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Università
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Kalman filter from the geodetic point of view

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History of the Symposia

Three-dimensional Geodesy: Venice 1959, Cortina 1963, Torino 1965

Hotine: Trieste 1969, Firenze 1972, Siena 1976, **Assisi 1978, Como 1981**

Hotine-Marussi: Rome 1985, Pisa 1989, L'Aquila 1994, Trento 1998, **Matera 2003, Wuhan 2006, Rome 2009, Rome 2013**, Rome 2018, Milan 2022, **Genova 2026**

The Kalman filter setup

A dynamical system governed by a first order linear differential equation:

$$\frac{d\mathbf{x}}{dt}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{w}(t)$$

↑ ↙
system state system noise
 (stochastic)

Discrete solution:

$$\mathbf{x}(t_k) = \mathbf{\Phi}(t_k, t_{k-1})\mathbf{x}(t_{k-1}) + \mathbf{w}(t_k) \quad \mathbf{w}(t_k) \sim N(\mathbf{0}, \mathbf{Q}_k)$$

transition
matrix

$$\mathbf{\Phi}(t_j, t_k)\mathbf{\Phi}(t_k, t_m) = \mathbf{\Phi}(t_j, t_m), \quad t_j \leq t_k \leq t_m$$
$$\mathbf{\Phi}(t_j, t_j) = \mathbf{I}$$

The Kalman filter setup

$$\begin{aligned}\mathbf{x}_k &= \Phi_{k,k-1} \mathbf{x}_{k-1} + \mathbf{w}_k & \mathbf{w}_k &\sim N(\mathbf{0}, \mathbf{Q}_k) & k &= 1, 2, \dots, n \\ \mathbf{x}_0 &\sim N(\mathbf{0}, \mathbf{C}_{\mathbf{x}_0})\end{aligned}$$

State “estimation” from available observations:

$$\mathbf{b}_k = \mathbf{A}_k \mathbf{x}_k + \mathbf{e}_k \quad \mathbf{e}_k \sim N(\mathbf{0}, \mathbf{R}_k) \quad k = 1, 2, \dots, n$$

Filtering: $\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|\mathbf{b}_1, \dots, \mathbf{b}_k}$ (estimate based on data up to epoch t_k)

Smoothing: $\hat{\mathbf{x}}_{k|n} = \hat{\mathbf{x}}_{k|\mathbf{b}_1, \dots, \mathbf{b}_n}$ (estimate based on all data)

What is the difference between the “engineering” and the “geodetic” point of view with respect to Kalman filtering and smoothing?

*The “engineering” point of view
is looking at the tree*



An aerial photograph of a vast, dense forest of tall, green coniferous trees. The forest extends to the horizon under a clear sky. The text is overlaid in the lower center of the image.

*The “geodetic” point of view
is looking at the forest*

The “engineering” tree

One-epoch-at-a-time sequential “estimation”

Forward:

$$\hat{\mathbf{x}}_{k|k} \rightarrow \hat{\mathbf{x}}_{k+1|k+1}$$

Filtering

Backward:

$$\hat{\mathbf{x}}_{k|n} \leftarrow \hat{\mathbf{x}}_{k+1|n}$$

Smoothing



The “engineering” tree

One-epoch-at-a-time sequential “estimation”

Forward: $\hat{\mathbf{x}}_{k|k} \rightarrow \hat{\mathbf{x}}_{k+1|k+1}$ **Filtering**

The Kalman filtering algorithm:

$$\mathbf{C}_{\hat{\mathbf{x}}_{k|k}} = \left[\left(\Phi_{k,k-1} \mathbf{C}_{\hat{\mathbf{x}}_{k-1|k-1}} \Phi_{k,k-1}^T + \mathbf{Q}_k \right)^{-1} + \mathbf{A}_k^T \mathbf{R}_k^{-1} \mathbf{A}_k \right]^{-1}$$

$$\hat{\mathbf{x}}_{k|k} = \mathbf{C}_{\hat{\mathbf{x}}_{k|k}} \left[\left(\Phi_{k,k-1} \mathbf{C}_{\hat{\mathbf{x}}_{k-1|k-1}} \Phi_{k,k-1}^T + \mathbf{Q}_k \right)^{-1} \Phi_{k,k-1} \hat{\mathbf{x}}_{k-1|k-1} + \mathbf{A}_k^T \mathbf{R}_k^{-1} \mathbf{b}_k \right]$$



The “engineering” tree

One-epoch-at-a-time sequential “estimation”

Forward: $\hat{\mathbf{x}}_{k|k} \rightarrow \hat{\mathbf{x}}_{k+1|k+1}$ **Filtering**

The usual Kalman filtering algorithm served in pieces:

“Prediction”: $\tilde{\mathbf{x}}_{k|k-1} = \Phi_{k,k-1} \hat{\mathbf{x}}_{k-1|k-1}$ $\mathbf{C}_{\tilde{\mathbf{x}}_{k|k-1}} = \Phi_{k,k-1} \mathbf{C}_{\hat{\mathbf{x}}_{k-1|k-1}} \Phi_{k,k-1}^T + \mathbf{Q}_k$
 $\tilde{\mathbf{e}}_k = \mathbf{b}_k - \mathbf{A}_k \tilde{\mathbf{x}}_{k|k-1}$ $\mathbf{C}_{\tilde{\mathbf{e}}_k} = \mathbf{R}_k + \mathbf{A}_k \mathbf{C}_{\tilde{\mathbf{x}}_{k|k-1}} \mathbf{A}_k^T$

Kalman gain: $\mathbf{K}_k = \mathbf{C}_{\tilde{\mathbf{x}}_{k|k-1}} \mathbf{A}_k^T \mathbf{C}_{\tilde{\mathbf{e}}_k}^{-1}$

“Estimation”: $\hat{\mathbf{x}}_{k|k} = \tilde{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \tilde{\mathbf{e}}_k$ $\mathbf{C}_{\hat{\mathbf{x}}_{k|k}} = (\mathbf{I} - \mathbf{K}_k \mathbf{A}_k) \mathbf{C}_{\tilde{\mathbf{x}}_{k|k-1}}$



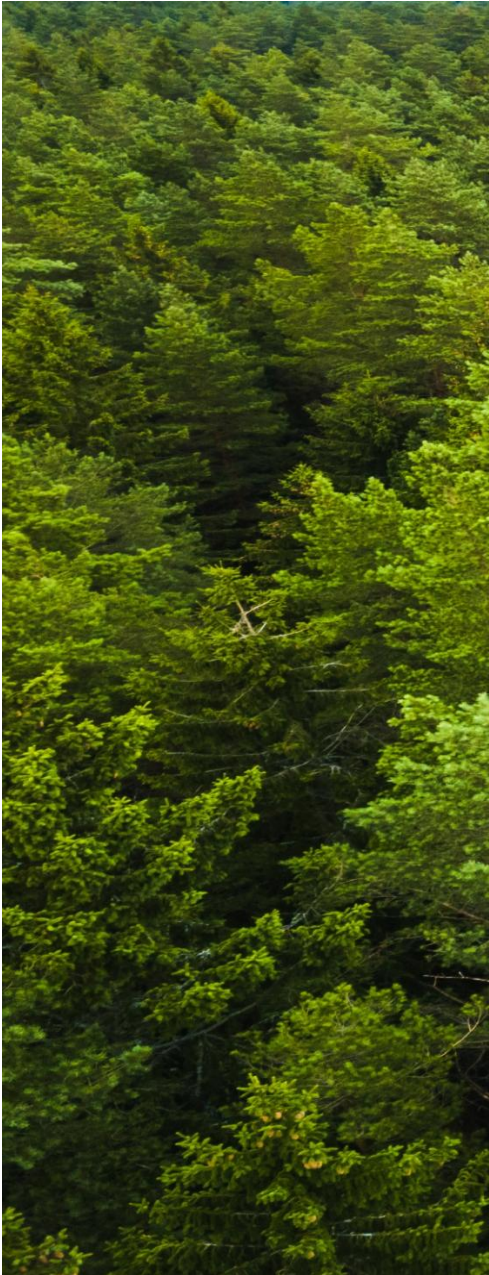
The “geodetic” forest

Get **in one step** the states at **all** epochs: $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{bmatrix}$

by relating them to the system noise vector: $\mathbf{w} = \begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_n \end{bmatrix}$

utilizing all available observations: $\mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_n \end{bmatrix}$

A global (usually not sequential) solution is sought!



*The basic model for
the state prediction problem*

The basic model for the state prediction problem

$$\mathbf{x}_k = \Phi_{k,k-1} \mathbf{x}_{k-1} + \mathbf{w}_k \quad \longrightarrow \quad \mathbf{w}_k = \mathbf{x}_k - \Phi_{k,k-1} \mathbf{x}_{k-1}, \quad k = 1, 2, \dots, n:$$

$$\begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \\ \vdots \\ \mathbf{w}_{n-1} \\ \mathbf{w}_n \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 - \Phi_{1,0} \mathbf{x}_0 \\ \mathbf{x}_2 - \Phi_{2,1} \mathbf{x}_1 \\ \mathbf{x}_3 - \Phi_{3,2} \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_{n-1} - \Phi_{n-1,n-2} \mathbf{x}_{n-2} \\ \mathbf{x}_n - \Phi_{n,n-1} \mathbf{x}_{n-1} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ -\Phi_{2,1} & \mathbf{I} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\Phi_{3,2} & \mathbf{I} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & -\Phi_{n,n-1} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \vdots \\ \mathbf{x}_{n-1} \\ \mathbf{x}_n \end{bmatrix} + \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{x}_0$$

$$\mathbf{w} = \mathbf{F} \mathbf{x} - \mathbf{J}_n \Phi_{1,0} \mathbf{x}_0$$

F is sparse!

The basic model for the state prediction problem

$$\mathbf{w} = \mathbf{F}\mathbf{x} - \mathbf{J}_n \Phi_{1,0} \mathbf{x}_0$$



$$\mathbf{x} = \mathbf{F}^{-1} \mathbf{w} + \mathbf{F}^{-1} \mathbf{J}_n \Phi_{1,0} \mathbf{x}_0 = \mathbf{F}^{-1} \mathbf{w} + \Phi_0 \mathbf{x}_0$$

$$\mathbf{F}^{-1} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \Phi_{2,1} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \Phi_{3,1} & \Phi_{3,2} & \mathbf{I} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \Phi_{n-1,1} & \Phi_{n-1,2} & \Phi_{n-1,3} & \cdots & \mathbf{I} & \mathbf{0} \\ \Phi_{n,1} & \Phi_{n,2} & \Phi_{n,3} & \cdots & \Phi_{n,n-1} & \mathbf{I} \end{bmatrix}$$

$\mathbf{F}^{-1}$ is lower triangular!

The basic model for the state prediction problem

$$\mathbf{w} = \mathbf{F}\mathbf{x} - \mathbf{J}_n \Phi_{1,0} \mathbf{x}_0 \quad \longrightarrow \quad \mathbf{x} = \mathbf{F}^{-1} \mathbf{w} + \mathbf{F}^{-1} \mathbf{J}_n \Phi_{1,0} \mathbf{x}_0 = \mathbf{F}^{-1} \mathbf{w} + \Phi_0 \mathbf{x}_0$$

$$\Phi_0 = \mathbf{F}^{-1} \mathbf{J}_n \Phi_{1,0} = \begin{bmatrix} \mathbf{I} \\ \Phi_{2,1} \\ \Phi_{3,1} \\ \vdots \\ \Phi_{n-1,1} \\ \Phi_{n,1} \end{bmatrix} \Phi_{1,0} = \begin{bmatrix} \Phi_{1,0} \\ \Phi_{2,1} \Phi_{1,0} \\ \Phi_{3,1} \Phi_{1,0} \\ \vdots \\ \Phi_{n-1,1} \Phi_{1,0} \\ \Phi_{n,1} \Phi_{1,0} \end{bmatrix} = \begin{bmatrix} \Phi_{1,0} \\ \Phi_{2,0} \\ \Phi_{3,0} \\ \vdots \\ \Phi_{n-1,0} \\ \Phi_{n,0} \end{bmatrix}$$

The basic model for the state prediction problem

$$\mathbf{w} = \mathbf{F}\mathbf{x} - \mathbf{J}_n \Phi_{1,0} \mathbf{x}_0$$



$$\mathbf{x} = \mathbf{F}^{-1} \mathbf{w} + \Phi_0 \mathbf{x}_0$$

$$\mathbf{b} = \mathbf{A}\mathbf{x} + \mathbf{e}$$

← used for computing the covariance matrix $\mathbf{C}_x$

← used to predict the stochastic parameter $\mathbf{x} \sim (\mathbf{0}, \mathbf{C}_x)$

It remains to clarify the role of $\mathbf{x}_0$

Differences between the Kalman and the geodetic approach

Global model:

$$\mathbf{x} = \mathbf{F}^{-1}\mathbf{w} + \mathbf{\Phi}_0\mathbf{x}_0,$$

$$\mathbf{b} = \mathbf{A}\mathbf{x} + \mathbf{e},$$

Engineering:

$$\mathbf{w} \sim N(\mathbf{0}, \mathbf{Q}), \quad \mathbf{x}_0 \sim N(\mathbf{0}, \mathbf{C}_{\mathbf{x}_0})$$

$$\Rightarrow \mathbf{x} \sim N(\mathbf{0}, \mathbf{C}_{\mathbf{x}})$$

“estimation” of $\mathbf{x}$: $\hat{\mathbf{x}}$

Differences between the Kalman and the geodetic approach

Global model:

$$\mathbf{x} = \mathbf{F}^{-1}\mathbf{w} + \Phi_0\mathbf{x}_0,$$

$$\mathbf{b} = \mathbf{A}\mathbf{x} + \mathbf{e},$$

Engineering:

$$\mathbf{w} \sim \cancel{N}(\mathbf{0}, \mathbf{Q}), \quad \mathbf{x}_0 \sim \cancel{N}(\mathbf{0}, \mathbf{C}_{\mathbf{x}_0})$$

$$\Rightarrow \mathbf{x} \sim \cancel{N}(\mathbf{0}, \mathbf{C}_{\mathbf{x}})$$

“estimation” of $\mathbf{x}$: $\hat{\mathbf{x}}$

Geodesy:

$$\mathbf{w} \sim (\mathbf{0}, \mathbf{Q}), \quad \mathbf{x}_0 \sim (\mathbf{0}, \mathbf{C}_{\mathbf{x}_0})$$

$$\Rightarrow \mathbf{x} \sim (\mathbf{0}, \mathbf{C}_{\mathbf{x}})$$

Removal of normality assumption!

