

BLUE-BLUP Iteration Method for Partially Observed Models

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Let us recall the BLUE & BLUP

$$E[\underline{y}] = Ax \quad , \quad D[\underline{y}] = Q$$

Best Linear Unbiased Estimator (BLUE):

$$\hat{x} = (A^T Q^{-1} A)^{-1} A^T Q^{-1} y$$

$$\begin{array}{l} \text{Observed} \rightarrow \\ \text{Predicted} \leftarrow \end{array} E \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} x \quad , \quad D \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$$

Best Linear Unbiased Predictor (BLUP):

$$\hat{y}_2 = A_2 \hat{x}_1 + Q_{21} Q_{11}^{-1} (\underline{y}_1 - A_1 \hat{x}_1)$$

Fully observed model

$$E[\underline{y}] = A\underline{x} \quad , \quad D[\underline{y}] = Q$$

$$\hat{\underline{x}} = \underbrace{(A^T Q^{-1} A)^{-1} A^T Q^{-1}}_{A^+} \underline{y}$$

Partially observed model

$$E[\underline{y}_L] = A_L \underline{x} \quad , \quad D[\underline{y}] = Q_L$$

$$(\quad \underline{y}_L = L^T \underline{y} \quad A_L = L^T A \quad Q_L = L^T Q L \quad)$$

$$\hat{\underline{x}}_L = \underbrace{(A_L^T Q_L^{-1} A_L)^{-1} A_L^T Q_L^{-1}}_{A_L^+} \underline{y}_L$$

$$\text{BLUP:} \quad \tilde{\underline{y}} = A \hat{\underline{x}}_L + Q L Q_L^{-1} (\underline{y}_L - A_L \hat{\underline{x}}_L)$$

$$\text{BLUE:} \quad \hat{\underline{x}}_L = A^+ \tilde{\underline{y}}$$

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$$\text{BLUE: } \hat{\underline{x}}_L = A^+ \tilde{\underline{y}}$$

BLUE-BLUP iteration:

$$x^{(t-1)} \xrightarrow{\text{BLUP-completion}} \tilde{y}^{(t)} \xrightarrow{\text{full-BLUE}} x^{(t)}$$

$$\tilde{y}^{(t)} = A x^{(t-1)} + Q L Q_L^{-1} [y_L - A_L x^{(t-1)}] \quad x^{(t)} = A^+ \tilde{y}^{(t)}$$

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$$x^{(t)} = x^{(t-1)} + A^+ P (y - A x^{(t-1)}) \quad P = Q L (L^T Q L)^{-1} L^T$$

BB-iteration

Fully observed model

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BLUE-BLUP iteration: $x^{(t-1)} \xrightarrow{\text{BLUP-completion}} \tilde{y}^{(t)} \xrightarrow{\text{full-BLUE}} x^{(t)}$

