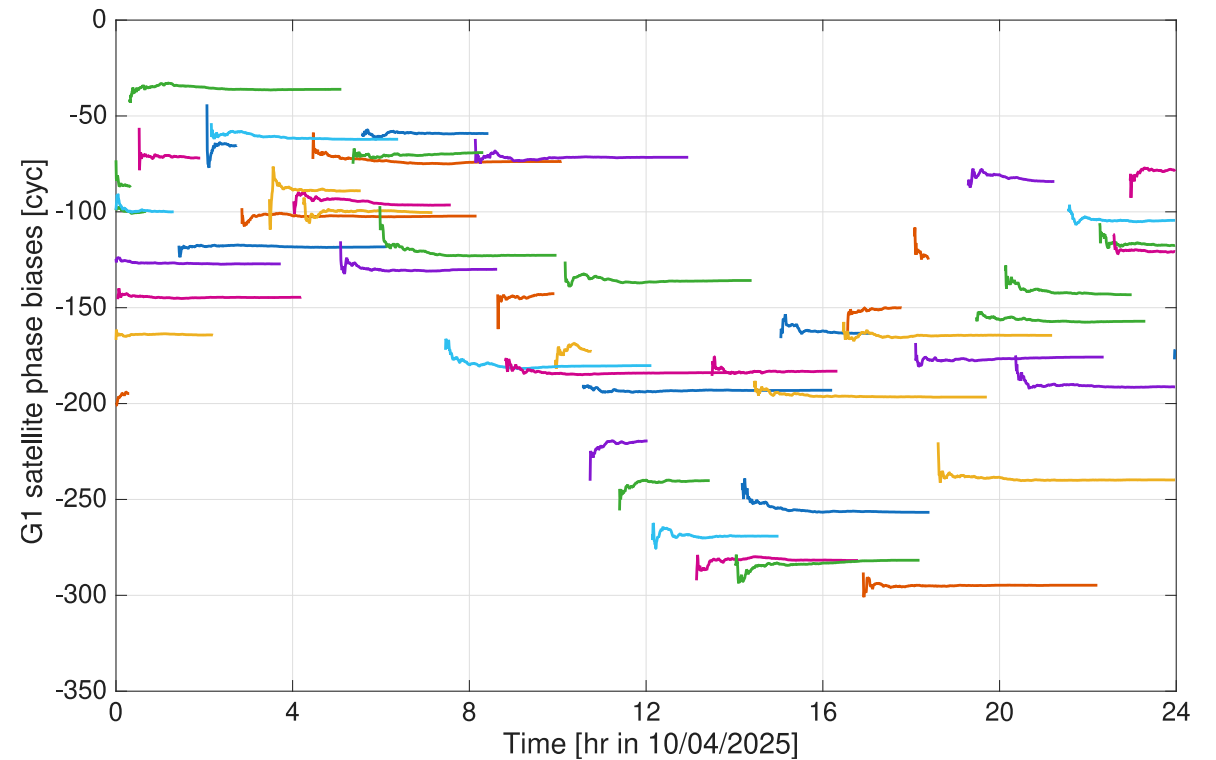


Real-world data processing | Satellite phase biases

GPS sat. phase biases

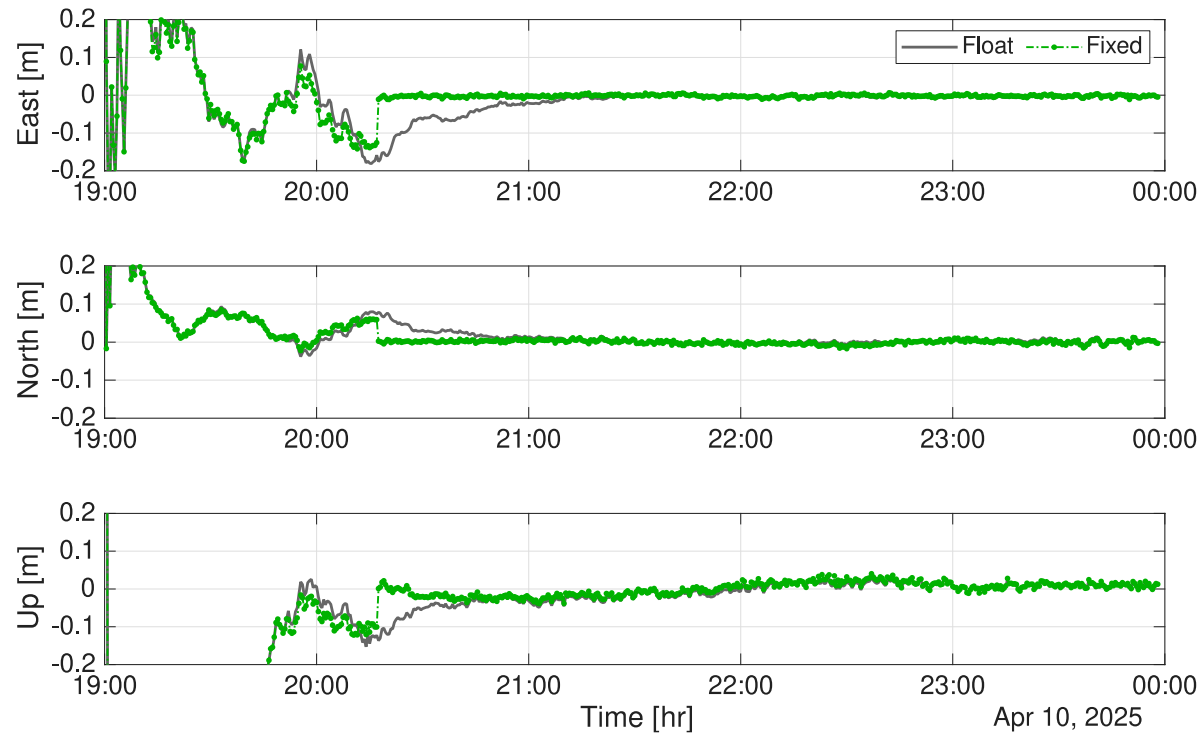