prediction of $\mathbf{x}$: $\tilde{\mathbf{x}}$

*The alternative
global geodetic solutions*

The alternative geodetic solutions

Depending on the treatment of the (before the observation epochs) initial state $\mathbf{x}_0$:

- (1) initial state $\mathbf{x}_0$: unknown random variable to be predicted
- (2) initial state $\mathbf{x}_0$: unknown deterministic variable to be estimated
- (3) No $\mathbf{x}_0$! First observation epoch $\mathbf{x}_1$: unknown deterministic variable to be estimated
- (4) initial state $\mathbf{x}_0$: random variable to be predicted with a **known outcome**
(Kalman compatible solution)

The alternative geodetic solutions

Depending on the treatment of the (before the observation epochs) initial state $\mathbf{x}_0$:

(1) initial state $\mathbf{x}_0$: unknown random variable to be predicted

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(Kalman compatible solution)

Prediction with a random effects model

The alternative geodetic solutions

Depending on the treatment of the (before the observation epochs) initial state $\mathbf{x}_0$:

(1) initial state $\mathbf{x}_0$: unknown random variable to be predicted

(2) initial state $\mathbf{x}_0$: unknown deterministic variable to be estimated

(3) No $\mathbf{x}_0$! First observation epoch $\mathbf{x}_1$: unknown deterministic variable to be estimated

(4) initial state $\mathbf{x}_0$: random variable to be predicted with a **known outcome**
(Kalman compatible solution)

Estimation and prediction with a mixed effects model

The alternative geodetic solutions

$$\mathbf{b} = \mathbf{A}\mathbf{x} + \mathbf{e}$$

$$\mathbf{e} \sim (\mathbf{0}, \mathbf{R})$$

$$\mathbf{x} = \mathbf{F}^{-1}\mathbf{w} + \mathbf{\Phi}_0\mathbf{x}_0$$

$$\mathbf{w} \sim (\mathbf{0}, \mathbf{Q})$$

$\mathbf{x}_0$ random: $\mathbf{x}_0 \sim (\mathbf{0}, \mathbf{C}_{\mathbf{x}_0})$

$$\mathbf{x} \sim (\mathbf{0}, \mathbf{C}_{\mathbf{x}})$$

$$\mathbf{C}_{\mathbf{x}} = \mathbf{F}^{-1}\mathbf{Q}\mathbf{F}^{-T} + \mathbf{\Phi}_0\mathbf{C}_{\mathbf{x}_0}\mathbf{\Phi}_0^T$$

$\mathbf{x}_0$: deterministic

$$\mathbf{x} \sim (\mathbf{\Phi}_0\mathbf{x}_0, \mathbf{C}_{\mathbf{x}})$$

$$\mathbf{C}_{\mathbf{x}} = \mathbf{F}^{-1}\mathbf{Q}\mathbf{F}^{-T}$$

$$\mathbf{b} = \mathbf{A}\mathbf{x} + \mathbf{e}$$

random effects model

$$\mathbf{b} = (\mathbf{A}\mathbf{\Phi}_0)\mathbf{x}_0 + (\mathbf{A}\mathbf{F}^{-1})\mathbf{w} + \mathbf{e}$$

mixed effects model

The alternative geodetic solutions

$$\mathbf{b} = \mathbf{A}\mathbf{x} + \mathbf{e}$$

$$\mathbf{x} = \mathbf{F}^{-1}\mathbf{w} + \mathbf{\Phi}_0\mathbf{x}_0$$

$$\mathbf{e} \sim (\mathbf{0}, \mathbf{R})$$

$$\mathbf{x}_0 \text{ random: } \mathbf{x}_0 \sim (\mathbf{0}, \mathbf{C}_{\mathbf{x}_0})$$

$$\mathbf{x} \sim (\mathbf{0}, \mathbf{C}_{\mathbf{x}})$$

$$\mathbf{C}_{\mathbf{x}} = \mathbf{F}^{-1}\mathbf{Q}\mathbf{F}^{-T} + \mathbf{\Phi}_0\mathbf{C}_{\mathbf{x}_0}\mathbf{\Phi}_0^T$$

$\mathbf{F}^{-1}$ not sparse implies $\mathbf{F}^{-1}\mathbf{Q}\mathbf{F}^{-T}$ not sparse!

We need algorithms implementing
its inverse sparse matrix

$$\mathbf{F}^T\mathbf{Q}^{-1}\mathbf{F}^{-T}$$

$$\mathbf{b} = \mathbf{A}\mathbf{x} + \mathbf{e}$$

random effects model

$$\mathbf{b} = (\mathbf{A}\mathbf{\Phi}_0)\mathbf{x}_0 + (\mathbf{A}\mathbf{F}^{-1})\mathbf{w} + \mathbf{e}$$

mixed effects model

The first straightforward geodetic solution

$$\mathbf{w} = \mathbf{F}\mathbf{x} - \mathbf{J}_n \Phi_{1,0} \mathbf{x}_0$$



$$\mathbf{x} = \mathbf{F}^{-1}\mathbf{w} + \mathbf{F}^{-1}\mathbf{J}_n \Phi_{1,0} \mathbf{x}_0$$

stochastic

stochastic

stochastic

Stochastic model:

$$\mathbf{x} \sim (\mathbf{0}, \mathbf{C}_x)$$

Observations: $\mathbf{b} = \mathbf{A}\mathbf{x} + \mathbf{e}$, $\mathbf{e} \sim N(\mathbf{0}, \mathbf{R})$

A random effects model

$$\mathbf{w} \sim (\mathbf{0}, \mathbf{Q})$$

$$\mathbf{x}_0 \sim (\mathbf{0}, \mathbf{C}_{x_0})$$

$\mathbf{w}$, $\mathbf{x}_0$, $\mathbf{e}$: all uncorrelated

Best (= minimum mean square error) linear unbiased prediction of $\mathbf{x}$:

$$\tilde{\mathbf{x}} = \mathbf{C}_{xb} \mathbf{C}_b^{-1} \mathbf{b} \quad \mathbf{C}_{\tilde{\mathbf{x}}-\mathbf{x}} = \mathbf{C}_x - \mathbf{C}_{xb} \mathbf{C}_b^{-1} \mathbf{C}_{bx}$$

Need to find: $\mathbf{C}_x$, $\mathbf{C}_b$, $\mathbf{C}_{xb}$

The first straightforward geodetic solution

Covariance propagation:

$$\mathbf{x} = \mathbf{F}^{-1}(\mathbf{w} + \mathbf{J}_n \Phi_{1,0} \mathbf{x}_0): \quad \mathbf{C}_{\mathbf{x}} = \mathbf{F}^{-1}(\mathbf{Q} + \mathbf{J}_n \Phi_{1,0} \mathbf{C}_{\mathbf{x}_0} \Phi_{1,0}^T \mathbf{J}_n^T) \mathbf{F}^{-T} = \mathbf{F}^{-1} \bar{\mathbf{Q}} \mathbf{F}^{-T}$$

$$\mathbf{C}_{\mathbf{x}_0 \mathbf{x}} = \mathbf{F}^{-1} \mathbf{J} \Phi_{1,0} \mathbf{C}_{\mathbf{x}_0}$$

$$\mathbf{b} = \mathbf{A} \mathbf{x} + \mathbf{e}: \quad \mathbf{C}_{\mathbf{b}} = \mathbf{A} \mathbf{C}_{\mathbf{x}} \mathbf{A}^T + \mathbf{R} \quad \mathbf{C}_{\mathbf{bx}} = \mathbf{A} \mathbf{C}_{\mathbf{x}} \quad \mathbf{C}_{\mathbf{bx}_0} = \mathbf{A} \mathbf{C}_{\mathbf{xx}_0}$$

Best linear unbiased prediction:

$$\tilde{\mathbf{x}} = \mathbf{C}_{\mathbf{xb}} \mathbf{C}_{\mathbf{b}}^{-1} \mathbf{b} = \mathbf{C}_{\mathbf{x}} \mathbf{A}^T (\mathbf{A} \mathbf{C}_{\mathbf{x}} \mathbf{A}^T + \mathbf{R})^{-1} \mathbf{b}$$

$$\mathbf{C}_{\tilde{\mathbf{x}}-\mathbf{x}} = \mathbf{C}_{\mathbf{x}} - \mathbf{C}_{\mathbf{x}} \mathbf{A}^T (\mathbf{A} \mathbf{C}_{\mathbf{x}} \mathbf{A}^T + \mathbf{R})^{-1} \mathbf{A} \mathbf{C}_{\mathbf{x}}$$

and after some algebraic manipulation ...

The first straightforward geodetic solution

$$\mathbf{x}_0 = (\mathbf{C}_x^{-1} + \mathbf{N})^{-1} \mathbf{u}$$

sparse!

$$\mathbf{N} = \mathbf{A}^T \mathbf{R}^{-1} \mathbf{A}$$

$$\mathbf{C}_{\mathbf{x}_0} = (\mathbf{C}_x^{-1} + \mathbf{N})^{-1}$$

$$\mathbf{u} = \mathbf{A}^T \mathbf{R}^{-1} \mathbf{b}$$

A reasonably-modeled perfectly sound solution **BUT ...**

Not compatible with the Kalman filtering algorithm
which needs a known value $\mathbf{x}_0$ as starting point

No $\mathbf{x}_0$ in above relations!

Here $\mathbf{x}_0$ is modelled as a stochastic unknown
that needs to be predicted $\mathbf{x}_0 \sim (\mathbf{0}, \mathbf{C}_{\mathbf{x}_0})$

Equivalent to $\mathbf{x}_0 \sim (\boldsymbol{\mu}_0, \mathbf{C}_{\mathbf{x}_0})$, $\boldsymbol{\mu}_0 = \text{known, through}$ $\delta \mathbf{x}_0 = \mathbf{x}_0 - \boldsymbol{\mu}_0 \sim (\mathbf{0}, \mathbf{C}_{\mathbf{x}_0}) \rightarrow \mathbf{x}_0$

The second (even more) straightforward geodetic solution

$$\mathbf{x} = \mathbf{F}^{-1}\mathbf{w} + \mathbf{\Phi}_0\mathbf{x}_0, \quad \mathbf{b} = \mathbf{A}\mathbf{x} + \mathbf{e},$$

$$\mathbf{w} \sim (\mathbf{0}, \mathbf{Q}), \quad \mathbf{x}_0 = \text{a deterministic unknown}$$

$$\mathbf{b} = \mathbf{A}\mathbf{x} + \mathbf{e}:$$

$$\mathbf{b} = \mathbf{A}(\mathbf{F}^{-1}\mathbf{w} + \mathbf{\Phi}_0\mathbf{x}_0) + \mathbf{e} = \mathbf{A}\mathbf{\Phi}_0\mathbf{x}_0 + \mathbf{A}\mathbf{F}^{-1}\mathbf{w} + \mathbf{e} \equiv \mathbf{A}_0\mathbf{x}_0 + (\mathbf{A}\mathbf{F}^{-1})\mathbf{w} + \mathbf{e}$$

$$\mathbf{A}_0 = \mathbf{A}\mathbf{\Phi}_0 = \mathbf{A} \begin{bmatrix} \mathbf{\Phi}_{1,0} \\ \mathbf{\Phi}_{2,0} \\ \mathbf{\Phi}_{3,0} \\ \vdots \\ \mathbf{\Phi}_{n-1,0} \\ \mathbf{\Phi}_{n,0} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1\mathbf{\Phi}_{1,0} \\ \mathbf{A}_2\mathbf{\Phi}_{2,0} \\ \mathbf{A}_3\mathbf{\Phi}_{3,0} \\ \vdots \\ \mathbf{A}_{n-1}\mathbf{\Phi}_{n-1,0} \\ \mathbf{A}_n\mathbf{\Phi}_{n,0} \end{bmatrix}$$

The second (even more) straightforward geodetic solution

$$\mathbf{b} = \Phi_0 \mathbf{x}_0 + (\mathbf{A}\mathbf{F}^{-1})\mathbf{w} + \mathbf{e}$$

$$\mathbf{w} \sim (\mathbf{0}, \mathbf{Q}) \quad \mathbf{e} \sim (\mathbf{0}, \mathbf{R})$$

a mixed effects model

Solution

$$\mathbf{C}_x = \mathbf{F}^{-1} \mathbf{Q} \mathbf{F}^{-T}$$

$$\mathbf{C}_b = \mathbf{A} \mathbf{C}_x \mathbf{A}^T + \mathbf{R}$$

$$\hat{\mathbf{x}}_0 = (\mathbf{A}_0^T \mathbf{C}_b^{-1} \mathbf{A}_0)^{-1} \mathbf{A}_0^T \mathbf{C}_b^{-1} \mathbf{b}$$

$$\tilde{\mathbf{x}} = \mathbf{C}_x \mathbf{A}^T \mathbf{C}_b^{-1} (\mathbf{b} - \mathbf{A}_0 \hat{\mathbf{x}}_0)$$

$$\mathbf{H} = \mathbf{I} - \mathbf{A}_0 (\mathbf{A}_0^T \mathbf{C}_b^{-1} \mathbf{A}_0)^{-1} \mathbf{A}_0^T \mathbf{C}_b^{-1}$$

$$\mathbf{C}_{\tilde{\mathbf{x}}-\mathbf{x}} = \mathbf{C}_x - \mathbf{C}_x \mathbf{A}^T \mathbf{C}_b^{-1} \mathbf{H} \mathbf{A} \mathbf{C}_x$$

or after some matrix manipulation ...