$$x^{(t)} = x^{(t-1)} + A^+ P (y - A x^{(t-1)})$$

$$P = Q L (L^T Q L)^{-1} L^T$$

$$x^{(t)} - \hat{x}_L = [I_n - A^+ P A] (x^{(t-1)} - \hat{x}_L)$$

$$\lim_{t \rightarrow \infty} x^{(t)} = \hat{x}_L$$

convergence

Example: Short baseline code-based positioning

Observation equations

$$E(y) = [e_f \otimes G] x_1 + [e_f \otimes e_m] x_2 + [F \otimes e_m] x_3$$

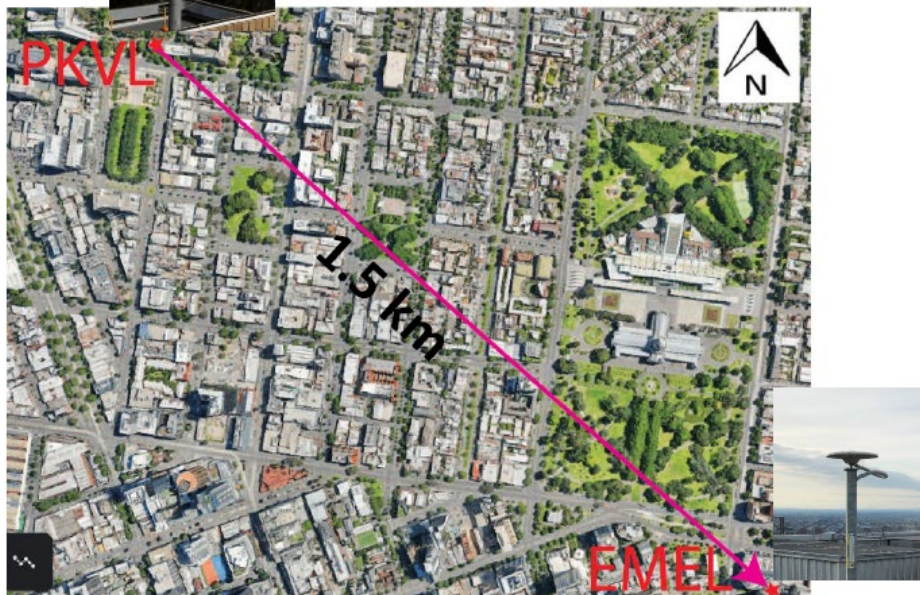
Position increments

Common clock offset

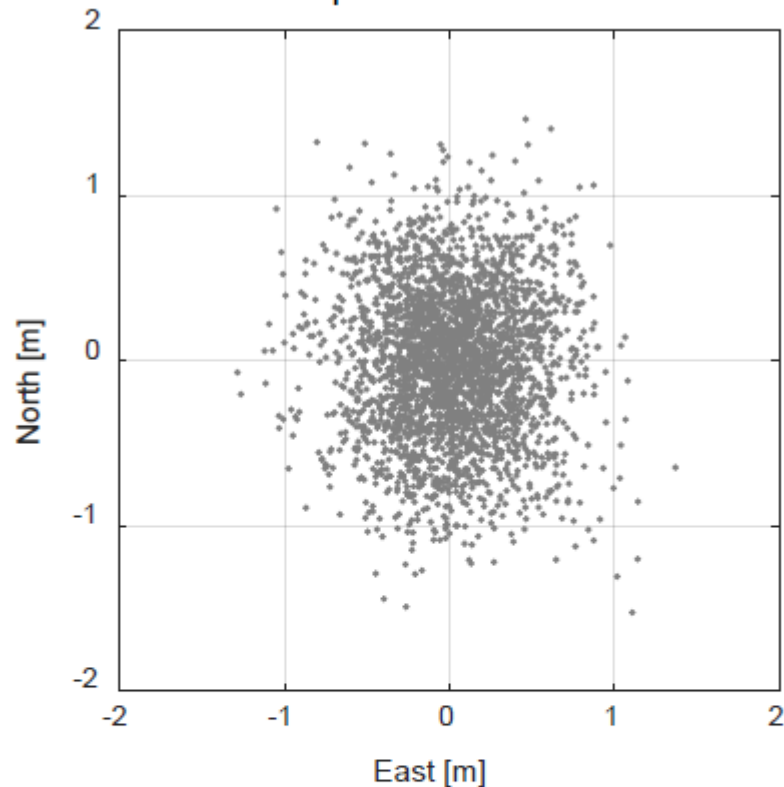
Code biases



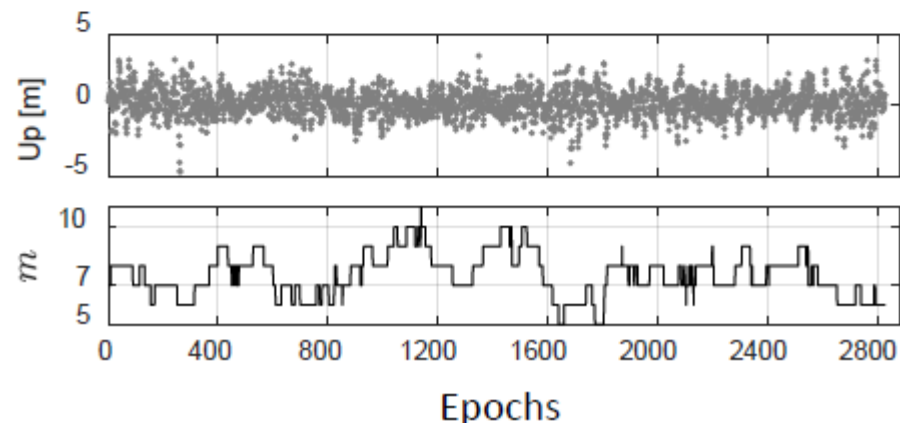
Short baseline: Parkville—East Melbourne
2nd of March 2026 (30-sec sampling rate)