GLONASS sat. phase biases

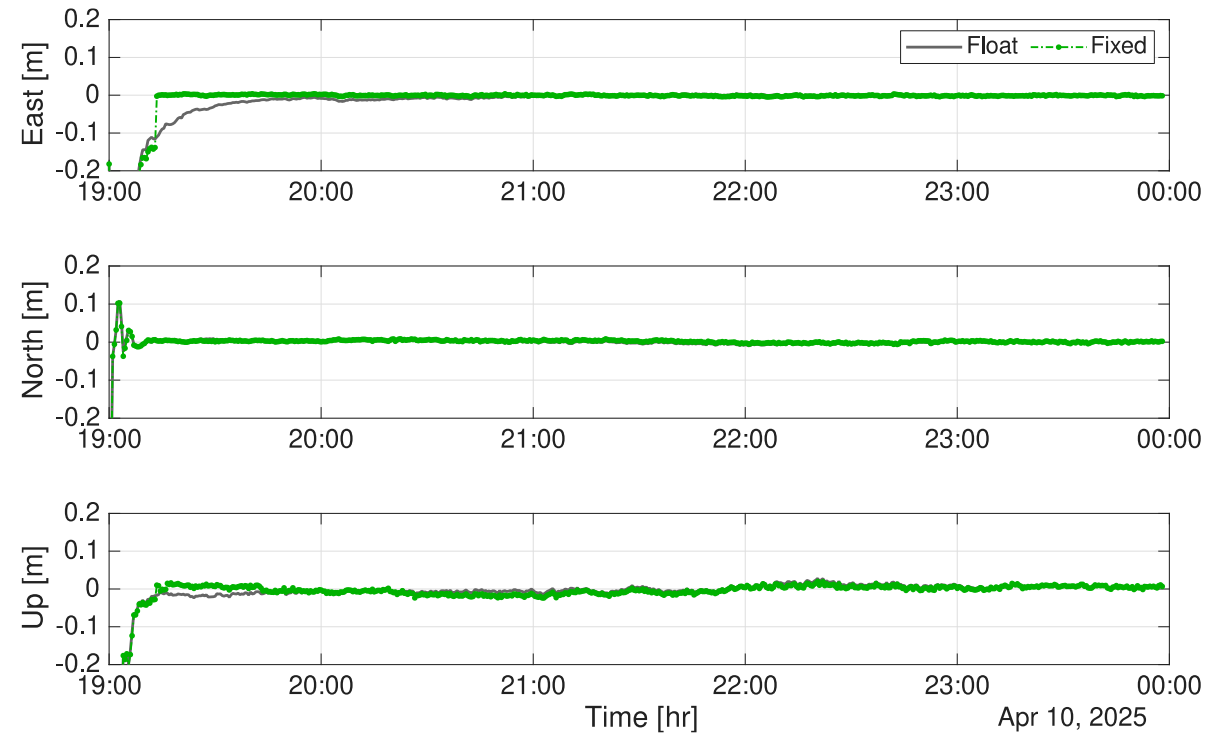


Real-world data processing | Positioning performance

GLONASS-only PPP-RTK