The second (even more) straightforward geodetic solution

... or after some matrix manipulation:

Solution

$$\mathbf{N} = \mathbf{A}^T \mathbf{R}^{-1} \mathbf{A}$$

$$\mathbf{u} = \mathbf{A}^T \mathbf{R}^{-1} \mathbf{b}$$

$$\hat{\mathbf{x}}_0 = \left[\mathbf{\Phi}_0^T (\mathbf{N}^{-1} + \mathbf{C}_x)^{-1} \mathbf{\Phi}_0 \right]^{-1} \mathbf{\Phi}_0^T (\mathbf{I} + \mathbf{N} \mathbf{C}_x)^{-1} \mathbf{u}$$

$$\mathbf{x}_0 = (\mathbf{C}_x^{-1} + \mathbf{N})^{-1} (\mathbf{u} - \mathbf{N} \mathbf{\Phi}_0 \hat{\mathbf{x}}_0)$$

sparse!

$$\mathbf{H} = \mathbf{I} - \mathbf{A} \mathbf{\Phi}_0 \left[\mathbf{\Phi}_0^T (\mathbf{N}^{-1} + \mathbf{C}_x)^{-1} \mathbf{\Phi}_0 \right]^{-1} \mathbf{\Phi}_0^T (\mathbf{I} + \mathbf{N} \mathbf{C}_x)^{-1} \mathbf{A}^T \mathbf{R}^{-1}$$

$$\mathbf{C}_{\mathbf{x}_0 - \mathbf{x}} = \mathbf{C}_x - (\mathbf{C}_x^{-1} + \mathbf{N})^{-1} \mathbf{A}^T \mathbf{R}^{-1} \mathbf{H} \mathbf{A} \mathbf{C}_x$$

The third straightforward geodetic solution (variation of second)

A variation: Consider state of first observation epoch as deterministic unknown

Renumber the epochs: $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$, become $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n$.

$$\mathbf{x} = \mathbf{F}^{-1}\mathbf{w} + \mathbf{\Phi}_0\mathbf{x}_0$$

$$\bar{\mathbf{b}} = \begin{bmatrix} \mathbf{b}_0 \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_0\mathbf{x}_0 + \mathbf{e}_0 \\ \mathbf{A}\mathbf{x} + \mathbf{e} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_0\mathbf{x}_0 + \mathbf{e}_0 \\ \mathbf{A}\mathbf{F}^{-1}\mathbf{w} + \mathbf{A}\mathbf{\Phi}_0\mathbf{x}_0 + \mathbf{e} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_0 \\ \mathbf{A}\mathbf{\Phi}_0 \end{bmatrix} \mathbf{x}_0 + \begin{bmatrix} \mathbf{0} \\ \mathbf{A}\mathbf{F}^{-1} \end{bmatrix} \mathbf{w} + \begin{bmatrix} \mathbf{e}_0 \\ \mathbf{e} \end{bmatrix}$$

$$\bar{\mathbf{b}} = \bar{\mathbf{A}}\mathbf{x}_0 + \mathbf{G}\mathbf{w} + \bar{\mathbf{e}} \quad \mathbf{w} \sim (\mathbf{0}, \mathbf{Q}) \quad \bar{\mathbf{e}} \sim (\mathbf{0}, \bar{\mathbf{R}}) \quad \bar{\mathbf{R}} = \begin{bmatrix} \mathbf{R}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}$$

a mixed effects model

The third straightforward geodetic solution (variation of second)

$$\mathbf{C}_{\mathbf{xw}} = \mathbf{C}_{\mathbf{x}} \mathbf{A}^T \quad \mathbf{C}_{\mathbf{x}} = \mathbf{F}^{-1} \mathbf{Q} \mathbf{F}^{-T} \quad \mathbf{C}_{\bar{\mathbf{b}}} = \begin{bmatrix} \mathbf{R}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{A} \mathbf{C}_{\mathbf{x}} \mathbf{A}^T + \mathbf{R} \end{bmatrix}$$

Solution

$$\hat{\mathbf{x}}_0 = (\bar{\mathbf{A}}^T \mathbf{C}_{\bar{\mathbf{b}}}^{-1} \bar{\mathbf{A}})^{-1} \bar{\mathbf{A}}^T \mathbf{C}_{\bar{\mathbf{b}}}^{-1} \bar{\mathbf{b}}$$

$$\mathbf{x}_0 = \mathbf{C}_{\mathbf{xw}} \mathbf{G}^T \mathbf{C}_{\bar{\mathbf{b}}}^{-1} (\bar{\mathbf{b}} - \bar{\mathbf{A}} \hat{\mathbf{x}}_0)$$

$$\mathbf{H} = \mathbf{I} - \bar{\mathbf{A}} (\bar{\mathbf{A}}^T \mathbf{C}_{\bar{\mathbf{b}}}^{-1} \bar{\mathbf{A}})^{-1} \bar{\mathbf{A}}^T \mathbf{C}_{\bar{\mathbf{b}}}^{-1}$$

$$\mathbf{C}_{\mathbf{x}_0 - \mathbf{x}} = \mathbf{C}_{\mathbf{x}} - \mathbf{C}_{\mathbf{xw}} \mathbf{G}^T \mathbf{C}_{\bar{\mathbf{b}}}^{-1} \mathbf{H} \mathbf{G} \mathbf{C}_{\mathbf{xw}}^T$$

or after some matrix manipulation ...

The third straightforward geodetic solution (variation of second)

... or after some matrix manipulation:

Solution

$$\mathbf{N}_0 = \mathbf{A}_0^T \mathbf{R}_0^{-1} \mathbf{A}_0 \quad \mathbf{u}_0 = \mathbf{A}_0^T \mathbf{R}_0^{-1} \mathbf{b}_0 \quad \mathbf{N} = \mathbf{A}^T \mathbf{R}^{-1} \mathbf{A} \quad \mathbf{u} = \mathbf{A}^T \mathbf{R}^{-1} \mathbf{b}$$

$$\hat{\mathbf{x}}_0 = \left[\mathbf{N}_0 + \Phi_0^T (\mathbf{N}^{-1} + \mathbf{C}_x)^{-1} \Phi_0 \right]^{-1} \left[\mathbf{u}_0 + \Phi_0^T \mathbf{C}_x^{-1} (\mathbf{C}_x^{-1} + \mathbf{N})^{-1} \mathbf{u} \right]$$

$$\mathbf{x}_0 = (\mathbf{C}_x^{-1} + \mathbf{N})^{-1} (\mathbf{u} - \mathbf{N} \Phi_0 \hat{\mathbf{x}}_0)$$

sparse!

$$\mathbf{H}_{22} = \mathbf{I} - \mathbf{A} \left[(\Phi_0 \mathbf{N}_0^{-1} \Phi_0^T)^{-1} + (\mathbf{N}^{-1} + \mathbf{C}_x)^{-1} \right] \mathbf{N} (\mathbf{N}^{-1} + \mathbf{C}_x)^{-1} \mathbf{A}^T \mathbf{R}^{-1}$$

$$\mathbf{C}_{\mathbf{x}_0-\mathbf{x}} = \mathbf{C}_x - (\mathbf{C}_x^{-1} + \mathbf{N})^{-1} \mathbf{A}^T \mathbf{R}^{-1} \mathbf{H}_{22} \mathbf{A} \mathbf{C}_x$$

*The Kalman compatible
global geodetic solution*

The fourth (not so) straightforward geodetic solution –
– Equivalent to Kalman filtering and smoothing

Addition of a trivial observation equation:

LHS:

random variable with a known
deterministic outcome

$$\mathbf{x}_0 = \mathbf{x}_0$$

RHS:

unknown random variable
to be predicted $\mathbf{x}_0 \sim (\mathbf{0}, \mathbf{C}_{\mathbf{x}_0})$

Total observations:

$$\boldsymbol{\beta} = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{A}\mathbf{x} + \mathbf{e} \end{bmatrix}$$

Prediction:

$$\hat{\boldsymbol{\eta}}_0 = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_0 \end{bmatrix} = \mathbf{C}_{\boldsymbol{\eta}\boldsymbol{\beta}} \mathbf{C}_{\boldsymbol{\beta}}^{-1} \boldsymbol{\beta} = \begin{bmatrix} \mathbf{C}_{\mathbf{x}_0} & \mathbf{C}_{\mathbf{x}_0\mathbf{b}} \\ \mathbf{C}_{\mathbf{x}\mathbf{x}_0} & \mathbf{C}_{\mathbf{x}\mathbf{b}} \end{bmatrix} \begin{bmatrix} \mathbf{C}_{\mathbf{x}_0} & \mathbf{C}_{\mathbf{x}_0\mathbf{b}} \\ \mathbf{C}_{\mathbf{x}_0\mathbf{b}}^T & \mathbf{C}_{\mathbf{b}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{b} \end{bmatrix}$$

and after some algebraic manipulation ...

The geodetic solution equivalent with Kalman filtering and smoothing

$$\hat{\mathbf{x}}_0 = \mathbf{x}_0 \quad !$$

$$\hat{\mathbf{x}}_0 = \mathbf{C}_{\mathbf{x}\mathbf{x}_0} \mathbf{C}_{\mathbf{x}_0}^{-1} \mathbf{x}_0 + \bar{\mathbf{C}}_{\mathbf{x}} \mathbf{A}^T (\mathbf{A} \bar{\mathbf{C}}_{\mathbf{x}} \mathbf{A}^T + \mathbf{R})^{-1} (\mathbf{b} - \mathbf{A} \mathbf{C}_{\mathbf{x}\mathbf{x}_0} \mathbf{C}_{\mathbf{x}_0}^{-1} \mathbf{x}_0)$$

$$\bar{\mathbf{C}}_{\mathbf{x}} = (\mathbf{F}^T \mathbf{Q}^{-1} \mathbf{F})^{-1}$$

versus

$$\mathbf{C}_{\mathbf{x}} = (\mathbf{F}^T \bar{\mathbf{Q}}^{-1} \mathbf{F})^{-1}$$

$$\bar{\mathbf{Q}} = \mathbf{Q} + \mathbf{J}_n \Phi_{1,0} \mathbf{C}_{\mathbf{x}_0} \Phi_{1,0}^T \mathbf{J}_n^T = \begin{bmatrix} \mathbf{Q}_1 + \Phi_{1,0} \mathbf{C}_{\mathbf{x}_0} \Phi_{1,0}^T & \mathbf{0} & \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_2 & \mathbf{L} & \mathbf{0} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ \mathbf{0} & \mathbf{0} & \mathbf{L} & \mathbf{Q}_n \end{bmatrix}$$

and after some algebraic manipulation ...

The geodetic solution equivalent with Kalman filtering and smoothing

... and after some algebraic manipulation

$$\mathbf{x}_0 = (\bar{\mathbf{C}}_x^{-1} + \mathbf{N})^{-1} (\mathbf{u} + \mathbf{J}\mathbf{Q}_1^{-1}\Phi_{1,0}\mathbf{x}_0) \equiv \mathbf{S}^{-1}\mathbf{d}$$

$$\mathbf{C}_\varepsilon = \mathbf{C}_{\mathbf{x}_0-\mathbf{x}} = \mathbf{S}^{-1} = (\bar{\mathbf{C}}_x^{-1} + \mathbf{N})^{-1}$$

Solution system

$$\mathbf{S}_{(n)} \mathbf{x}_{(n)|n} = \mathbf{d}_{(n)}$$

subscript (n)
denotes system
for epochs $t_1, t_2, \dots, t_n$

$$\mathbf{S}_{(n)} = \bar{\mathbf{C}}_x^{-1} + \mathbf{N} = \mathbf{F}^T \mathbf{Q}^{-1} \mathbf{F} + \mathbf{A}^T \mathbf{R}^{-1} \mathbf{A} = \mathbf{F}^T \mathbf{Q}^{-1} \mathbf{F} + \mathbf{N}$$

$$\mathbf{d}_{(n)} = \mathbf{A}^T \mathbf{R}^{-1} \mathbf{b} + \mathbf{J}_n \mathbf{Q}_1^{-1} \Phi_{1,0} \mathbf{x}_0 = \mathbf{u} + \mathbf{J}_n \mathbf{Q}_1^{-1} \Phi_{1,0} \mathbf{x}_0$$

sparse

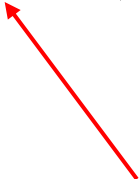
Solution system

$$\mathbf{S}_{(n)} \mathbf{x}_{(n)|n} = \mathbf{d}_{(n)}$$

$$\mathbf{S}_{(n)} = \bar{\mathbf{C}}_{\mathbf{x}}^{-1} + \mathbf{N} = (\mathbf{F}^T \mathbf{Q}^{-1} \mathbf{F}) + \mathbf{N} \quad \mathbf{d}_{(n)} = \mathbf{u} + \mathbf{J}_n \mathbf{Q}_1^{-1} \mathbf{\Phi}_{1,0} \mathbf{x}_0$$

Solution system

$$\mathbf{S}_{(n)} \mathbf{x}_{(n)|n} = \mathbf{d}_{(n)}$$

$$\mathbf{S}_{(n)} = \bar{\mathbf{C}}_{\mathbf{x}}^{-1} + \mathbf{N} = (\mathbf{F}^T \mathbf{Q}^{-1} \mathbf{F}) + \mathbf{N}$$


$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_2 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{N}_3 & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{N}_{n-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{N}_n \end{bmatrix}$$

Solution system

$$\mathbf{S}_{(n)} \mathbf{x}_{(n)|n} = \mathbf{d}_{(n)}$$

$$\mathbf{S}_{(n)} = \bar{\mathbf{C}}_{\mathbf{x}}^{-1} + \mathbf{N} = (\mathbf{F}^T \mathbf{Q}^{-1} \mathbf{F}) + \mathbf{N}$$

$$\mathbf{Q}^{-1} = \begin{bmatrix} \mathbf{Q}_1^{-1} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_2^{-1} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_3^{-1} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{Q}_{n-1}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{Q}_n^{-1} \end{bmatrix}$$