Float position errors



L1/L2
UD
sigma
20 cm



Example: Short baseline code-based positioning

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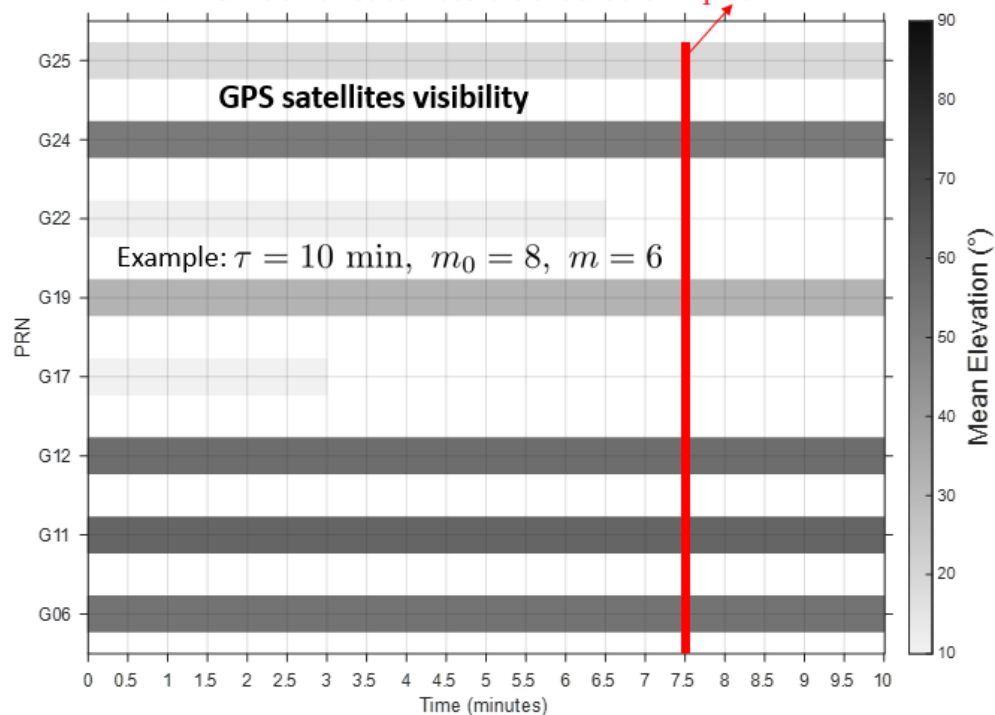
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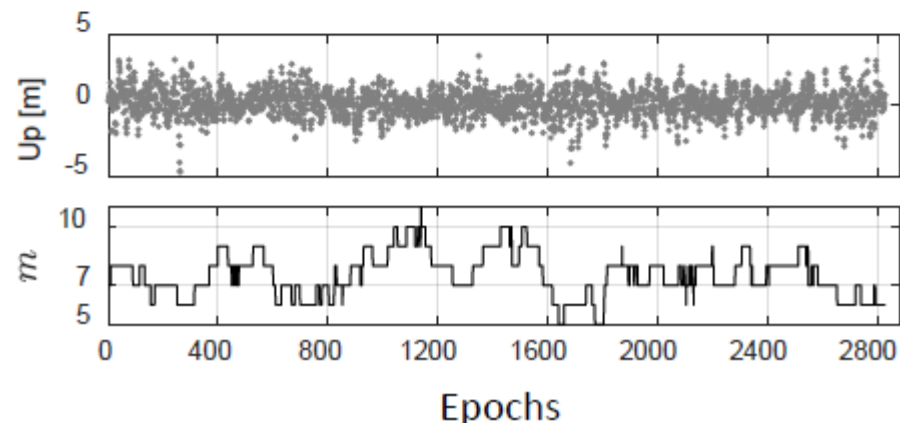
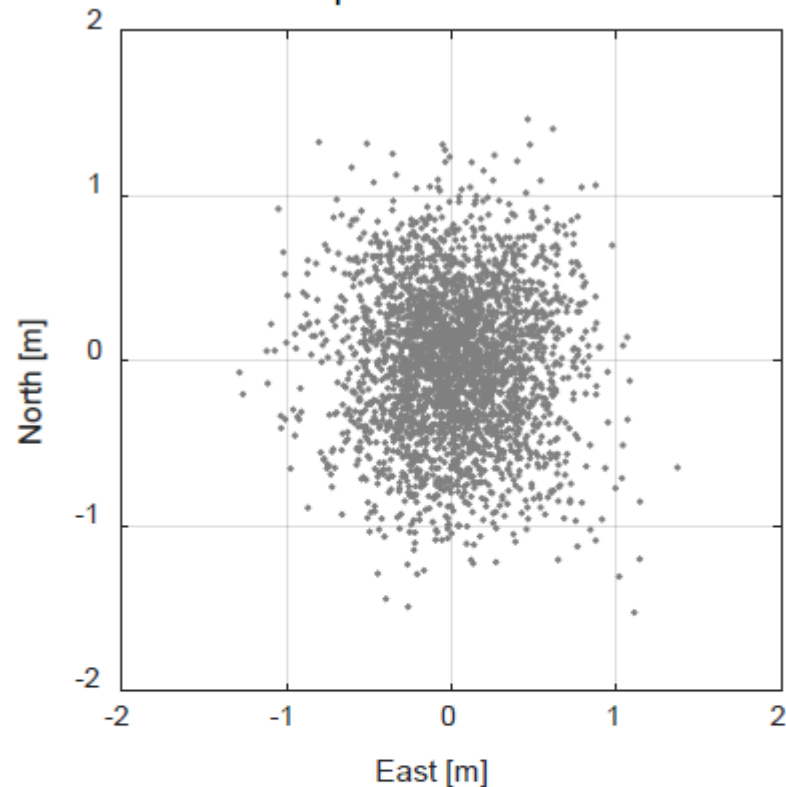
Code biases