GPS+GLONASS PPP-RTK

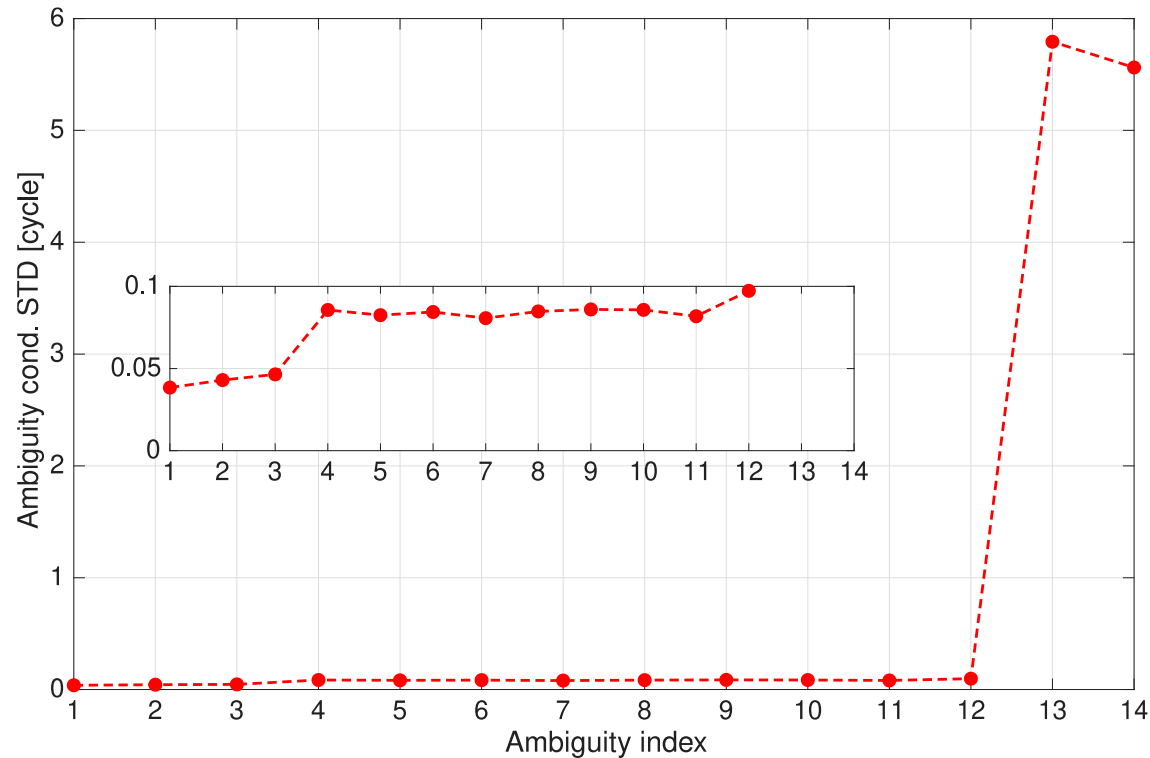


Successful GLONASS ambiguity resolution, but
not all GLONASS phase ambiguities can be resolved.