Solution system

$$\mathbf{S}_{(n)} \mathbf{x}_{(n)|n} = \mathbf{d}_{(n)}$$

$$\mathbf{S}_{(n)} = \bar{\mathbf{C}}_{\mathbf{x}}^{-1} + \mathbf{N} = (\mathbf{F}^T \mathbf{Q}^{-1} \mathbf{F}) + \mathbf{N}$$


$$\mathbf{F} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ -\Phi_{2,1} & \mathbf{I} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\Phi_{3,2} & \mathbf{I} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & -\Phi_{n,n-1} & \mathbf{I} \end{bmatrix}$$

Solution system

$$\mathbf{S}_{(n)} \mathbf{x}_{(n)|n} = \mathbf{d}_{(n)}$$

$$\mathbf{S}_{(n)} = \bar{\mathbf{C}}_{\mathbf{x}}^{-1} + \mathbf{N} = (\mathbf{F}^T \mathbf{Q}^{-1} \mathbf{F}) + \mathbf{N}$$



$$\bar{\mathbf{C}}_{\mathbf{x}}^{-1} = \begin{bmatrix} \mathbf{Q}_1^{-1} + \mathbf{\Phi}_{2,1}^T \mathbf{Q}_2^{-1} \mathbf{\Phi}_{2,1} & -\mathbf{\Phi}_{2,1}^T \mathbf{Q}_2^{-1} & \mathbf{L} & 0 & 0 \\ -\mathbf{Q}_2^{-1} \mathbf{\Phi}_{2,1} & \mathbf{Q}_2^{-1} + \mathbf{\Phi}_{3,2}^T \mathbf{Q}_3^{-1} \mathbf{\Phi}_{3,2} & \mathbf{L} & 0 & 0 \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{M} \\ 0 & 0 & \mathbf{L} & \mathbf{Q}_{n-1}^{-1} + \mathbf{\Phi}_{n,n-1}^T \mathbf{Q}_n^{-1} \mathbf{\Phi}_{n,n-1} & -\mathbf{\Phi}_{n,n-1}^T \mathbf{Q}_n^{-1} \\ 0 & 0 & \mathbf{L} & -\mathbf{Q}_n^{-1} \mathbf{\Phi}_{n,n-1} & \mathbf{Q}_n^{-1} \end{bmatrix}$$

Solution system

$$\mathbf{S}_{(n)} \mathbf{x}_{(n)|n} = \mathbf{d}_{(n)}$$

$$\mathbf{S}_{(n)} = \bar{\mathbf{C}}_{\mathbf{x}}^{-1} + \mathbf{N} = \begin{bmatrix} \mathbf{S}_1 & \mathbf{T}_1^T & \mathbf{0} & \mathbf{0} & \text{L} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{T}_1 & \mathbf{S}_2 & \mathbf{T}_2^T & \mathbf{0} & \text{L} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_2 & \mathbf{S}_3 & \mathbf{T}_3^T & \text{L} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \text{M} & \text{M} & \text{M} & \text{M} & \text{O} & \text{M} & \text{M} & \text{M} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \text{L} & \mathbf{S}_{n-2} & \mathbf{T}_{n-2}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \text{L} & \mathbf{T}_{n-2} & \mathbf{S}_{n-1} & \mathbf{T}_{n-1}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \text{L} & \mathbf{0} & \mathbf{T}_{n-1} & \mathbf{Q}_n^{-1} + \mathbf{N}_n \end{bmatrix}$$

$$\mathbf{S}_k = \mathbf{Q}_k^{-1} + \mathbf{N}_k + \mathbf{\Phi}_{k+1,k}^T \mathbf{Q}_{k+1}^{-1} \mathbf{\Phi}_{k+1,k}$$

$$\mathbf{T}_k = -\mathbf{Q}_{k+1}^{-1} \mathbf{\Phi}_{k+1,k}$$

$$k = 1, 2, \dots, n-1$$

Solution system

Geodetic solution system explicitly:

$$\mathbf{S}_1 \mathbf{x}_{1|n} + \mathbf{T}_1^T \mathbf{x}_{2|n} = \mathbf{u}_1 + \mathbf{Q}_1^{-1} \mathbf{\Phi}_{1,0} \mathbf{x}_0$$

$$\mathbf{T}_2 \mathbf{x}_{1|n} + \mathbf{S}_2 \mathbf{x}_{2|n} + \mathbf{T}_3^T \mathbf{x}_{3|n} = \mathbf{u}_2$$

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$$\mathbf{T}_{n-1} \mathbf{x}_{n-3|n} + \mathbf{S}_{n-2} \mathbf{x}_{n-2|n} + \mathbf{T}_{n-2}^T \mathbf{x}_{n-1|n} = \mathbf{u}_{n-2}$$

$$\mathbf{T}_{n-2} \mathbf{x}_{n-2|n} + \mathbf{S}_{n-1} \mathbf{x}_{n-1|n} - \mathbf{T}_{n-1}^T \mathbf{x}_{n|n} = \mathbf{u}_{n-1}$$

$$\mathbf{T}_{n-1} \mathbf{x}_{n-1|n} + (\mathbf{Q}_n^{-1} + \mathbf{N}_n) \mathbf{x}_{n|n} = \mathbf{u}_n$$

$$\mathbf{S}_k = \mathbf{Q}_k^{-1} + \mathbf{N}_k + \mathbf{\Phi}_{k+1,k}^T \mathbf{Q}_{k+1}^{-1} \mathbf{\Phi}_{k+1,k}$$

$$\mathbf{T}_k = -\mathbf{Q}_{k+1}^{-1} \mathbf{\Phi}_{k+1,k}$$

$$k = 1, 2, \dots, n-1$$

*From the global geodetic solution
to the Kalman filter*

From global geodetic solution to Kalman filter

$$\mathbf{S}_{(n)} \mathbf{x}_{(n)|n} = \mathbf{d}_{(n)}$$

$$\begin{bmatrix} \mathbf{S}_1 & \mathbf{T}_1^T & \mathbf{L} & 0 \\ \mathbf{T}_1 & \mathbf{S}_2 & \mathbf{L} & 0 \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ 0 & 0 & \mathbf{L} & \mathbf{Q}_{n-1}^{-1} + \mathbf{N}_{n-1} + \mathbf{\Phi}_{n,n-1}^T \mathbf{Q}_n^{-1} \mathbf{\Phi}_{n,n-1} \\ 0 & 0 & \mathbf{L} & \mathbf{T}_{n-1} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \mathbf{M} \\ \mathbf{T}_{n-1}^T \\ \mathbf{Q}_n^{-1} + \mathbf{N}_n \end{bmatrix} \begin{bmatrix} \mathbf{x}_{1|n} \\ \mathbf{x}_{2|n} \\ \mathbf{M} \\ \mathbf{x}_{n-1|n} \\ \mathbf{x}_{n|n} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{u}}_1 \\ \mathbf{u}_2 \\ \mathbf{M} \\ \mathbf{u}_{n-1} \\ \mathbf{u}_n \end{bmatrix}$$

$$\mathbf{N}_k = \mathbf{A}_k^T \mathbf{R}_k^{-1} \mathbf{A}_k$$

$$\mathbf{u}_k = \mathbf{A}_k^T \mathbf{R}_k^{-1} \mathbf{b}_k$$

$$\bar{\mathbf{u}}_1 = \mathbf{u}_1 + \mathbf{Q}_1^{-1} \mathbf{\Phi}_{1,0} \mathbf{x}_0$$

From global geodetic solution to Kalman filter

$$\mathbf{S}_{(n)} \mathbf{x}_{(n)|n} = \mathbf{d}_{(n)}$$

$$\begin{bmatrix} \mathbf{S}_1 & \mathbf{T}_1^T & \mathbf{L} & 0 \\ \mathbf{T}_1 & \mathbf{S}_2 & \mathbf{L} & 0 \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ 0 & 0 & \mathbf{L} & \mathbf{Q}_{n-1}^{-1} + \mathbf{N}_{n-1} + \mathbf{\Phi}_{n,n-1}^T \mathbf{Q}_n^{-1} \mathbf{\Phi}_{n,n-1} \\ 0 & 0 & \mathbf{L} & \mathbf{T}_{n-1} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \mathbf{M} \\ \mathbf{T}_{n-1}^T \\ \mathbf{Q}_n^{-1} + \mathbf{N}_n \end{bmatrix} \begin{bmatrix} \mathbf{x}_{1|n} \\ \mathbf{x}_{2|n} \\ \mathbf{M} \\ \mathbf{x}_{n-1|n} \\ \mathbf{x}_{n|n} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{u}}_1 \\ \mathbf{u}_2 \\ \mathbf{M} \\ \mathbf{u}_{n-1} \\ \mathbf{u}_n \end{bmatrix}$$

$$\mathbf{N}_k = \mathbf{A}_k^T \mathbf{R}_k^{-1} \mathbf{A}_k$$

$$\mathbf{u}_k = \mathbf{A}_k^T \mathbf{R}_k^{-1} \mathbf{b}_k$$

$$\bar{\mathbf{u}}_1 = \mathbf{u}_1 + \mathbf{Q}_1^{-1} \mathbf{\Phi}_{1,0} \mathbf{x}_0$$

$$\mathbf{S}_{(n)} \mathbf{x}_{(n)|n} = \begin{bmatrix} \mathbf{S}_{(n-1)} + \Delta \mathbf{S}_{n-1} & \mathbf{B}_{n-1} \\ \mathbf{B}_{n-1}^T & \mathbf{Q}_n^{-1} + \mathbf{N}_n \end{bmatrix} \begin{bmatrix} \mathbf{x}_{(n-1)|n} \\ \mathbf{x}_{n|n} \end{bmatrix} = \begin{bmatrix} \mathbf{d}_{(n-1)} \\ \mathbf{u}_n \end{bmatrix} = \mathbf{d}_{(n)}$$

From global geodetic solution to Kalman filter

$(n) \rightarrow (k)$

$$\mathbf{x}_{(k)|k} = \begin{bmatrix} \mathbf{x}_{(k-1)|k} \\ \mathbf{x}_{k|k} \end{bmatrix} = \mathbf{S}_{(k)}^{-1} \mathbf{d}_{(k)} = \begin{bmatrix} \mathbf{S}_{(k-1)} + \Delta \mathbf{S}_{k-1} & \mathbf{B}_{k-1} \\ \mathbf{B}_{k-1}^T & \mathbf{Q}_k^{-1} + \mathbf{N}_k \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{d}_{(k-1)} \\ \mathbf{u}_k \end{bmatrix}$$

subscript (k)
denotes system
for epochs $t_1, t_2, \dots, t_k$

$$\mathbf{S}_{(k-1)} \mathbf{x}_{(k-1)|k-1} = \mathbf{d}_{(k-1)}$$

$$\begin{bmatrix} \mathbf{x}_{(k-1)|k} \\ \mathbf{x}_{k|k} \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{(k-1)} + \Delta \mathbf{S}_{k-1} & \mathbf{B}_{k-1} \\ \mathbf{B}_{k-1}^T & \mathbf{Q}_k^{-1} + \mathbf{N}_k \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{S}_{(k-1)} \mathbf{x}_{(k-1)|k-1} \\ \mathbf{u}_k \end{bmatrix}$$

We solve the above equation for $\mathbf{x}_{k|k}$ to obtain a linear relation:

$$\mathbf{x}_{k|k} = \mathbf{L}(\mathbf{x}_{(k-1)|k-1}) \stackrel{!}{=} \mathbf{L}(\mathbf{x}_{k-1|k-1})$$

Kalman filter !

*From the global geodetic solution
to a new smoothing algorithm*

From the global geodetic solution to a new smoothing algorithm

Backward sequential smoothing:

$$\mathbf{S}_1 \mathbf{x}_{1|n} + \mathbf{T}_1^T \mathbf{x}_{2|n} = \mathbf{u}_1 + \mathbf{Q}_1^{-1} \mathbf{\Phi}_{1,0} \mathbf{x}_0$$

$$\mathbf{T}_2 \mathbf{x}_{1|n} + \mathbf{S}_2 \mathbf{x}_{2|n} + \mathbf{T}_3^T \mathbf{x}_{3|n} = \mathbf{u}_2$$

$$\mathbf{L} \quad \mathbf{L} \quad \mathbf{L} \quad \mathbf{L} \quad \mathbf{L}$$

$$\mathbf{T}_{n-1} \mathbf{x}_{n-3|n} + \mathbf{S}_{n-2} \mathbf{x}_{n-2|n} + \mathbf{T}_{n-2}^T \mathbf{x}_{n-1|n} = \mathbf{u}_{n-2}$$

$$\mathbf{T}_{n-2} \mathbf{x}_{n-2|n} + \mathbf{S}_{n-1} \mathbf{x}_{n-1|n} - \mathbf{T}_{n-1}^T \mathbf{x}_{n|n} = \mathbf{u}_{n-1}$$

$$\mathbf{T}_{n-1} \mathbf{x}_{n-1|n} + (\mathbf{Q}_n^{-1} + \mathbf{N}_n) \mathbf{x}_{n|n} = \mathbf{u}_n$$

Known from end of Kalman filter

Determined by solving equation

From the global geodetic solution to a new smoothing algorithm

Backward sequential smoothing:

$$\mathbf{S}_1 \mathbf{x}_{1|n} + \mathbf{T}_1^T \mathbf{x}_{2|n} = \mathbf{u}_1 + \mathbf{Q}_1^{-1} \mathbf{\Phi}_{1,0} \mathbf{x}_0$$

$$\mathbf{T}_2 \mathbf{x}_{1|n} + \mathbf{S}_2 \mathbf{x}_{2|n} + \mathbf{T}_3^T \mathbf{x}_{3|n} = \mathbf{u}_2$$