m_0 : number of satellites visible over the interval τ

m : number of satellites tracked at an epoch

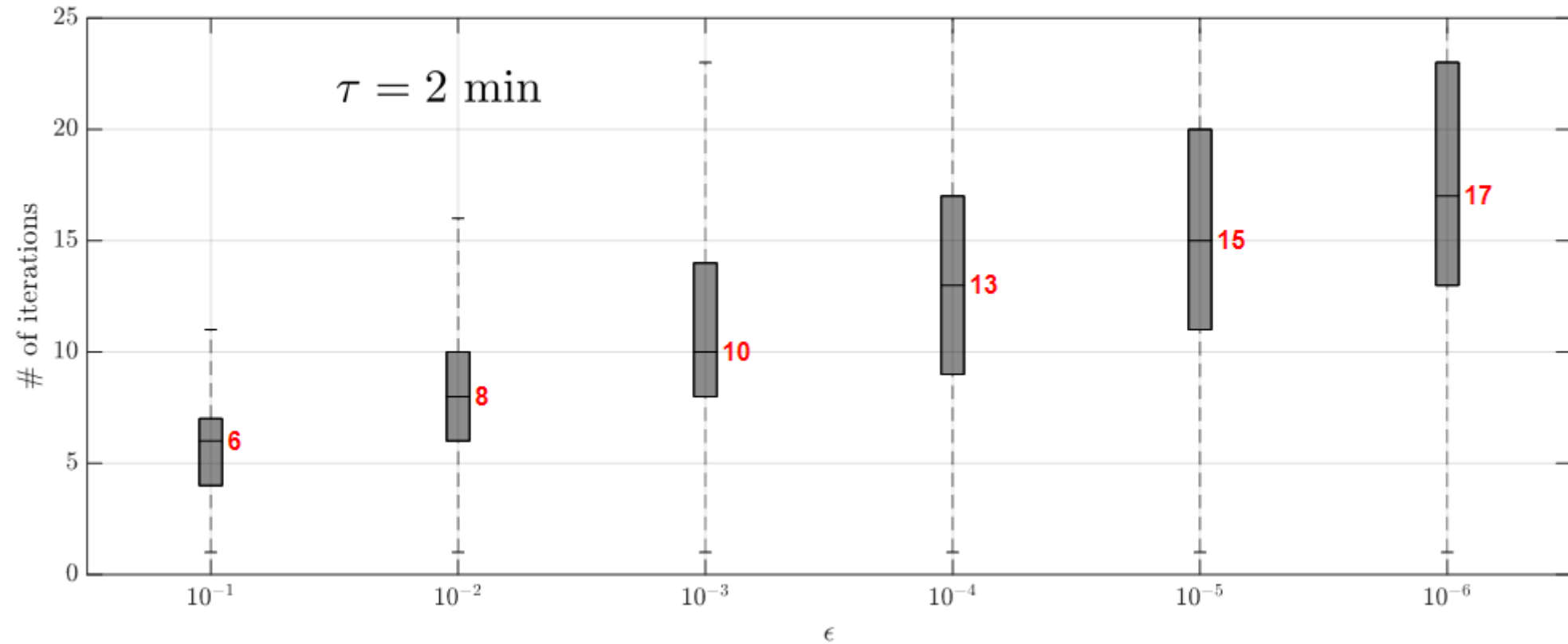


Float position errors



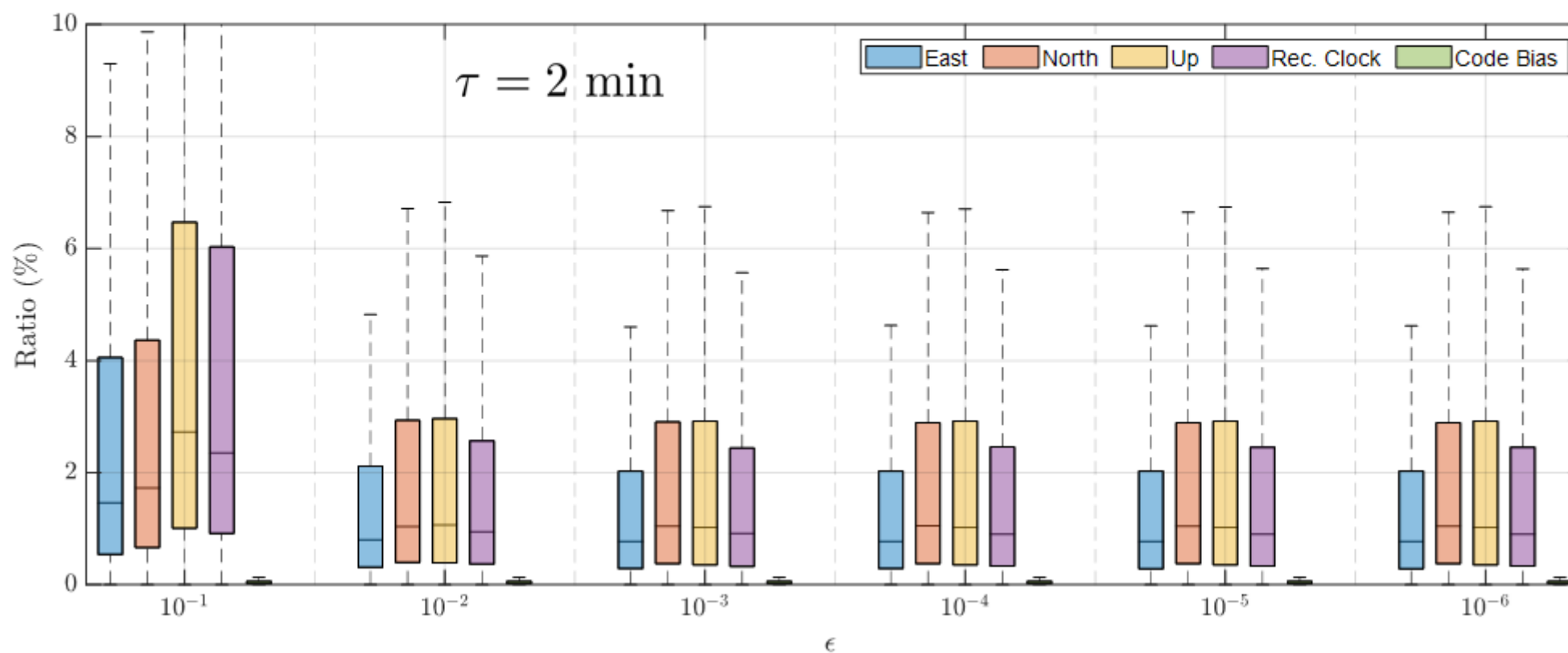
Number of iterations as function of stopping threshold