Real-world data processing | Insight into IAR

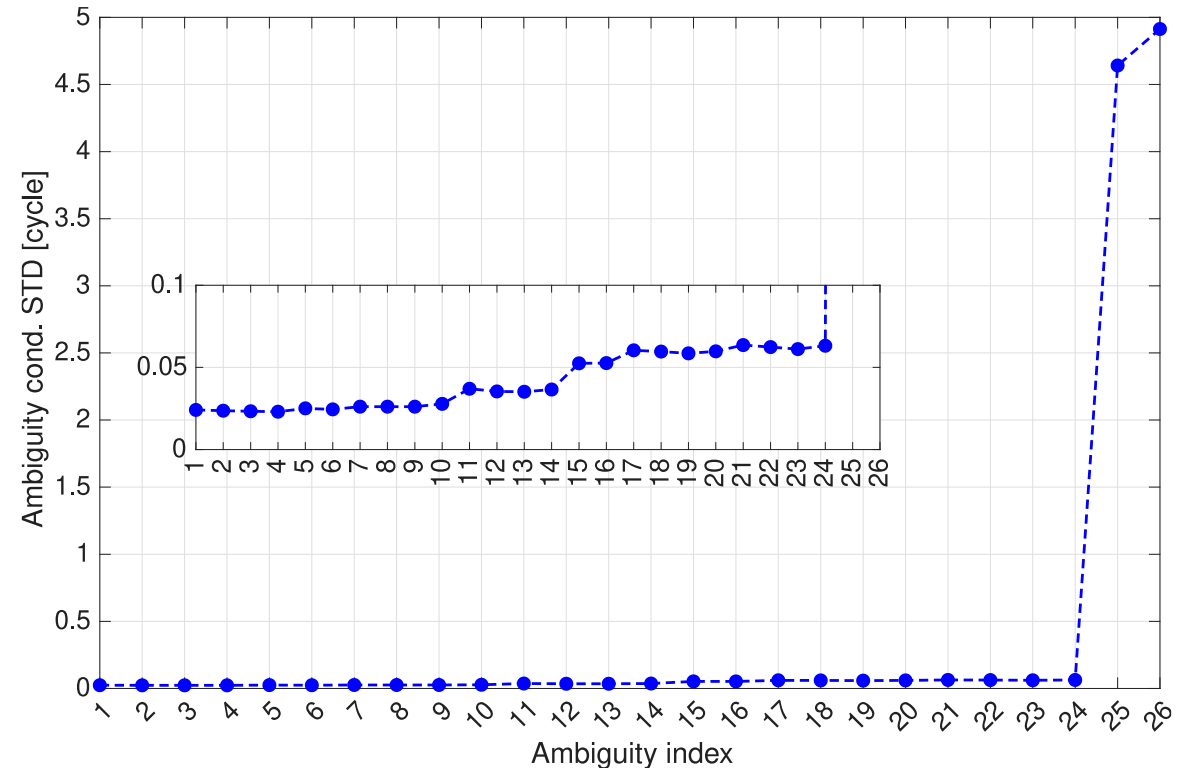
GLONASS-only

IB success rate: **PAR 100%** vs **FAR 0.5%**



GPS+GLONASS

IB success rate: **PAR 100%** vs **FAR 0.7%**



GLONASS-specific property: only $m - 2$ ambiguities per frequency can be best determined.

$$m: \# \text{ satellites} \quad | \quad m - 1: \# \text{ estimable ambiguities}$$

Is PPP-RTK always possible ?

PPP-RTK realizability condition:

Teunissen & Khodabandeh (2022)

$$\tilde{Z}_{u1}^T (P_u P^+) Z_2 \text{ must be integer}$$

Is PPP-RTK always possible ?

PPP-RTK realizability condition:

Teunissen & Khodabandeh (2022)

$$\tilde{Z}_{u1}^T (P_u P^+) Z_2 \text{ must be integer}$$

left inverse of the network's
reduced incidence matrix

e.g.,
$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ +1 & -1 & 0 \\ +1 & 0 & -1 \end{bmatrix}^+$$

Is PPP-RTK always possible ?

PPP-RTK realizability condition:

Teunissen & Khodabandeh (2022)

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network's
integer-inestimability space

Is PPP-RTK always possible ?

PPP-RTK realizability condition:

Teunissen & Khodabandeh (2022)

$$\tilde{Z}_{u1}^T (P_u P^+) Z_2 \text{ must be integer}$$

user's
satellite-only incidence matrix

e.g.,
$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Is PPP-RTK always possible ?

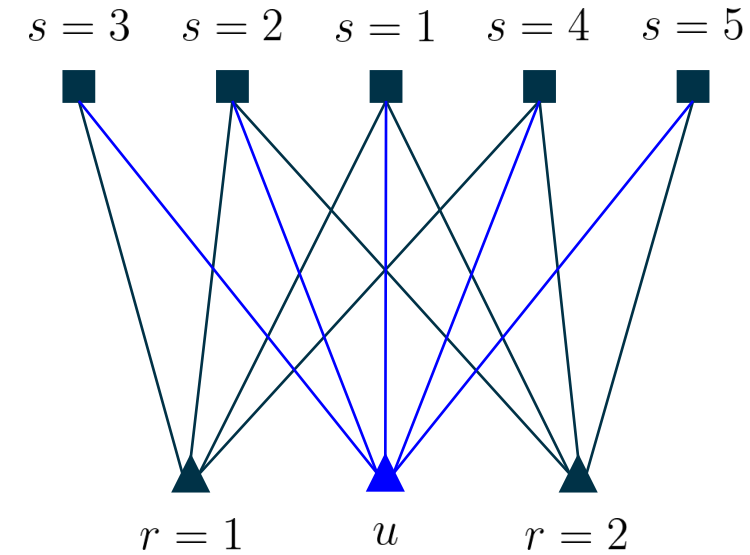
PPP-RTK realizability condition:

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$$\tilde{Z}_{u1}^T (P_u P^+) Z_2 \text{ must be integer}$$

user's
integer-estimability space

Tracking configuration #1



$$\kappa^1 = +1$$

$$\kappa^2 = -4$$

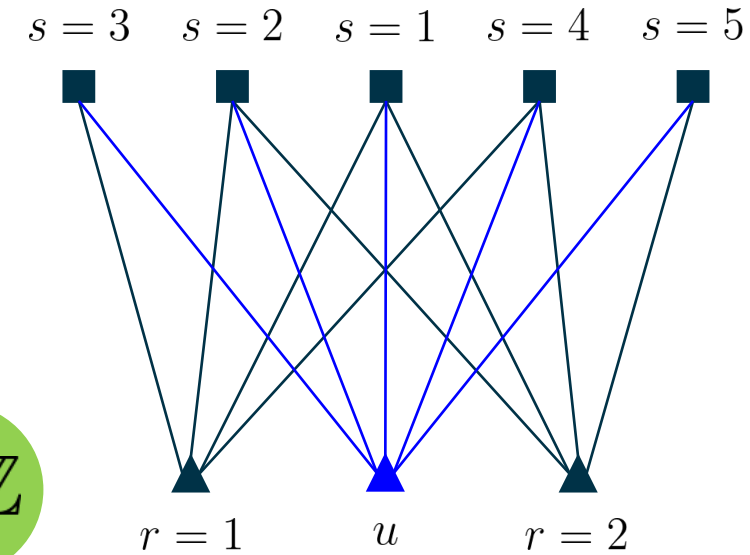
$$\kappa^3 = -7$$

$$\kappa^4 = +5$$

$$\kappa^5 = +6$$

Tracking configuration #1

$$\tilde{Z}_{u1}^T (P_u P^+) Z_2 = \begin{bmatrix} 0 & 0 & -947 & 948 & 0 & 0 \\ 0 & 1 & 6 & -5 & 0 & -2 \\ 0 & 0 & -4 & 3 & 1 & 0 \\ 0 & 0 & 13 & -10 & 0 & -3 \end{bmatrix} \in \mathbb{Z}$$