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$$\mathbf{T}_{n-1} \mathbf{x}_{n-3|n} + \mathbf{S}_{n-2} \mathbf{x}_{n-2|n} + \mathbf{T}_{n-2}^T \mathbf{x}_{n-1|n} = \mathbf{u}_{n-2}$$

$$\mathbf{T}_{n-2} \mathbf{x}_{n-2|n} + \mathbf{S}_{n-1} \mathbf{x}_{n-1|n} - \mathbf{T}_{n-1}^T \mathbf{x}_{n|n} = \mathbf{u}_{n-1}$$

Known from previous step

Determined by solving equation

$$\mathbf{T}_{n-1} \mathbf{x}_{n-1|n} + (\mathbf{Q}_n^{-1} + \mathbf{N}_n) \mathbf{x}_{n|n} = \mathbf{u}_n$$

From the global geodetic solution to a new smoothing algorithm

Backward sequential smoothing:

$$\mathbf{S}_1 \mathbf{x}_{1|n} + \mathbf{T}_1^T \mathbf{x}_{2|n} = \mathbf{u}_1 + \mathbf{Q}_1^{-1} \mathbf{\Phi}_{1,0} \mathbf{x}_0$$

$$\mathbf{T}_2 \mathbf{x}_{1|n} + \mathbf{S}_2 \mathbf{x}_{2|n} + \mathbf{T}_3^T \mathbf{x}_{3|n} = \mathbf{u}_2$$

L L L L L

$$\mathbf{T}_{n-1} \mathbf{x}_{n-3|n} + \mathbf{S}_{n-2} \mathbf{x}_{n-2|n} + \mathbf{T}_{n-2}^T \mathbf{x}_{n-1|n} = \mathbf{u}_{n-2}$$

Known from previous step

$$\mathbf{T}_{n-2} \mathbf{x}_{n-2|n} + \mathbf{S}_{n-1} \mathbf{x}_{n-1|n} - \mathbf{T}_{n-1}^T \mathbf{x}_{n|n} = \mathbf{u}_{n-1}$$

Determined by solving equation

$$\mathbf{T}_{n-1} \mathbf{x}_{n-1|n} + (\mathbf{Q}_n^{-1} + \mathbf{N}_n) \mathbf{x}_{n|n} = \mathbf{u}_n$$

From the global geodetic solution to a new smoothing algorithm

Backward sequential smoothing:

$$\mathbf{S}_1 \cancel{\mathbf{x}}_{1|n} + \mathbf{T}_1^T \cancel{\mathbf{x}}_{2|n} = \mathbf{u}_1 + \mathbf{Q}_1^{-1} \Phi_{1,0} \mathbf{x}_0$$

$$\mathbf{T}_2 \cancel{\mathbf{x}}_{1|n} + \mathbf{S}_2 \cancel{\mathbf{x}}_{2|n} + \mathbf{T}_3^T \cancel{\mathbf{x}}_{3|n} = \mathbf{u}_2$$

Known from previous step

Determined by solving equation

L L L L L

$$\mathbf{T}_{n-1} \cancel{\mathbf{x}}_{n-3|n} + \mathbf{S}_{n-2} \cancel{\mathbf{x}}_{n-2|n} + \mathbf{T}_{n-2}^T \cancel{\mathbf{x}}_{n-1|n} = \mathbf{u}_{n-2}$$

$$\mathbf{T}_{n-2} \cancel{\mathbf{x}}_{n-2|n} + \mathbf{S}_{n-1} \cancel{\mathbf{x}}_{n-1|n} - \mathbf{T}_{n-1}^T \cancel{\mathbf{x}}_{n|n} = \mathbf{u}_{n-1}$$

$$\mathbf{T}_{n-1} \cancel{\mathbf{x}}_{n-1|n} + (\mathbf{Q}_n^{-1} + \mathbf{N}_n) \cancel{\mathbf{x}}_{n|n} = \mathbf{u}_n$$

From the global geodetic solution to a new smoothing algorithm

Backward sequential smoothing:

$$\mathbf{S}_1 \cancel{\mathbf{x}_{0|n}} + \mathbf{T}_1^T \cancel{\mathbf{x}_{2|n}} = \mathbf{u}_1 + \mathbf{Q}_1^{-1} \Phi_{1,0} \mathbf{x}_0$$

Known from previous step

$$\mathbf{T}_2 \cancel{\mathbf{x}_{1|n}} + \mathbf{S}_2 \cancel{\mathbf{x}_{2|n}} + \mathbf{T}_3^T \cancel{\mathbf{x}_{3|n}} = \mathbf{u}_2$$

Compute $\mathbf{x}_0$ and compare to check computational error accumulation

$$\mathbf{L} \quad \mathbf{L} \quad \mathbf{L} \quad \mathbf{L} \quad \mathbf{L}$$

$$\mathbf{T}_{n-1} \cancel{\mathbf{x}_{n-3|n}} + \mathbf{S}_{n-2} \cancel{\mathbf{x}_{n-2|n}} + \mathbf{T}_{n-2}^T \cancel{\mathbf{x}_{n-1|n}} = \mathbf{u}_{n-2}$$

$$\mathbf{T}_{n-2} \cancel{\mathbf{x}_{n-2|n}} + \mathbf{S}_{n-1} \cancel{\mathbf{x}_{n-1|n}} - \mathbf{T}_{n-1}^T \cancel{\mathbf{x}_{n|n}} = \mathbf{u}_{n-1}$$

$$\mathbf{T}_{n-1} \cancel{\mathbf{x}_{n-1|n}} + (\mathbf{Q}_n^{-1} + \mathbf{N}_n) \cancel{\mathbf{x}_{n|n}} = \mathbf{u}_n$$

From the global geodetic solution to a new smoothing algorithm

Backward sequential smoothing for covariance matrices:

$$\mathbf{C}_{\boldsymbol{\varepsilon}} = \mathbf{C}_{\mathbf{x}/\mathbf{x}} = \mathbf{S}_{(n)}^{-1} = (\bar{\mathbf{C}}_{\mathbf{x}}^{-1} + \mathbf{N})^{-1}$$

$$\mathbf{S}_{(n)} \mathbf{C}_{\boldsymbol{\varepsilon}} = \mathbf{I}$$

$$\mathbf{S}_{(n)} \begin{bmatrix} \mathbf{C}_{\boldsymbol{\varepsilon},1} & \mathbf{C}_{\boldsymbol{\varepsilon},2} & \text{L} & \mathbf{C}_{\boldsymbol{\varepsilon},n-1} & \mathbf{C}_{\boldsymbol{\varepsilon},n} \end{bmatrix} = \begin{bmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \text{L} & \mathbf{E}_{n-1} & \mathbf{E}_n \end{bmatrix}$$

$$\mathbf{C}_{\boldsymbol{\varepsilon},k} = \begin{bmatrix} \mathbf{C}_{\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_k} \\ \mathbf{C}_{\boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_k} \\ \text{M} \\ \mathbf{C}_{\boldsymbol{\varepsilon}_{n-1}, \boldsymbol{\varepsilon}_k} \\ \mathbf{C}_{\boldsymbol{\varepsilon}_n, \boldsymbol{\varepsilon}_k} \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{\mathbf{x}_1/\mathbf{x}_1, \mathbf{x}_k/\mathbf{x}_k} \\ \mathbf{C}_{\mathbf{x}_2/\mathbf{x}_2, \mathbf{x}_k/\mathbf{x}_k} \\ \text{M} \\ \mathbf{C}_{\mathbf{x}_{n-1}/\mathbf{x}_{n-1}, \mathbf{x}_k/\mathbf{x}_k} \\ \mathbf{C}_{\mathbf{x}_n/\mathbf{x}_n, \mathbf{x}_k/\mathbf{x}_k} \end{bmatrix} \quad \mathbf{E}_k = \begin{bmatrix} \mathbf{0} \\ \text{M} \\ \mathbf{I} \\ \text{M} \\ \mathbf{0} \end{bmatrix} \quad \leftarrow \text{in } k^{\text{th}} \text{ sub-block}$$

From the global geodetic solution to a new smoothing algorithm

Backward sequential smoothing for covariance matrices:

$$\mathbf{C}_{\boldsymbol{\varepsilon}} = \mathbf{C}_{\mathbf{x}|\mathbf{y}} = \mathbf{S}_{(n)}^{-1} = (\bar{\mathbf{C}}_{\mathbf{x}}^{-1} + \mathbf{N})^{-1}$$

$$\mathbf{S}_{(n)} \mathbf{C}_{\boldsymbol{\varepsilon}} = \mathbf{I}$$

$$\mathbf{S}_{(n)} \begin{bmatrix} \mathbf{C}_{\boldsymbol{\varepsilon},1} & \mathbf{C}_{\boldsymbol{\varepsilon},2} & \text{L} & \mathbf{C}_{\boldsymbol{\varepsilon},n-1} & \mathbf{C}_{\boldsymbol{\varepsilon},n} \end{bmatrix} = \begin{bmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \text{L} & \mathbf{E}_{n-1} & \mathbf{E}_n \end{bmatrix}$$

$$\mathbf{S}_{(n)} \mathbf{C}_{\boldsymbol{\varepsilon},k} = \mathbf{E}_k, \quad k = 1, 2, \dots, n \quad \text{Solve for } \mathbf{C}_{\boldsymbol{\varepsilon},k}$$

We can use the same algorithm as in the case of state prediction ($\mathbf{S}_{(n)} \mathbf{x}_{(n)|n} = \mathbf{d}_{(n)}$)!

From the global geodetic solution to a new smoothing algorithm

Backward sequential smoothing for covariance matrices:

$$\mathbf{C}_{\boldsymbol{\varepsilon}} = \mathbf{C}_{\mathbf{x}|\boldsymbol{\varepsilon}} = \mathbf{S}_{(n)}^{-1} = (\bar{\mathbf{C}}_{\mathbf{x}}^{-1} + \mathbf{N})^{-1}$$

$$\mathbf{S}_{(n)} \mathbf{C}_{\boldsymbol{\varepsilon}} = \mathbf{I}$$

$$\mathbf{S}_{(n)} \begin{bmatrix} \mathbf{C}_{\boldsymbol{\varepsilon},1} & \mathbf{C}_{\boldsymbol{\varepsilon},2} & \text{L} & \mathbf{C}_{\boldsymbol{\varepsilon},n-1} & \mathbf{C}_{\boldsymbol{\varepsilon},n} \end{bmatrix} = \begin{bmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \text{L} & \mathbf{E}_{n-1} & \mathbf{E}_n \end{bmatrix}$$

$$\mathbf{S}_{(n)} \mathbf{C}_{\boldsymbol{\varepsilon},k} = \mathbf{E}_k, \quad k = 1, 2, \dots, n \quad \text{Solve for } \mathbf{C}_{\boldsymbol{\varepsilon},k}$$

Solve sequentially backwards

$$\mathbf{S}_{(n)} \mathbf{C}_{\boldsymbol{\varepsilon},n} = \mathbf{E}_n, \quad \mathbf{S}_{(n)} \mathbf{C}_{\boldsymbol{\varepsilon},n-1} = \mathbf{E}_{n-1}, \quad \text{L L} \quad \mathbf{S}_{(n)} \mathbf{C}_{\boldsymbol{\varepsilon},2} = \mathbf{E}_2, \quad \mathbf{S}_{(n)} \mathbf{C}_{\boldsymbol{\varepsilon},1} = \mathbf{E}_1$$

From the global geodetic solution to a new smoothing algorithm

Backward sequential smoothing for covariance matrices:

$$\mathbf{S}_{(n)} \mathbf{C}_{\varepsilon, k} = \mathbf{E}_k$$

Use same backward smoothing algorithm as for unknowns:

$$\mathbf{S}_{(n)} \mathbf{x}_{(n)} = \mathbf{d}_{(n)}$$

$$\mathbf{C}_{\varepsilon} = \mathbf{C}_{\mathbf{x} \otimes \mathbf{x}} = \begin{bmatrix} \mathbf{C}_{\varepsilon_1, \varepsilon_1} & \mathbf{C}_{\varepsilon_1, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_1, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_1, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_1, \varepsilon_n} \\ \mathbf{C}_{\varepsilon_2, \varepsilon_1} & \mathbf{C}_{\varepsilon_2, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_2, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_2, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_2, \varepsilon_n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{M} \\ \mathbf{C}_{\varepsilon_k, \varepsilon_1} & \mathbf{C}_{\varepsilon_k, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_k, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_k, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_k, \varepsilon_n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{M} \\ \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_1} & \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_n} \\ \mathbf{C}_{\varepsilon_n, \varepsilon_k} & \mathbf{C}_{\varepsilon_n, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_n, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_n, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_n, \varepsilon_n} \end{bmatrix}$$

Known by symmetry

Computed by backward algorithm

Known from end of Kalman filter

From the global geodetic solution to a new smoothing algorithm

Backward sequential smoothing for covariance matrices:

$$\mathbf{S}_{(n)} \mathbf{C}_{\varepsilon, k} = \mathbf{E}_k$$

Use same backward smoothing algorithm as for unknowns:

$$\mathbf{S}_{(n)} \mathbf{x}_{(n)} = \mathbf{d}_{(n)}$$

$$\mathbf{C}_{\varepsilon} = \mathbf{C}_{\mathbf{x} \otimes \mathbf{x}} = \begin{bmatrix} \mathbf{C}_{\varepsilon_1, \varepsilon_1} & \mathbf{C}_{\varepsilon_1, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_1, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_1, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_1, \varepsilon_n} \\ \mathbf{C}_{\varepsilon_2, \varepsilon_1} & \mathbf{C}_{\varepsilon_2, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_2, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_2, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_2, \varepsilon_n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{M} \\ \mathbf{C}_{\varepsilon_k, \varepsilon_1} & \mathbf{C}_{\varepsilon_k, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_k, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_k, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_k, \varepsilon_n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{M} \\ \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_1} & \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_n} \\ \mathbf{C}_{\varepsilon_n, \varepsilon_k} & \mathbf{C}_{\varepsilon_n, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_n, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_n, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_n, \varepsilon_n} \end{bmatrix}$$