$$\|x^{(t)} - x^{(t-1)}\|_{Q_{\hat{x}\hat{x}}} \leq \epsilon, \quad \epsilon \in \{10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}\}$$



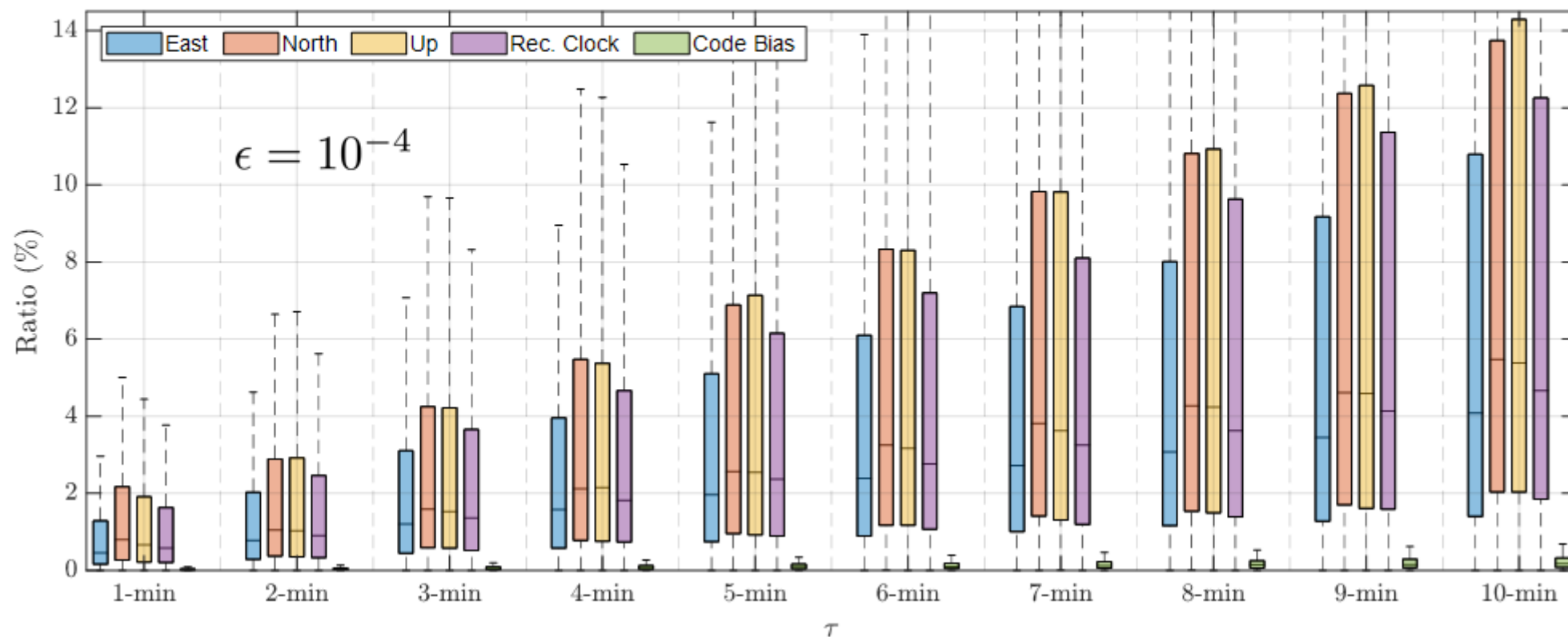
BBI-BLUE error as function of stopping threshold

$$\text{Ratio} = \frac{|x_i^{(t)} - \hat{x}_i|}{|x_i - \hat{x}_i|} : \text{BBI-BLUE error } (x_i^{(t)} - \hat{x}_i) \text{ relative to the BLUE error } (x_i - \hat{x}_i)$$