$$\kappa^1 = +1$$

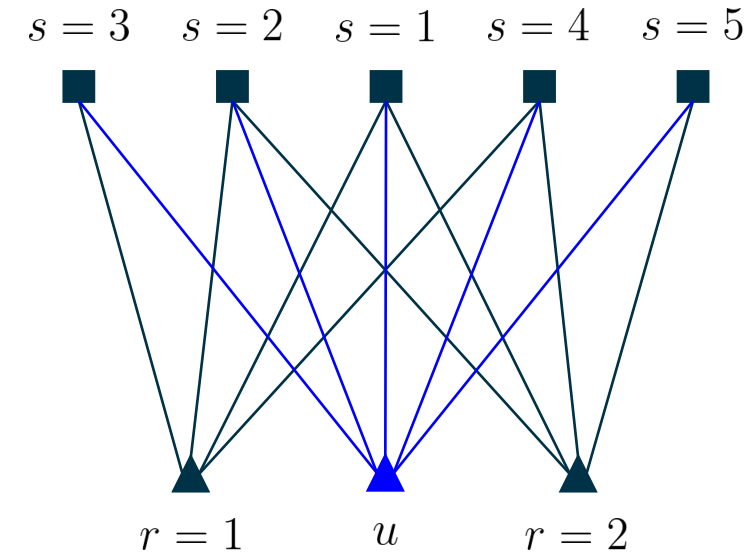
$$\kappa^2 = -4$$

$$\kappa^3 = -7$$

$$\kappa^4 = +5$$

$$\kappa^5 = +6$$

Tracking configuration #2

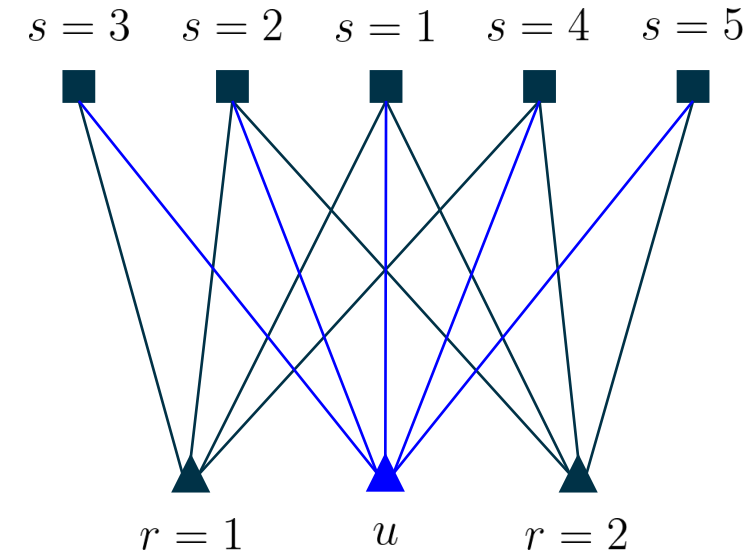


$$\begin{array}{l} \kappa^1 = +1 \\ \kappa^2 = -4 \\ \kappa^3 = -7 \\ \kappa^4 = +5 \\ \kappa^5 = +6 \end{array} \quad \left. \begin{array}{l} \leftarrow \\ \leftarrow \end{array} \right\} \text{swap}$$

Tracking configuration #2

$$\tilde{Z}_{u1}^T (P_u P^+) Z_2 = \begin{bmatrix} 0 & 948 & -947 & 0 & 0 & 0 \\ 0 & -5 & 6 & 1 & 0 & \frac{2}{3} \\ 0 & 3 & -4 & 0 & 1 & 0 \\ 0 & -10 & 13 & 0 & 0 & 1 \end{bmatrix} \notin \mathbb{Z}$$

Measurement setups and frequency choices can render PPP-RTK infeasible.

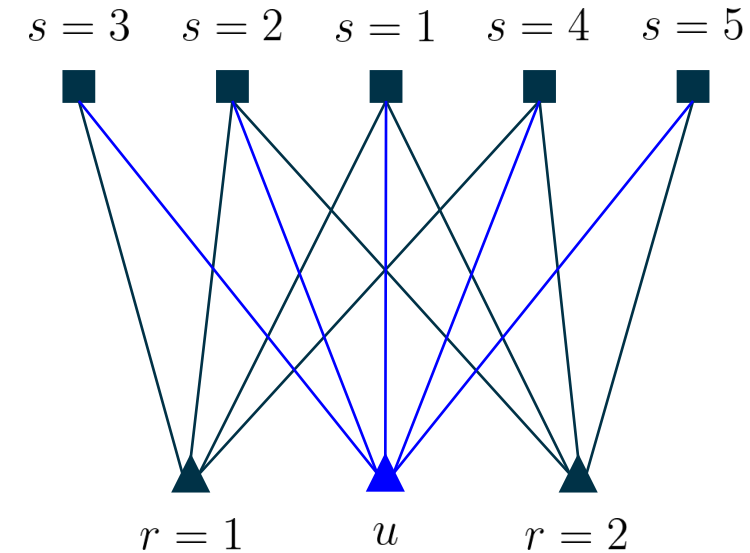


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Tracking configuration #2

- **Solution 1:** we need 1 network Rx to track all satellites

$$\tilde{Z}_{u1}^T (P_u P^+) Z_2 = \begin{bmatrix} 0 & 948 & -947 & 0 & 0 & 0 \\ 0 & -5 & 6 & 1 & 0 & -2 \\ 0 & 3 & -4 & 0 & 1 & 0 \\ 0 & -10 & 13 & 0 & 0 & -3 \end{bmatrix} \in \mathbb{Z}$$



$$\begin{aligned} \kappa^1 &= +1 \\ \kappa^2 &= -4 \\ \kappa^3 &= -7 \\ \kappa^4 &= +5 \\ \kappa^5 &= +6 \end{aligned} \quad \left. \begin{array}{l} \kappa^1 \\ \kappa^2 \\ \kappa^3 \end{array} \right\} \text{swap}$$