Known by symmetry

Computed by backward algorithm

Known from next column

From the global geodetic solution to a new smoothing algorithm

Backward sequential smoothing for covariance matrices:

$$\mathbf{S}_{(n)} \mathbf{C}_{\boldsymbol{\varepsilon},k} = \mathbf{E}_k$$

Use same backward smoothing algorithm as for unknowns:

$$\mathbf{S}_{(n)} \mathbf{x}_{(n)} = \mathbf{d}_{(n)}$$

$$\mathbf{C}_{\boldsymbol{\varepsilon}} = \mathbf{C}_{\mathbf{x} \otimes \mathbf{x}} = \begin{bmatrix} \mathbf{C}_{\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_1} & \mathbf{C}_{\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2} & \mathbf{L} & \mathbf{C}_{\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_k} & \mathbf{L} & \mathbf{C}_{\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_{n-1}} & \mathbf{C}_{\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_n} \\ \mathbf{C}_{\boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_1} & \mathbf{C}_{\boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_2} & \mathbf{L} & \mathbf{C}_{\boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_k} & \mathbf{L} & \mathbf{C}_{\boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_{n-1}} & \mathbf{C}_{\boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{M} \\ \mathbf{C}_{\boldsymbol{\varepsilon}_k, \boldsymbol{\varepsilon}_1} & \mathbf{C}_{\boldsymbol{\varepsilon}_k, \boldsymbol{\varepsilon}_2} & \mathbf{L} & \mathbf{C}_{\boldsymbol{\varepsilon}_k, \boldsymbol{\varepsilon}_k} & \mathbf{L} & \mathbf{C}_{\boldsymbol{\varepsilon}_k, \boldsymbol{\varepsilon}_{n-1}} & \mathbf{C}_{\boldsymbol{\varepsilon}_k, \boldsymbol{\varepsilon}_n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{M} \\ \mathbf{C}_{\boldsymbol{\varepsilon}_{n-1}, \boldsymbol{\varepsilon}_1} & \mathbf{C}_{\boldsymbol{\varepsilon}_{n-1}, \boldsymbol{\varepsilon}_2} & \mathbf{L} & \mathbf{C}_{\boldsymbol{\varepsilon}_{n-1}, \boldsymbol{\varepsilon}_k} & \mathbf{L} & \mathbf{C}_{\boldsymbol{\varepsilon}_{n-1}, \boldsymbol{\varepsilon}_{n-1}} & \mathbf{C}_{\boldsymbol{\varepsilon}_{n-1}, \boldsymbol{\varepsilon}_n} \\ \mathbf{C}_{\boldsymbol{\varepsilon}_n, \boldsymbol{\varepsilon}_1} & \mathbf{C}_{\boldsymbol{\varepsilon}_n, \boldsymbol{\varepsilon}_2} & \mathbf{L} & \mathbf{C}_{\boldsymbol{\varepsilon}_n, \boldsymbol{\varepsilon}_k} & \mathbf{L} & \mathbf{C}_{\boldsymbol{\varepsilon}_n, \boldsymbol{\varepsilon}_{n-1}} & \mathbf{C}_{\boldsymbol{\varepsilon}_n, \boldsymbol{\varepsilon}_n} \end{bmatrix}$$

Computation order

From the global geodetic solution to a new smoothing algorithm

Backward sequential smoothing for covariance matrices:

$$\mathbf{S}_{(n)} \mathbf{C}_{\varepsilon, k} = \mathbf{E}_k$$

Use same backward smoothing algorithm as for unknowns:

$$\mathbf{S}_{(n)} \mathbf{x}_{(n)} = \mathbf{d}_{(n)}$$

$$\mathbf{C}_{\varepsilon} = \mathbf{C}_{\mathbf{x} \otimes \mathbf{x}} = \begin{bmatrix} \mathbf{C}_{\varepsilon_1, \varepsilon_1} & \mathbf{C}_{\varepsilon_1, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_1, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_1, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_1, \varepsilon_n} \\ \mathbf{C}_{\varepsilon_2, \varepsilon_1} & \mathbf{C}_{\varepsilon_2, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_2, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_2, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_2, \varepsilon_n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{M} \\ \mathbf{C}_{\varepsilon_k, \varepsilon_1} & \mathbf{C}_{\varepsilon_k, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_k, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_k, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_k, \varepsilon_n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{M} \\ \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_1} & \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_n} \\ \mathbf{C}_{\varepsilon_n, \varepsilon_k} & \mathbf{C}_{\varepsilon_n, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_n, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_n, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_n, \varepsilon_n} \end{bmatrix}$$

Computation order

From the global geodetic solution to a new smoothing algorithm

Backward sequential smoothing for covariance matrices:

$$\mathbf{S}_{(n)} \mathbf{C}_{\varepsilon, k} = \mathbf{E}_k$$

Use same backward smoothing algorithm as for unknowns:

$$\mathbf{S}_{(n)} \mathbf{x}_{(n)} = \mathbf{d}_{(n)}$$

$$\mathbf{C}_{\varepsilon} = \mathbf{C}_{\mathbf{x} \otimes \mathbf{x}} = \begin{bmatrix} \mathbf{C}_{\varepsilon_1, \varepsilon_1} & \mathbf{C}_{\varepsilon_1, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_1, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_1, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_1, \varepsilon_n} \\ \mathbf{C}_{\varepsilon_2, \varepsilon_1} & \mathbf{C}_{\varepsilon_2, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_2, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_2, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_2, \varepsilon_n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{M} \\ \mathbf{C}_{\varepsilon_k, \varepsilon_1} & \mathbf{C}_{\varepsilon_k, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_k, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_k, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_k, \varepsilon_n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{M} \\ \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_1} & \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_n} \\ \mathbf{C}_{\varepsilon_n, \varepsilon_k} & \mathbf{C}_{\varepsilon_n, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_n, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_n, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_n, \varepsilon_n} \end{bmatrix}$$

Computation order

From the global geodetic solution to a new smoothing algorithm

Backward sequential smoothing for covariance matrices:

$$\mathbf{S}_{(n)} \mathbf{C}_{\varepsilon, k} = \mathbf{E}_k$$

Use same backward smoothing algorithm as for unknowns:

$$\mathbf{S}_{(n)} \mathbf{x}_{(n)} = \mathbf{d}_{(n)}$$

$$\mathbf{C}_{\varepsilon} = \mathbf{C}_{\mathbf{x} \otimes \mathbf{x}} = \begin{bmatrix} \mathbf{C}_{\varepsilon_1, \varepsilon_1} & \mathbf{C}_{\varepsilon_1, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_1, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_1, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_1, \varepsilon_n} \\ \mathbf{C}_{\varepsilon_2, \varepsilon_1} & \mathbf{C}_{\varepsilon_2, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_2, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_2, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_2, \varepsilon_n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{M} \\ \mathbf{C}_{\varepsilon_k, \varepsilon_1} & \mathbf{C}_{\varepsilon_k, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_k, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_k, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_k, \varepsilon_n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{M} \\ \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_1} & \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_n} \\ \mathbf{C}_{\varepsilon_n, \varepsilon_k} & \mathbf{C}_{\varepsilon_n, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_n, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_n, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_n, \varepsilon_n} \end{bmatrix}$$

Computation order

From the global geodetic solution to a new smoothing algorithm

Backward sequential smoothing for covariance matrices:

$$\mathbf{S}_{(n)} \mathbf{C}_{\varepsilon, k} = \mathbf{E}_k$$

Use same backward smoothing algorithm as for unknowns:

$$\mathbf{S}_{(n)} \mathbf{x}_{(n)} = \mathbf{d}_{(n)}$$

$$\mathbf{C}_{\varepsilon} = \mathbf{C}_{\mathbf{x} \otimes \mathbf{x}} = \begin{bmatrix} \mathbf{C}_{\varepsilon_1, \varepsilon_1} & \mathbf{C}_{\varepsilon_1, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_1, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_1, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_1, \varepsilon_n} \\ \mathbf{C}_{\varepsilon_2, \varepsilon_1} & \mathbf{C}_{\varepsilon_2, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_2, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_2, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_2, \varepsilon_n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{M} \\ \mathbf{C}_{\varepsilon_k, \varepsilon_1} & \mathbf{C}_{\varepsilon_k, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_k, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_k, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_k, \varepsilon_n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{M} \\ \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_1} & \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_{n-1}, \varepsilon_n} \\ \mathbf{C}_{\varepsilon_n, \varepsilon_k} & \mathbf{C}_{\varepsilon_n, \varepsilon_2} & \mathbf{L} & \mathbf{C}_{\varepsilon_n, \varepsilon_k} & \mathbf{L} & \mathbf{C}_{\varepsilon_n, \varepsilon_{n-1}} & \mathbf{C}_{\varepsilon_n, \varepsilon_n} \end{bmatrix}$$

Computation order

*From the global geodetic solution
to the Rauch-Sting-Stiebel
smoothing algorithm*

Backward smoothing as an update of forward Kalman filter

$$\mathbf{S}_{(n)} \mathbf{x}_{(n)|n} = \mathbf{d}_{(n)}$$

$$\mathbf{S}_{(n)} \mathbf{x}_{(n)|n} = \begin{bmatrix} \mathbf{S}_1 & \mathbf{T}_1^T & \mathbf{L} & \mathbf{0} & \mathbf{0} & \mathbf{L} & \mathbf{0} \\ \mathbf{T}_1 & \mathbf{S}_2 & \mathbf{L} & \mathbf{0} & \mathbf{0} & \mathbf{L} & \mathbf{0} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ \mathbf{0} & \mathbf{0} & \mathbf{L} & \mathbf{Q}_k^{-1} + \mathbf{N}_k + \mathbf{\Phi}_{k+1,k}^T \mathbf{Q}_k^{-1} \mathbf{\Phi}_{k+1,k} & \mathbf{T}_k^T & \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{L} & \mathbf{T}_k & \mathbf{S}_{k+1} & \mathbf{L} & \mathbf{0} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ \mathbf{0} & \mathbf{0} & \mathbf{L} & \mathbf{0} & \mathbf{0} & \mathbf{L} & \mathbf{Q}_n^{-1} + \mathbf{N}_n \end{bmatrix} \begin{bmatrix} \mathbf{x}_{1|n} \\ \mathbf{x}_{2|n} \\ \mathbf{M} \\ \mathbf{x}_{k|n} \\ \mathbf{x}_{k+1|n} \\ \mathbf{M} \\ \mathbf{x}_{n|n} \end{bmatrix} = \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{x}_2 \\ \mathbf{M} \\ \mathbf{d}_k \\ \mathbf{d}_{k+1} \\ \mathbf{M} \\ \mathbf{d}_n \end{bmatrix} = \mathbf{d}_{(n)}$$

Backward smoothing as an update of forward Kalman filter

$$\mathbf{S}_{(n)} \mathbf{x}_{(n)|n} = \mathbf{d}_{(n)}$$

$$\mathbf{S}_{(n)} \mathbf{x}_{(n)|n} = \begin{bmatrix} \mathbf{S}_1 & \mathbf{T}_1^T & \mathbf{L} & \mathbf{0} \\ \mathbf{T}_1 & \mathbf{S}_2 & \mathbf{L} & \mathbf{0} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ \mathbf{0} & \mathbf{0} & \mathbf{L} & \mathbf{Q}_k^{-1} + \mathbf{N}_k + \mathbf{\Phi}_{k+1,k}^T \mathbf{Q}_k^{-1} \mathbf{\Phi}_{k+1,k} \\ \mathbf{0} & \mathbf{0} & \mathbf{L} & \mathbf{T}_k \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ \mathbf{0} & \mathbf{0} & \mathbf{L} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{1|n} \\ \mathbf{x}_{2|n} \\ \mathbf{M} \\ \mathbf{x}_{k|n} \\ \mathbf{x}_{k+1|n} \\ \mathbf{M} \\ \mathbf{x}_{n|n} \end{bmatrix} = \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{x}_2 \\ \mathbf{M} \\ \mathbf{d}_k \\ \mathbf{d}_{k+1} \\ \mathbf{M} \\ \mathbf{d}_n \end{bmatrix} = \mathbf{d}_{(n)}$$

$$\begin{bmatrix} \mathbf{S}_{(k)} + \Delta \mathbf{S}_k & \mathbf{B}_k \\ \mathbf{B}_k^T & \mathbf{S}_{(k+1_n)} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{(k)|n} \\ \mathbf{x}_{(k+1_n)|n} \end{bmatrix} = \begin{bmatrix} \mathbf{d}_{(k)} \\ \mathbf{d}_{(k+1_n)} \end{bmatrix}$$

Backward smoothing as an update of forward Kalman filter

$$\mathbf{S}_{(n)} \mathbf{x}_{(n)|n} = \mathbf{d}_{(n)}$$

$$\mathbf{S}_{(k)} \mathbf{x}_{(k)|k} = \mathbf{d}_{(k)}$$

$$\begin{bmatrix} \mathbf{S}_{(k)} + \Delta \mathbf{S}_k & \mathbf{B}_k \\ \mathbf{B}_k^T & \mathbf{S}_{(k+1_n)} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{(k)|n} \\ \mathbf{x}_{(k+1_n)|n} \end{bmatrix} = \begin{bmatrix} \mathbf{d}_{(k)} \\ \mathbf{d}_{(k+1_n)} \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{(k)} \mathbf{x}_{(k)|k} \\ \mathbf{d}_{(k+1_n)} \end{bmatrix}$$

