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More results on Convergence in Collocation Theory

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One year ago, I was supposed to present in Rimini new results on **convergence in Collocation Theory**.

Alas, at the time I had a fever and I could not participate. Meanwhile, new-new results have come, so I will shortly illustrate the matter to the audience of this **Hotine-Marussi Symposium**.

Warning: the presentation is just an illustration without proofs. Not all results will be presented.

The main results of 'The convergence problem of collocation theory: a return'

Setup of the problem:

We have a sequence of observations $\{Y_n\}$ on the gravity anomalous potential $u(\underline{x})$, ($\underline{x} \in \Omega$, Ω being the exterior of the Earth, u harmonic in Ω), following the linear(ized) observation equations:

$$Y_n = L_n(u) + \varepsilon_n$$

(L_n linear functional).

Here we treat only the noiseless case $\varepsilon_n = 0$.

The main results of 'The convergence problem of collocation theory: a return'

H_p:

We assume $u \in H$, a specific Hilbert space of functions harmonic in Ω , with scalar product $\langle, \rangle_H$.

We consider also another Hilbert space H' of harmonic functions, harmonic in $\Omega' \supset \supset \Omega$,
e.g., harmonic down to a Bjerrhammar sphere, with scalar product $\langle, \rangle_{H'}$.

We search for an "approximation" of u by a $\hat{u} \in H'$;

more precisely,

with R representing the restriction to Ω of functions in H' ,

we want to know whether $R\hat{u}$ tends to u , in some sense, when considering the first N measurements and letting $N \rightarrow \infty$.

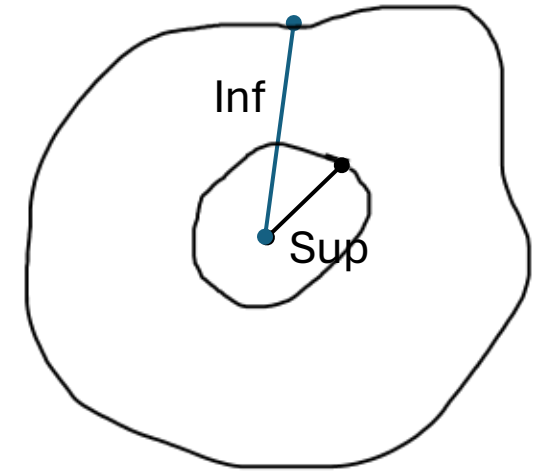
Note:

We know that R is an operator $H' \rightarrow H$

- R is one-to-one: by the unique continuation property, $(\exists R^{-1})$,
- $\text{range}(R)$ is dense in H , by Krarup's theorem
- thus, any $\hat{u} \in H'$ can also be considered as a function in H
- R is continuous and even compact if

$$\sup_{\underline{x} \in (\Omega')^c} |\underline{x}| < \inf_{\underline{x} \in \Omega} |\underline{x}|$$

i.e. the Bjerhammar sphere is strictly contained into Ω^c .



Hp:

We assume L_n to be a continuous functional on H ; otherwise Y_n would not be a 'measurement' ,

so, by the Riesz representation theorem

$$\forall u \in H \quad L_n(u) = \langle h_n, u \rangle_H, \quad h_n \in H;$$

since R is continuous $H' \rightarrow H$, one has

$$\forall \hat{u} \in H' \quad \|\hat{u}\|_H \leq c \|\hat{u}\|_{H'}$$

therefore, there are $h'_n \in H'$ such that

$$L_n(\hat{u}) = \langle h'_n, \hat{u} \rangle_{H'} \equiv \langle h_n, u \rangle_H$$

Def:

The collocation approximation $\hat{u}$ of u in the 'smoothing' sense, based on the first N observations, is the solution of the variational problem

$$\left\{ \begin{array}{l} \hat{u}(\lambda, N) \equiv \operatorname{argmin}_{\hat{u} \in H'} \left\{ \lambda \|\hat{u}\|_{H'}^2 + \sum_{n=1}^N (\langle h_n, \hat{u} \rangle_H - Y_n)^2 \right\} \\ \text{with } Y_n = \langle h_n, u \rangle_H, \quad \lambda > 0 \end{array} \right.$$

Note

Would $u \in H'$ too, (i.e., if u can be downward continued to the Bjerhammar sphere) then the solution of the above principle is just a smoothed orthogonal projection (in H') of u onto $\operatorname{span}\{h'_n, n = 1, 2, \dots, N\}$ and the convergence theorem becomes trivial. So we will assume that

$$u \in H \text{ but } u \notin H'$$

as is the case for almost all functions in H !

The ‘New’ result is the following theorem.

Theorem:

assume that $\{h_n\}$ is **total in H** , i.e., there is no other function than 0 that is orthogonal to all h_n ,
then, there is a ‘strategy’

$$M \rightarrow +\infty \quad N_M \rightarrow +\infty, \quad \lambda_M \rightarrow 0$$

such that

$$\|\hat{u}(\lambda, N) - u\|_H \xrightarrow{M \rightarrow +\infty} 0$$

Here the important point of the result is that $\hat{u}(\lambda, N) \rightarrow u$ in H , namely in the smaller domain Ω , as everybody would expect.