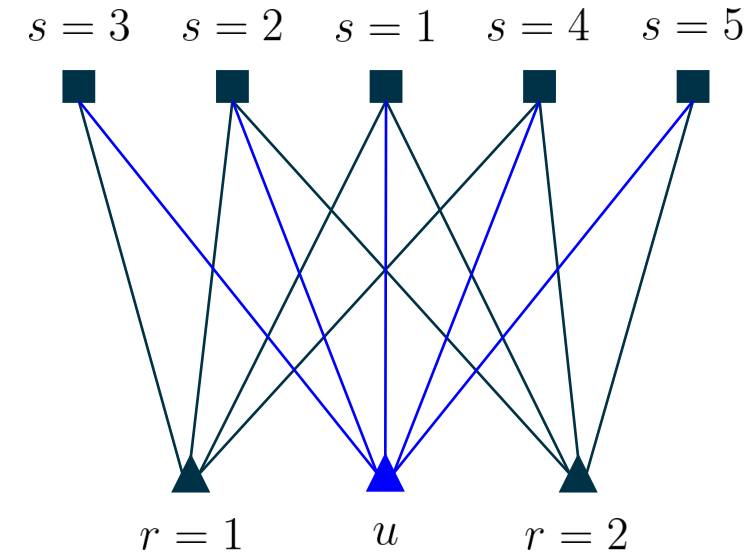
Tracking configuration #2

- **Solution 1:** we need 1 network Rx to track all satellites

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- **Solution 2:** introduce an extra receiver phase bias in the user model

$$\tilde{Z}_{u1}^T(P_u P^+)Z_2 = \begin{bmatrix} 0 & 5 & -8 & 3 & 0 & 0 \\ 0 & 948 & -947 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2854 & 951 \end{bmatrix} \in \mathbb{Z}$$



$$\begin{array}{l} \kappa^1 = +1 \\ \kappa^2 = -4 \\ \kappa^3 = -7 \\ \kappa^4 = +5 \\ \kappa^5 = +6 \end{array} \left. \begin{array}{l} \leftarrow \\ \leftarrow \end{array} \right\} \text{swap}$$

Conclusions

Integer-estimability is not guaranteed by merely asking of the estimable functions that they are integer.

Integer-sweeping is a generalized algorithmic means of obtaining a **full-rank integer-estimable model**.

GLONASS PPP-RTK ambiguity resolution is possible, under conditions.

The **PPP-RTK realizability condition** shall be the **first point of analysis** when designing a system or a network-user tracking configuration.