First k equations become:

$$(\mathbf{S}_{(k)} + \Delta \mathbf{S}_k) \mathbf{x}_{(k)|n} + \mathbf{B}_k \mathbf{x}_{(k+1_n)|n} = \mathbf{S}_{(k)} \mathbf{x}_{(k)|k}$$

or using $\mathbf{C}_{\mathbf{e}_{(k)}|k} = \mathbf{S}_{(k)}^{-1}$:

$$\mathbf{x}_{(k)|n} - \mathbf{x}_{(k)|k} = -\mathbf{S}_{(k)}^{-1} (\Delta \mathbf{S}_k \mathbf{x}_{(k)|n} + \mathbf{B}_k \mathbf{x}_{(k+1_n)|n}) = -\mathbf{C}_{\mathbf{e}_{(k)}|k} (\Delta \mathbf{S}_k \mathbf{x}_{(k)|n} + \mathbf{B}_k \mathbf{x}_{(k+1_n)|n})$$

Backward smoothing as an update of forward Kalman filter

$$\mathbf{x}_{(k)|n} - \mathbf{x}_{(k)|k} = -\mathbf{C}_{\mathbf{e}_{(k)}|k} (\Delta \mathbf{S}_k \mathbf{x}_{(k)|n} + \mathbf{B}_k \mathbf{x}_{(k+1|n)|n})$$

The last element of $\mathbf{x}_{(k)|n} - \mathbf{x}_{(k)|k}$ becomes:

$$\mathbf{x}_{k|n} - \mathbf{x}_{k|k} = \mathbf{C}_{\mathbf{e}_k|k} \mathbf{\Phi}_{k+1,k}^T (\mathbf{Q}_{k+1} + \mathbf{\Phi}_{k+1,k} \mathbf{C}_{\mathbf{e}_k|k} \mathbf{\Phi}_{k+1,k}^T)^{-1} (\mathbf{x}_{k+1|n} - \mathbf{\Phi}_{k+1,k} \mathbf{x}_{k|k})$$

Backward smoothing as an update of forward Kalman filter

Backward smoothing for covariances:

$$\mathbf{C}_{\boldsymbol{\varepsilon}_{(n)}|n} = \begin{bmatrix} \mathbf{C}_{\boldsymbol{\varepsilon}_{(k)}|n} & \mathbf{C}_{\boldsymbol{\varepsilon}_{(k)}\boldsymbol{\varepsilon}_{(k+1_n)}|n} \\ \mathbf{C}_{\boldsymbol{\varepsilon}_{(k+1_n)}\boldsymbol{\varepsilon}_{(k)}|n} & \mathbf{C}_{\boldsymbol{\varepsilon}_{(k+1_n)}|n} \end{bmatrix} = \mathbf{S}_{(n)}^{-1} = \begin{bmatrix} \mathbf{S}_{(k)} & \mathbf{B}_k \\ \mathbf{B}_k^T & \mathbf{S}_{(k+1_n)} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{C}_{\boldsymbol{\varepsilon}_{(k)}|k}^{-1} & \mathbf{B}_k \\ \mathbf{B}_k^T & \mathbf{S}_{(k+1_n)} \end{bmatrix}^{-1}$$

We can solve for $\mathbf{C}_{\boldsymbol{\varepsilon}_{(k)}|n}$ or just for its last element $\mathbf{C}_{\boldsymbol{\varepsilon}_k|n}$ to obtain:

$$\begin{aligned} \mathbf{C}_{\boldsymbol{\varepsilon}_k|n} = & \mathbf{C}_{\boldsymbol{\varepsilon}_k|k} + \mathbf{C}_{\boldsymbol{\varepsilon}_k|k} \boldsymbol{\Phi}_{k+1,k}^T (\mathbf{Q}_{k+1} + \boldsymbol{\Phi}_{k+1,k} \mathbf{C}_{\boldsymbol{\varepsilon}_k|k} \boldsymbol{\Phi}_{k+1,k}^T)^{-1} \times \\ & \times (\mathbf{C}_{\boldsymbol{\varepsilon}_{k+1}|n} - \mathbf{Q}_{k+1} - \boldsymbol{\Phi}_{k+1,k} \mathbf{C}_{\boldsymbol{\varepsilon}_k|k} \boldsymbol{\Phi}_{k+1,k}^T) (\mathbf{Q}_{k+1} + \boldsymbol{\Phi}_{k+1,k} \mathbf{C}_{\boldsymbol{\varepsilon}_k|k} \boldsymbol{\Phi}_{k+1,k}^T)^{-1} \boldsymbol{\Phi}_{k+1,k} \mathbf{C}_{\boldsymbol{\varepsilon}_k|k} \end{aligned}$$

Backward smoothing as an update of forward Kalman filter

$$\mathbf{x}_{k|n} - \mathbf{x}_{k|k} = \mathbf{C}_{\varepsilon_k|k} \mathbf{\Phi}_{k+1,k}^T (\mathbf{Q}_{k+1} + \mathbf{\Phi}_{k+1,k} \mathbf{C}_{\varepsilon_k|k} \mathbf{\Phi}_{k+1,k}^T)^{-1} (\mathbf{x}_{k+1|n} - \mathbf{\Phi}_{k+1,k} \mathbf{x}_{k|k})$$

$$\begin{aligned} \mathbf{C}_{\varepsilon_k|n} = & \mathbf{C}_{\varepsilon_k|k} + \mathbf{C}_{\varepsilon_k|k} \mathbf{\Phi}_{k+1,k}^T (\mathbf{Q}_{k+1} + \mathbf{\Phi}_{k+1,k} \mathbf{C}_{\varepsilon_k|k} \mathbf{\Phi}_{k+1,k}^T)^{-1} \times \\ & \times (\mathbf{C}_{\varepsilon_{k+1}|n} - \mathbf{Q}_{k+1} - \mathbf{\Phi}_{k+1,k} \mathbf{C}_{\varepsilon_k|k} \mathbf{\Phi}_{k+1,k}^T) (\mathbf{Q}_{k+1} + \mathbf{\Phi}_{k+1,k} \mathbf{C}_{\varepsilon_k|k} \mathbf{\Phi}_{k+1,k}^T)^{-1} \mathbf{\Phi}_{k+1,k} \mathbf{C}_{\varepsilon_k|k} \end{aligned}$$

This is a well-known smoothing algorithm!
To identify it we must translate it
to the Kalman “language”!

Backward smoothing as an update of forward Kalman filter

$$\cancel{\mathbf{x}}_{k|n} - \cancel{\mathbf{x}}_{k|k} = \mathbf{C}_{\mathbf{\varepsilon}_k|k} \mathbf{\Phi}_{k+1,k}^T (\mathbf{Q}_{k+1} + \mathbf{\Phi}_{k+1,k} \mathbf{C}_{\mathbf{\varepsilon}_k|k} \mathbf{\Phi}_{k+1,k}^T)^{-1} (\cancel{\mathbf{x}}_{k+1|n} - \mathbf{\Phi}_{k+1,k} \cancel{\mathbf{x}}_{k|k})$$

$$\begin{aligned} \mathbf{C}_{\mathbf{\varepsilon}_k|n} &= \mathbf{C}_{\mathbf{\varepsilon}_k|k} + \mathbf{C}_{\mathbf{\varepsilon}_k|k} \mathbf{\Phi}_{k+1,k}^T (\mathbf{Q}_{k+1} + \mathbf{\Phi}_{k+1,k} \mathbf{C}_{\mathbf{\varepsilon}_k|k} \mathbf{\Phi}_{k+1,k}^T)^{-1} \times \\ &\times (\mathbf{C}_{\mathbf{\varepsilon}_{k+1}|n} - \mathbf{Q}_{k+1} - \mathbf{\Phi}_{k+1,k} \mathbf{C}_{\mathbf{\varepsilon}_k|k} \mathbf{\Phi}_{k+1,k}^T) (\mathbf{Q}_{k+1} + \mathbf{\Phi}_{k+1,k} \mathbf{C}_{\mathbf{\varepsilon}_k|k} \mathbf{\Phi}_{k+1,k}^T)^{-1} \mathbf{\Phi}_{k+1,k} \mathbf{C}_{\mathbf{\varepsilon}_k|k} \end{aligned}$$

From the forward Kalman Filter:

$$\cancel{\mathbf{x}}_{k+1|k} = \mathbf{\Phi}_{k+1,k} \cancel{\mathbf{x}}_{k|k}$$

(predicted state)

$$\mathbf{C}_{\mathbf{\varepsilon}_{k+1}|k} = \mathbf{Q}_{k+1} + \mathbf{\Phi}_{k+1,k} \mathbf{C}_{\mathbf{\varepsilon}_k|k} \mathbf{\Phi}_{k+1,k}^T$$

(predicted error covariance)

Backward smoothing as an update of forward Kalman filter

$$\mathbf{x}_{k|n} - \mathbf{x}_{k|k} = \mathbf{C}_{\mathbf{\varepsilon}_k|k} \mathbf{\Phi}_{k+1,k}^T \mathbf{C}_{\mathbf{\varepsilon}_{k+1}|k}^{-1} (\mathbf{x}_{k+1|n} - \mathbf{x}_{k+1|k})$$

$$\mathbf{C}_{\mathbf{\varepsilon}_k|n} = \mathbf{C}_{\mathbf{\varepsilon}_k|k} + \mathbf{C}_{\mathbf{\varepsilon}_k|k} \mathbf{\Phi}_{k+1,k}^T \mathbf{C}_{\mathbf{\varepsilon}_{k+1}|k}^{-1} (\mathbf{C}_{\mathbf{\varepsilon}_{k+1}|n} - \mathbf{C}_{\mathbf{\varepsilon}_{k+1}|k}) \mathbf{C}_{\mathbf{\varepsilon}_{k+1}|k}^{-1} \mathbf{\Phi}_{k+1,k} \mathbf{C}_{\mathbf{\varepsilon}_k|k}$$

Backward smoothing as an update of forward Kalman filter

$$\mathbf{x}_{k|n} - \mathbf{x}_{k|k} = \mathbf{C}_{\mathbf{\varepsilon}_k|k} \mathbf{\Phi}_{k+1,k}^T \mathbf{C}_{\mathbf{\varepsilon}_{k+1}|k}^{-1} (\mathbf{x}_{k+1|n} - \mathbf{x}_{k+1|k})$$

$$\mathbf{C}_{\mathbf{\varepsilon}_k|n} = \mathbf{C}_{\mathbf{\varepsilon}_k|k} + \mathbf{C}_{\mathbf{\varepsilon}_k|k} \mathbf{\Phi}_{k+1,k}^T \mathbf{C}_{\mathbf{\varepsilon}_{k+1}|k}^{-1} (\mathbf{C}_{\mathbf{\varepsilon}_{k+1}|n} - \mathbf{C}_{\mathbf{\varepsilon}_{k+1}|k}) \mathbf{C}_{\mathbf{\varepsilon}_{k+1}|k}^{-1} \mathbf{\Phi}_{k+1,k} \mathbf{C}_{\mathbf{\varepsilon}_k|k}$$

Introduce:

$$\mathbf{D}_k = \mathbf{C}_{\mathbf{\varepsilon}_k|k} \mathbf{\Phi}_{k+1,k}^T \mathbf{C}_{\mathbf{\varepsilon}_{k+1}|k}^{-1}$$

Backward smoothing as an update of forward Kalman filter

$$\mathbf{x}_{k|n} - \mathbf{x}_{k|k} = \mathbf{D}_k (\mathbf{x}_{k+1|n} - \mathbf{x}_{k+1|k})$$

$$\mathbf{C}_{\mathbf{\varepsilon}_k|n} = \mathbf{C}_{\mathbf{\varepsilon}_k|k} + \mathbf{D}_k (\mathbf{C}_{\mathbf{\varepsilon}_{k+1}|n} - \mathbf{C}_{\mathbf{\varepsilon}_{k+1}|k}) \mathbf{D}_k^T$$

$$\mathbf{D}_k = \mathbf{C}_{\mathbf{\varepsilon}_k|k} \mathbf{\Phi}_{k+1,k}^T \mathbf{C}_{\mathbf{\varepsilon}_{k+1}|k}^{-1}$$

Backward smoothing as an update of forward Kalman filter

$$\mathbf{D}_k = \mathbf{C}_{\mathbf{x}_k|k} \Phi_{k+1,k}^T \mathbf{C}_{\mathbf{x}_{k+1}|k}^{-1}$$

$$\mathbf{x}_{k|n} - \mathbf{x}_{k|k} = \mathbf{D}_k (\mathbf{x}_{k+1|n} - \mathbf{x}_{k+1|k})$$

$$\mathbf{C}_{\mathbf{x}_k|n} = \mathbf{C}_{\mathbf{x}_k|k} + \mathbf{D}_k (\mathbf{C}_{\mathbf{x}_{k+1}|n} - \mathbf{C}_{\mathbf{x}_{k+1}|k}) \mathbf{D}_k^T$$

Rauch-Tung-Striebel smoothing algorithm!