BBI-BLUE error as function of time interval

$$\text{Ratio} = \frac{|x_i^{(t)} - \hat{x}_i|}{|x_i - \hat{x}_i|} : \text{BBI-BLUE error } (x_i^{(t)} - \hat{x}_i) \text{ relative to the BLUE error } (\hat{x}_i - x_i)$$



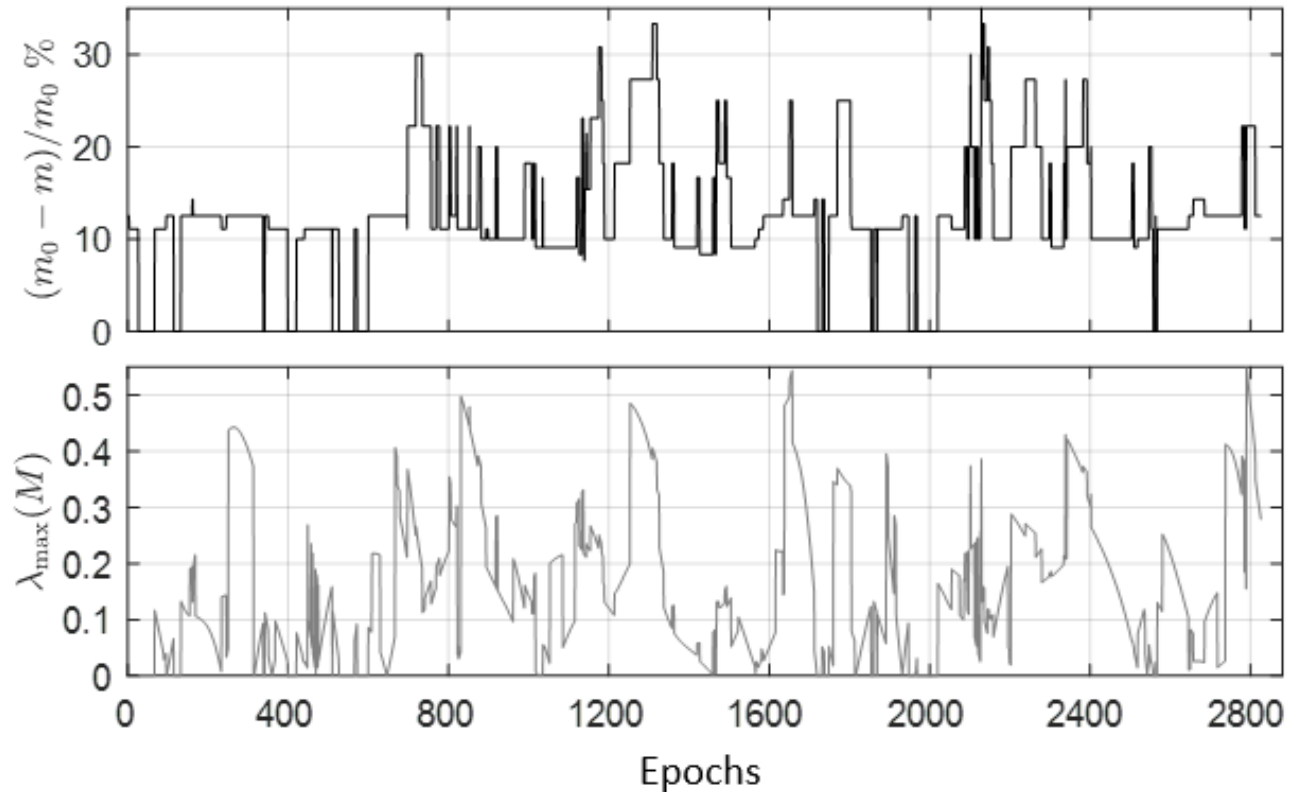
Missing satellites dictating rate of convergence