The theorem has been applied **as an example** to a sequence of 'observed' $\delta g(\underline{x}_n)$ (gravity disturbances) at points $\{\underline{x}_n\}$ filling densely the surface of the Earth, $S \equiv \partial\Omega$.

One easily gets a convergence result by using the theory of Geodetic BVP's with $\delta g \in L^2(\sigma)$ ($\sigma \equiv (\theta, \lambda)$ point on the unit sphere) by imposing restrictive conditions on S

$$S = \{r = R_\sigma\} \text{ (starshaped); } \quad R_\sigma \in H^{2,\infty}(\sigma) \text{ (bounded curvature)}$$

$u|_S \in H^{3,2}(\sigma)$: this is required to make the functionals

$$L_n(u) = r \frac{\partial u}{\partial r} \Big|_{\underline{x} \in S} \quad \text{continuous.}$$

In fact σ is 2-dimensional, so by Sobolev imbedding theorem

$$v \in H^{2,2}(\sigma) \rightarrow v \in C(\sigma);$$

therefore

$$u \in H^{3,2}(\sigma) \rightarrow r \frac{\partial}{\partial r} u(\sigma) \in H^{2,2} \Rightarrow r \frac{\partial}{\partial r} u \in C(\sigma).$$

Criticism

Can we prove convergence under conditions **closer** to physical reality?

This question has stimulated a **new-new result** which consists of two parts.

Part 1

Physically acceptable regularity conditions could be

- for S ; $S = \{r = R_\sigma\}$ (star-shaped); R_σ Lipschitz, i.e.,

$|R_\sigma - R_{\sigma'}| \leq c \psi_{\sigma\sigma'}$ (where $\psi_{\sigma\sigma'}$ is the spherical distance between σ and σ');

this is equivalent to saying that S has an inclination on $\underline{e}_r$ less than $\pi/2$.

- for u ; since u is generated by a bounded set, namely the Earth, filled with an essentially bounded mass density ρ , we have

$$u \in H^{2,p}(\Omega), \quad \forall p > 3 \quad \left(\int_{\Omega} \left\{ |u|^p + |\nabla u|^p + \sum |\partial_{ik} u|^p \right\} dx < +\infty \right)$$

In fact, by a famous theorem by Calderon and Zygmund

$$\rho \in L^{\infty}(\Omega) \rightarrow \rho \in L^p(\Omega) \rightarrow u \in H^{2,p}(\Omega) \quad \forall p > 1$$

and by imbedding theorems

$$2 \times p > \text{dimension of } \Omega \Rightarrow H^{2,p}(\Omega) \subset C(\Omega).$$

Part 2

If one wants to remain within the theory of BVP for functions in **Hilbert spaces**, and in particular if one wants $\delta g(\underline{x})$ to be continuous on S , as it is physically reasonable, one is **led** to the following result.

Theorem:

If $\{\underline{x}_n\}$ is a sequence of points dense on S , if S is star-shaped and Lipschitz, if $u \in H^{3,2}(\Omega)$ then convergence follows according to the first Theorem.

In fact, by the famous Sobolev's imbedding theorem

$$H^{3,2}(\Omega) \subset C^{1,\lambda}(\Omega) \quad 0 \leq \lambda \leq 1/2$$

$$H^{3,2}(\Omega) \subset H^{2,q}(\Omega) \quad 2 \leq q \leq 6;$$

note that $u \in H^{3,2}(\Omega)$ is a sufficient condition, but it is not necessary because

$$\forall q > 3, u \in H^{2,q}(\Omega) \not\Rightarrow u \in H^{3,2}(\Omega)$$

Comments 1

Can we get rid of the ugly - and excessive - requirement $u \in H^{3,2}(\Omega)$?

Yes, if we develop an $H^{2,p}(\Omega)$ theory for the geodetic BVP

and

an $L^p - H^{2,p}$ theory of collocation.

Note that, to build the $H^{2,p}(\Omega)$ theory for GBVP, it is not enough to develop it for the spherical approximation of the boundary operator, which is trivial.

Rather, we need to evaluate explicitly the majorization constant required to switch from the radial derivative to the derivative , for example, in the direction of the ellipsoidal normal $\underline{\nu}$

Comments 2

Geodesy, with collocation theory, anticipated what is called Statistical Learning - the father of which is Vladimir Vapnik (a 90-years-old Russian researcher working in the USA), nowadays called Machine Learning.

As far as Hilbert space theory is concerned, Geodesy was ahead of Mathematics and Statistics, until about 10 years ago.

Yet, within the Machine Learning community, researchers have introduced the theory of Reproducing Kernel Banach Spaces and developed a Banach-space theory for Machine Learning, mainly to make numerical problems in image analysis more tractable. So we are lagging behind.

There is still a lot of work to do!

(Fortunately!)