*Application of the general theory
of sequential prediction
to the global geodetic solution*

From global geodetic solution to Kalman filter

The Kalman filter can be also obtained as an application of the general theory of **sequential prediction**:

The prediction $\hat{s}_{|b}, C_{\hat{s}_{|b}-s}$ based on **initial data b** is updated to a prediction $\hat{s}_{|b,b'}, C_{\hat{s}_{|b,b'}-s}$ based on **both old data b and new data b'** :

From global geodetic solution to Kalman filter

The Kalman filter can be also obtained as an application of the general theory of **sequential prediction**:

The prediction $\hat{\mathbf{s}}_{|\mathbf{b}}, \mathbf{C}_{\hat{\mathbf{s}}_{|\mathbf{b}}-\mathbf{s}}$ based on **initial data** $\mathbf{b}$ is updated to a prediction $\hat{\mathbf{s}}_{|\mathbf{b},\mathbf{b}'}, \mathbf{C}_{\hat{\mathbf{s}}_{|\mathbf{b},\mathbf{b}'}-\mathbf{s}}$ based on **both old data** $\mathbf{b}$ **and new data** $\mathbf{b}'$:

Initial data prediction: $\hat{\mathbf{s}}_{|\mathbf{b}} = \mathbf{C}_{\mathbf{s}\mathbf{b}} \mathbf{C}_{\mathbf{b}}^{-1} \mathbf{b} \qquad \mathbf{C}_{\hat{\mathbf{s}}_{|\mathbf{b}}-\mathbf{s}} = \mathbf{C}_{\mathbf{s}} - \mathbf{C}_{\mathbf{s}\mathbf{b}} \mathbf{C}_{\mathbf{b}}^{-1} \mathbf{C}_{\mathbf{b}\mathbf{s}}$

From global geodetic solution to Kalman filter

The Kalman filter can be also obtained as an application of the general theory of **sequential prediction**:

The prediction $\hat{\mathbf{s}}_{|\mathbf{b}}, \mathbf{C}_{\hat{\mathbf{s}}_{|\mathbf{b}}-\mathbf{s}}$ based on **initial data** $\mathbf{b}$ is updated to a prediction $\hat{\mathbf{s}}_{|\mathbf{b},\mathbf{b}'}, \mathbf{C}_{\hat{\mathbf{s}}_{|\mathbf{b},\mathbf{b}'}-\mathbf{s}}$ based on **both old data** $\mathbf{b}$ **and new data** $\mathbf{b}'$:

Initial data prediction: $\hat{\mathbf{s}}_{|\mathbf{b}} = \mathbf{C}_{\mathbf{s}\mathbf{b}} \mathbf{C}_{\mathbf{b}}^{-1} \mathbf{b} \quad \mathbf{C}_{\hat{\mathbf{s}}_{|\mathbf{b}}-\mathbf{s}} = \mathbf{C}_{\mathbf{s}} - \mathbf{C}_{\mathbf{s}\mathbf{b}} \mathbf{C}_{\mathbf{b}}^{-1} \mathbf{C}_{\mathbf{b}\mathbf{s}}$

Prediction of future data: $\hat{\mathbf{b}}_{|\mathbf{b}} = \mathbf{C}_{\mathbf{b}'\mathbf{b}} \mathbf{C}_{\mathbf{b}}^{-1} \mathbf{b}$
 $\mathbf{C}_{\hat{\mathbf{b}}_{|\mathbf{b}}-\mathbf{b}'} = \mathbf{C}_{\mathbf{b}'\mathbf{b}} - \mathbf{C}_{\mathbf{b}'\mathbf{b}} \mathbf{C}_{\mathbf{b}}^{-1} \mathbf{C}_{\mathbf{b}\mathbf{b}'} \quad \mathbf{C}_{\hat{\mathbf{s}}_{|\mathbf{b}}-\mathbf{s}, \hat{\mathbf{b}}_{|\mathbf{b}}-\mathbf{b}'} = \mathbf{C}_{\mathbf{s}\mathbf{b}'} - \mathbf{C}_{\mathbf{s}\mathbf{b}} \mathbf{C}_{\mathbf{b}}^{-1} \mathbf{C}_{\mathbf{b}\mathbf{b}'}$

From global geodetic solution to Kalman filter

The Kalman filter can be also obtained as an application of the general theory of **sequential prediction**:

The prediction $\hat{\mathbf{s}}_{|\mathbf{b}}, \mathbf{C}_{\hat{\mathbf{s}}_{|\mathbf{b}}-s}$ based on **initial data** $\mathbf{b}$ is updated to a prediction $\hat{\mathbf{s}}_{|\mathbf{b},\mathbf{b}'}, \mathbf{C}_{\hat{\mathbf{s}}_{|\mathbf{b},\mathbf{b}'}-s}$ based on **both old data** $\mathbf{b}$ **and new data** $\mathbf{b}'$:

Initial data prediction: $\hat{\mathbf{s}}_{|\mathbf{b}} = \mathbf{C}_{sb} \mathbf{C}_b^{-1} \mathbf{b} \quad \mathbf{C}_{\hat{\mathbf{s}}_{|\mathbf{b}}-s} = \mathbf{C}_s - \mathbf{C}_{sb} \mathbf{C}_b^{-1} \mathbf{C}_{bs}$

Prediction of future data: $\hat{\mathbf{b}}_{|\mathbf{b}}^o = \mathbf{C}_{b'b} \mathbf{C}_b^{-1} \mathbf{b}$
 $\mathbf{C}_{\hat{\mathbf{b}}_{|\mathbf{b}}^o-b'} = \mathbf{C}_{b'b} - \mathbf{C}_{b'b} \mathbf{C}_b^{-1} \mathbf{C}_{bb'} \quad \mathbf{C}_{\hat{\mathbf{s}}_{|\mathbf{b}}-s, \hat{\mathbf{b}}_{|\mathbf{b}}^o-b'} = \mathbf{C}_{sb'} - \mathbf{C}_{sb} \mathbf{C}_b^{-1} \mathbf{C}_{bb'}$

All-data prediction: $\hat{\mathbf{s}}_{|\mathbf{b},\mathbf{b}'} = \hat{\mathbf{s}}_{|\mathbf{b}} + \mathbf{C}_{\hat{\mathbf{s}}_{|\mathbf{b}}-s, \hat{\mathbf{b}}_{|\mathbf{b}}^o-b'} \mathbf{C}_{\hat{\mathbf{b}}_{|\mathbf{b}}^o-b'}^{-1} (\mathbf{b}' - \hat{\mathbf{b}}_{|\mathbf{b}}^o)$
 $\mathbf{C}_{\hat{\mathbf{s}}_{|\mathbf{b},\mathbf{b}'}-s} = \mathbf{C}_{\hat{\mathbf{s}}_{|\mathbf{b}}-s} + \mathbf{C}_{\hat{\mathbf{s}}_{|\mathbf{b}}-s, \hat{\mathbf{b}}_{|\mathbf{b}}^o-b'} \mathbf{C}_{\hat{\mathbf{b}}_{|\mathbf{b}}^o-b'}^{-1} \mathbf{C}_{\hat{\mathbf{s}}_{|\mathbf{b}}-s, \hat{\mathbf{b}}_{|\mathbf{b}}^o-b'}^T$

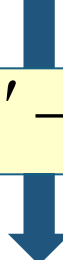
collocation
like !

From global geodetic solution to Kalman filter

The Kalman filter can be also obtained as an application of the general theory of **sequential prediction**:

$$\mathbf{s}_{\mathbf{b}, \mathbf{b}'} = \mathbf{s}_{\mathbf{b}} + \mathbf{C}_{\mathbf{s}_{\mathbf{b}} - \mathbf{s}, \mathbf{b}_{\mathbf{b}} - \mathbf{b}'} \mathbf{C}_{\mathbf{b}_{\mathbf{b}} - \mathbf{b}'}^{-1} (\mathbf{b}' - \mathbf{b}_{\mathbf{b}})$$

$$\mathbf{C}_{\mathbf{s}_{\mathbf{b}, \mathbf{b}'} - \mathbf{s}} = \mathbf{C}_{\mathbf{s}_{\mathbf{b}} - \mathbf{s}} + \mathbf{C}_{\mathbf{s}_{\mathbf{b}} - \mathbf{s}, \mathbf{b}_{\mathbf{b}} - \mathbf{b}'} \mathbf{C}_{\mathbf{b}_{\mathbf{b}} - \mathbf{b}'}^{-1} \mathbf{C}_{\mathbf{s}_{\mathbf{b}} - \mathbf{s}, \mathbf{b}_{\mathbf{b}} - \mathbf{b}'}$$


$$\mathbf{b} \rightarrow \mathbf{b}_{(k-1)}, \quad \mathbf{b}' \rightarrow \mathbf{b}_k, \quad \mathbf{s} \rightarrow \mathbf{x}_k,$$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{C}_{\mathbf{x}_{k|k-1}, \mathbf{b}_{k|k-1} - \mathbf{b}_k} \mathbf{C}_{\mathbf{b}_{k|k-1} - \mathbf{b}_k}^{-1} (\mathbf{b}_k - \mathbf{b}_{k|k-1})$$

$$\mathbf{C}_{\mathbf{x}_{k|k} - \mathbf{x}_k} = \mathbf{C}_{\mathbf{x}_{k|k-1} - \mathbf{x}_k} + \mathbf{C}_{\mathbf{x}_{k|k-1} - \mathbf{x}_k, \mathbf{b}_{k|k-1} - \mathbf{b}_k} \mathbf{C}_{\mathbf{b}_{k|k-1} - \mathbf{b}_k}^{-1} \mathbf{C}_{\mathbf{x}_{k|k-1} - \mathbf{x}_k, \mathbf{b}_{k|k-1} - \mathbf{b}_k}^T$$

Kalman filter!

Sequential form of global geodetic solutions

$$\mathbf{b} \rightarrow \mathbf{b}_{(k-1)}, \mathbf{b}' \rightarrow \mathbf{b}_k$$

All global solutions accept at least a **forward sequential form**: $\mathbf{s} \rightarrow \mathbf{x}_{(k)}$

At each epoch t_k the observations $\mathbf{b}_k$ are used to compute $\tilde{\mathbf{x}}_{k|k}$ and update $[\tilde{\mathbf{x}}_{1|k-1}, \dots, \tilde{\mathbf{x}}_{k-1|k-1}]$ to $[\tilde{\mathbf{x}}_{1|k}, \dots, \tilde{\mathbf{x}}_{k-1|k}]$.

At the last epoch t_n we have the global solution $[\tilde{\mathbf{x}}_{1|n}, \dots, \tilde{\mathbf{x}}_{n|n}]$.

Versus **Kalman sequential form**: $\mathbf{s} \rightarrow \mathbf{x}_k$

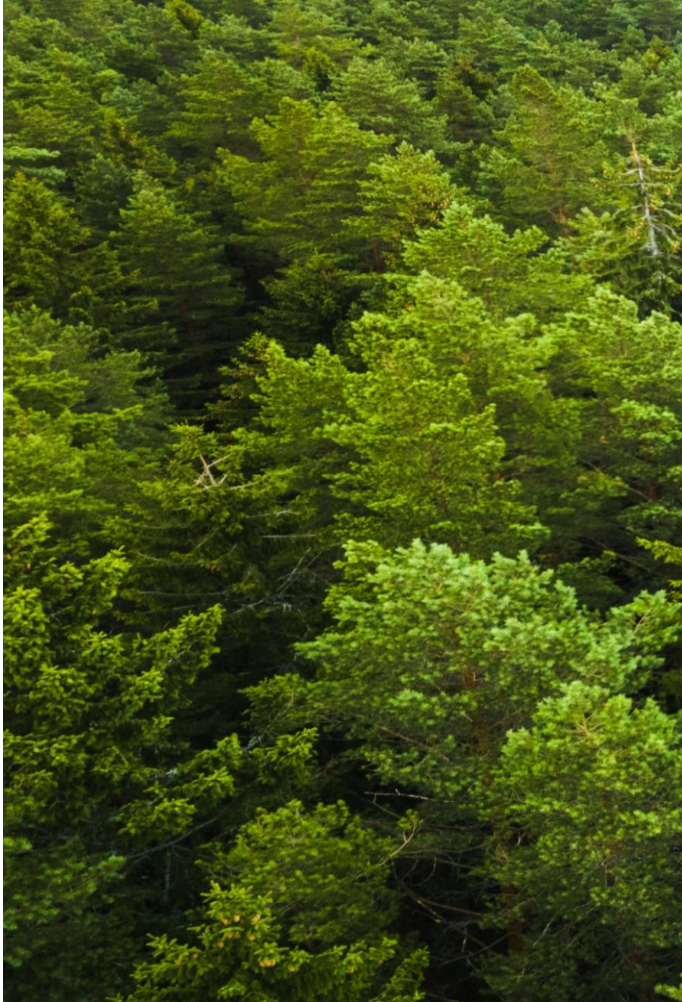
Forward: At each epoch t_k the observations $\mathbf{b}_k$ are used to compute only $\tilde{\mathbf{x}}_{k|k}$

Backward: At each epoch t_k we update $\tilde{\mathbf{x}}_{k|k}$ to $\tilde{\mathbf{x}}_{k|n}$

Conclusions

What can we gain by looking at the forest instead of the tree?

Geodesy



Engineering



OR

What can we gain by looking at the forest instead of the tree?



We may find a better tree!

What can we gain by looking at the forest instead of the tree?

... or at least several new interesting trees!



The new trees:

- *Four global “geodetic” solutions – only one equivalent to the Kalman solution*
 - (1) $\mathbf{x}_0$ = *unknown random variable to be predicted*
 - (2) $\mathbf{x}_0$ = *deterministic variable to be estimated*
 - (3) $\mathbf{x}_1$ (first observation epoch state) = *deterministic variable to be estimated*
 - (4) $\mathbf{x}_0$ = *unknown random variable (to be predicted) with a known outcome (?)*
(equivalent to Kalman filtering and smoothing)

None of above requires knowledge of probability distributions !

The new trees:

- *A new (Kalman equivalent) smoothing algorithm
providing all state-prediction-error correlations between different epochs*
- *From the 4th solution above we have derived the **Kalman filtering** algorithm
and the **Rauch–Tung–Stiebel smoothing** algorithm by simple matrix algebra
- no statistical tools used!*
- *We have derived Kalman filtering as a simple application of sequential prediction*
- *We have derived sequential estimation algorithms for all four geodetic solutions
where all epoch state predictions are updated at each new data epoch.
No backward smoothing algorithm is necessary!*

Thanks for your attention!

Further reading:

Sansò F. (2006) ***Navigazione Geodetica e Posizionamento Cinematico.***
Polipress, Politecnico di Milano. (in Italian)

Sansò F., C. Pileggi, L. Biagi (2024): ***A tutorial on Kalman filtering.***
The linear, extended and unscented filters.

Dermanis A., D. Rossikopoulos (>2026): ***The analysis of observations.***
Estimation and prediction of unknown parameters and functions (in preparation)

This presentation will appear in:

<https://dermanis.webpages.auth.gr/>