$$\|x^{(t)} - \hat{x}\| \leq \lambda_{\max}(M) \|x^{(t-1)} - \hat{x}\|;$$

m_0 : number of **visible** satellites
in the fully observed model

m : number of **tracked** satellites in
the **partially** observed model

$\lambda_{\max}(M)$: maximum eigenvalue of M



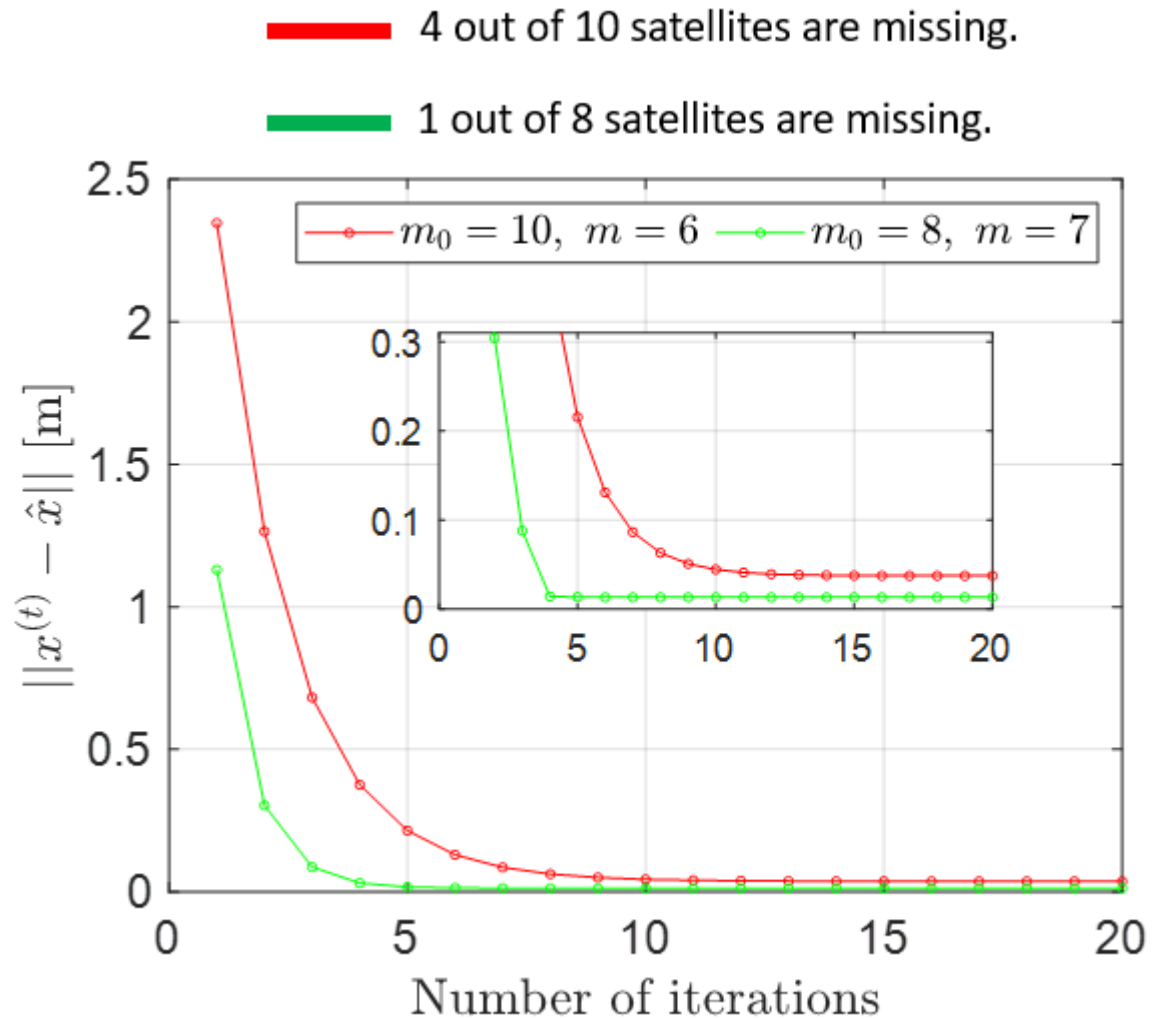
$$M = Q_{\hat{x}_0 \hat{x}_0} (Q_{\hat{x}_0 \hat{x}_0}^{-1} - Q_{\hat{x} \hat{x}}^{-1})$$

$Q_{\hat{x}_0 \hat{x}_0}$: vc-matrix of the BLUE under the fully observed model
 $Q_{\hat{x} \hat{x}}$: vc-matrix of the BLUE under the partially observed model

Missing satellites dictating rate of convergence

m_0 : number of **visible** satellites
in the fully observed model

m : number of **tracked** satellites in
the **partially** observed model



BLUE–BLUP Iteration: Exploit the Complete Model to Solve the Incomplete Model

Advantages

- **No need to factorize a new reduced normal matrix** for every missing-data pattern.
- **Preserves complete-model structure:** analytical inverses, sparse/block/Kronecker structure, existing software, etc.
- **Cheap implementation:** only applications of products $N^{-1}x$ and $N_L x$ are required.
- **Warm starts are possible:** previous epoch/solution can initialize the next adjustment.
- **Residuals are generated naturally**, making overall model testing inexpensive to add.
- **Converged to exact solution**, not approximate

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Direct solution favourable if:

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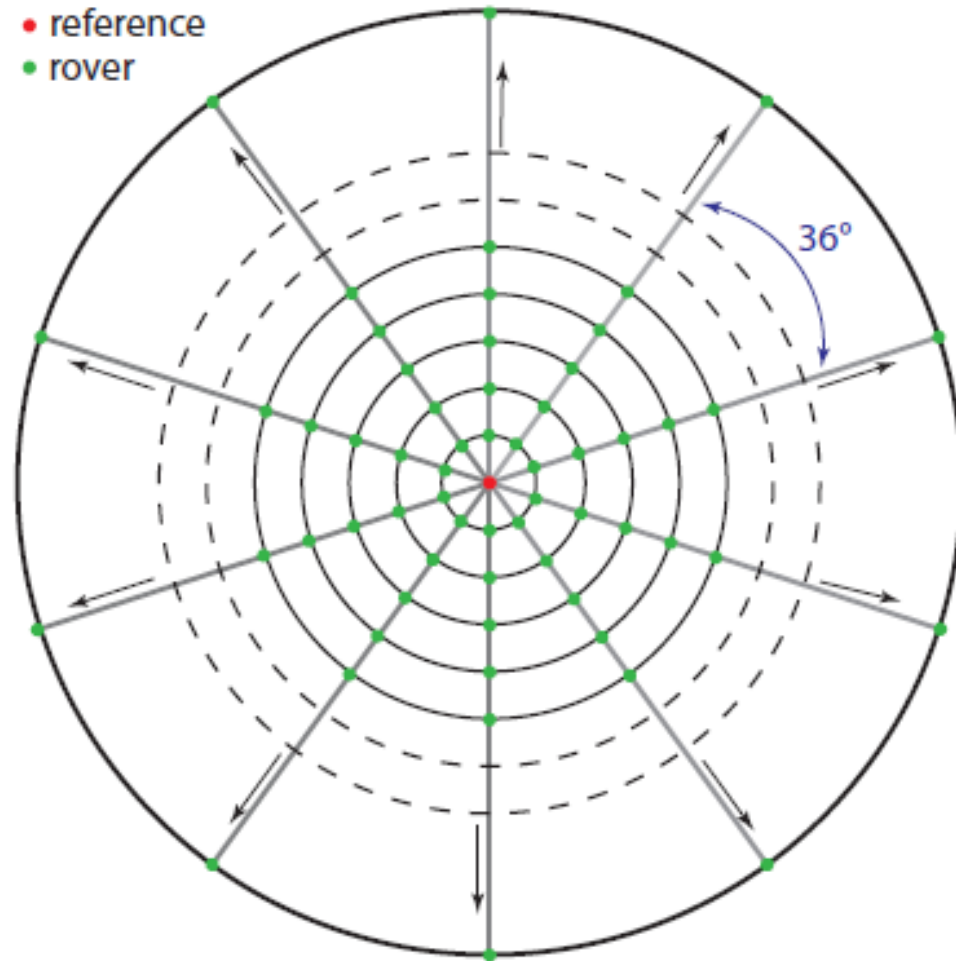
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BBI converts a difficult changing model into repeated use of an easy fixed model

BBI is most useful when structure is stable, but data availability is not

Example: Array code-based positioning



Planar array, with 10 arms, each having 100 antennas, 1 metre apart

Table 1 Averaged execution times (in seconds) required for delivering the network (BLUE) solutions, cf. Figure 1. The computation is done in MATLAB using a laptop with CPU of 2GHz and RAM of 16GB. For each scenario, 1,000 simulated datasets are considered.

baselines with missing data	1%	5%	10%	15%	20%	25%
Step 1 (forming y , A and Qy)	13.69	13.86	13.94	14.01	14.11	14.16
Step 2 (Cholesky factor of Qy)	2.48	2.62	2.59	2.52	2.58	2.56
Step 3 (forward/backward substitution)	3.66	4.33	4.43	4.38	4.34	4.26
Total (G -matrix computation plus Steps 1-3)	19.91	20.90	21.06	21.02	21.14	21.07

Table 3 Averaged execution times (in seconds) required for delivering the BB-iterated solutions, cf. Figure 1. The computation is done in MATLAB using a laptop with CPU of 2GHz and RAM of 16GB. The iteration is initialised by $b^{(0)} = 0$. For each scenario, 1,000 simulated datasets are considered.

baselines with missing data	1%	5%	10%	15%	20%	25%
Step 1 (forming L and its Cholesky factor)	0.00	0.01	0.03	0.06	0.11	0.17
Step 2 (iteration of $b^{(t)}$)	0.04	0.32	0.82	1.61	2.47	3.85
Total (G -matrix computation plus Steps 1-2)	0.05	0.34	0.86	1.68	2.59	4.03

	BB	EM	MM
Starting point	Linear estimation and prediction	Incomplete-data likelihood	Optimization objective
First operation	BLUP missing observations	Conditional expectation	Construct a surrogate
Second operation	Full-model BLUE	Maximize expected complete-data criterion	Optimize surrogate
Gaussianity needed?	No for the BLUE fixed-point result	Yes for the particular Gaussian equivalence used here	No general requirement
Missing data essential?	They motivate the present construction	Latent/incomplete data are central to EM	No
Present Gaussian model	Exact BB iteration	Exactly the same iteration	An MM realization of the same likelihood ascent
Natural uncertainty language	BLUE/BLUP covariance and prediction theory	Observed information; extra machinery may be needed computationally	Not supplied by MM alone

EM=Expectation-Maximization; MM= Minorization-Maximization