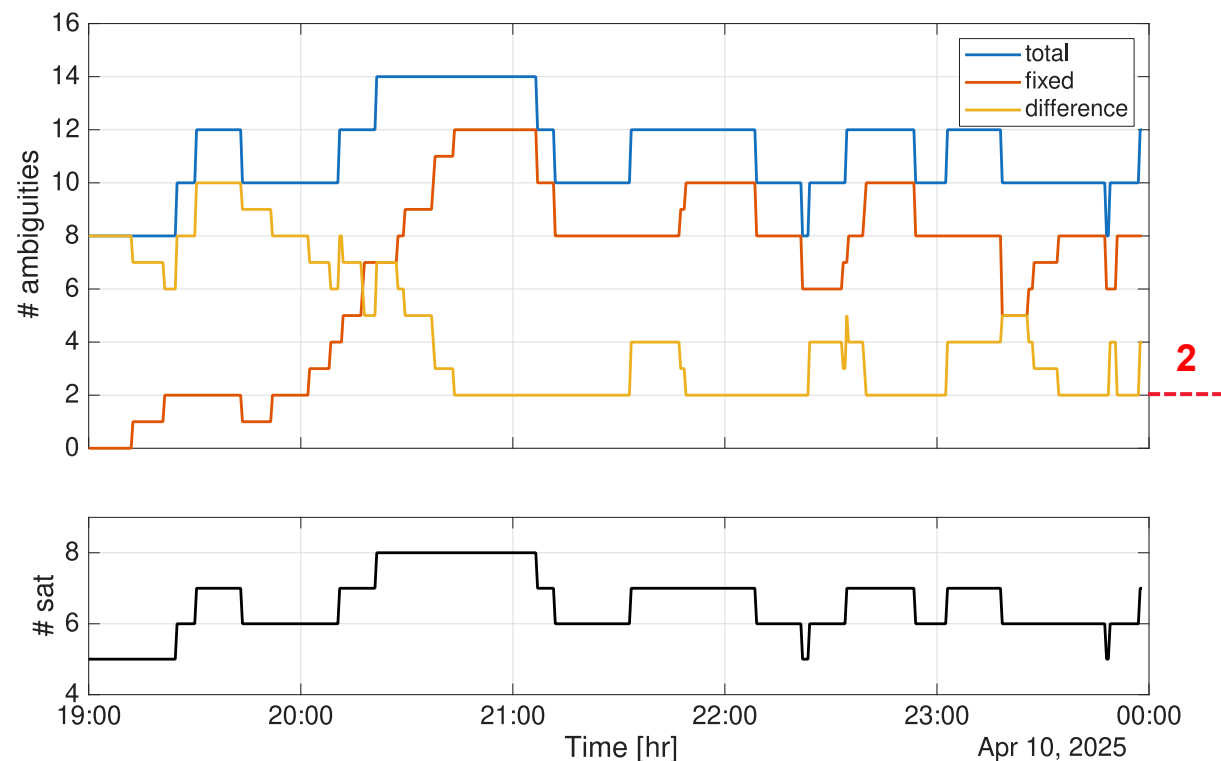
EUROPEAN SPACE AGENCY

THANK YOU!

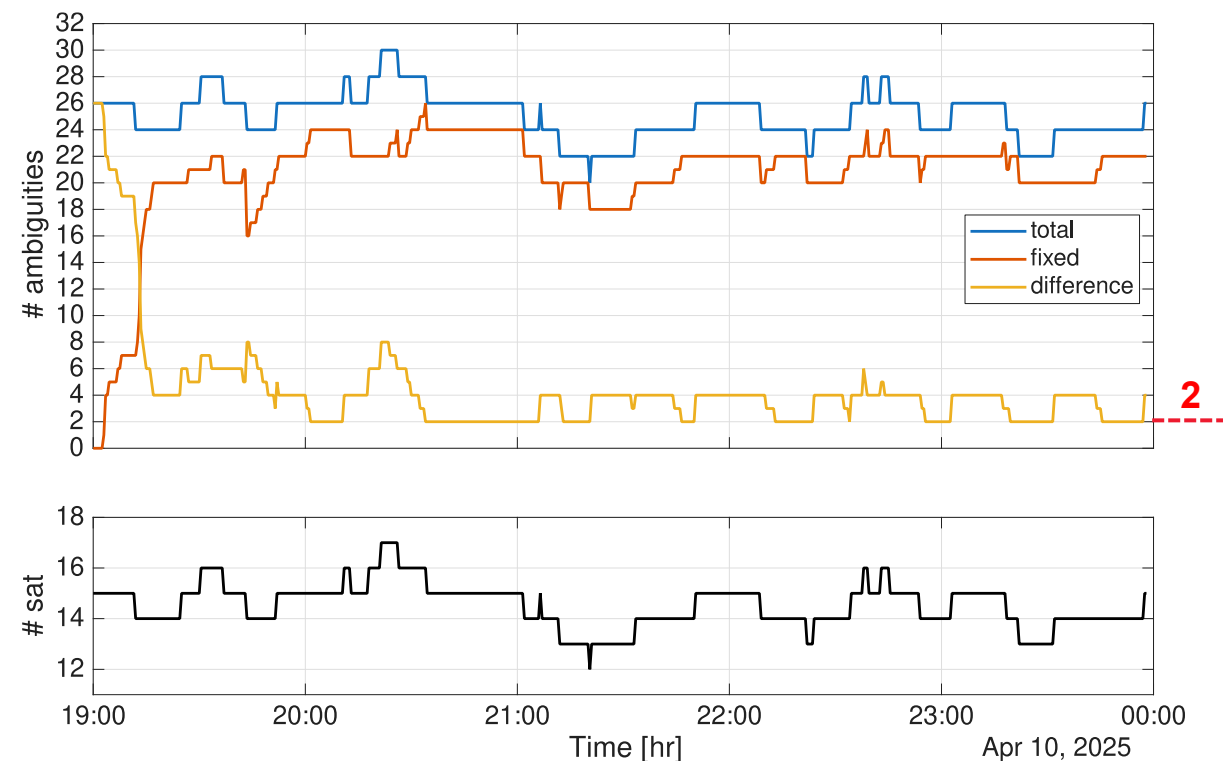
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Real-world data processing | Insight into IAR

GLONASS-only solution



GPS+GLONASS solution



Not all GLONASS phase ambiguities can be resolved.