

Geoid and height anomaly: a possible summary on theory and estimation methods

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The Molodensky Geodetic Boundary Value Problems (GBVP)

- ✓ *The vectorial Free GBVP (Vectorial Molodensky problem)*

Given on the Earth surface S the values of $\|\underline{g}\|$, (Φ, Λ) and of the potential W find the estimate of W outside S and of the surface S

- ✓ *The scalar Free GBVP (Scalar Molodensky problem)*

Given on the Earth surface S the values of the $\|\underline{g}\|$, (φ, λ) and of the potential W find the estimate of W outside S and of the height $h(\varphi, \lambda)$ of points on S

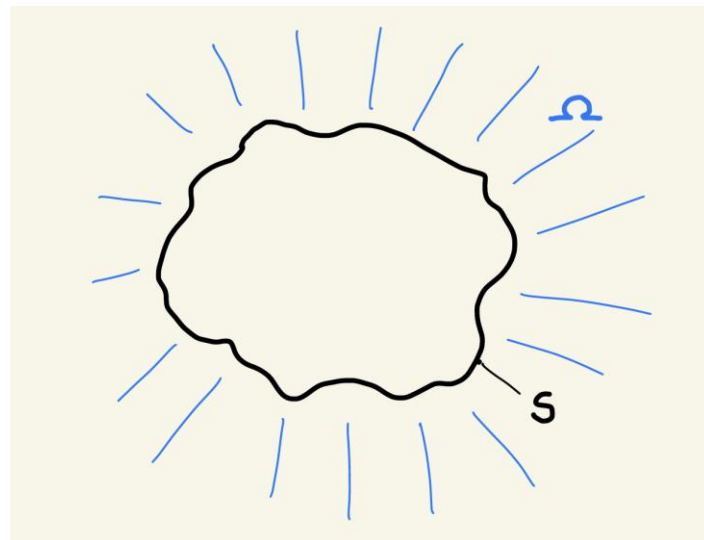
- ✓ *The fixed GBVP (Bakus problem)*

Given on the known Earth surface S the values of the $\|\underline{g}\|$ find the estimate of W outside S

(a numerical solution of this problem has been proposed recently in a paper by Macák et al, 2025)

GBVP: data and unknowns

GBVP	DATA	UNKNOWNNS
Vectorial Free	$\ \underline{g}\ _S (\Phi, \Lambda) W _S$	$(\varphi, \lambda, h) _S$ and W in Ω
Scalar Free	$\ \underline{g}\ _S (\varphi, \lambda) W _S$	$h(\varphi, \lambda) _S$ and W in Ω
Fixed	$\ \underline{g}\ _S (\varphi, \lambda, h)$	W in Ω



The Scalar Free Molodensky Problem

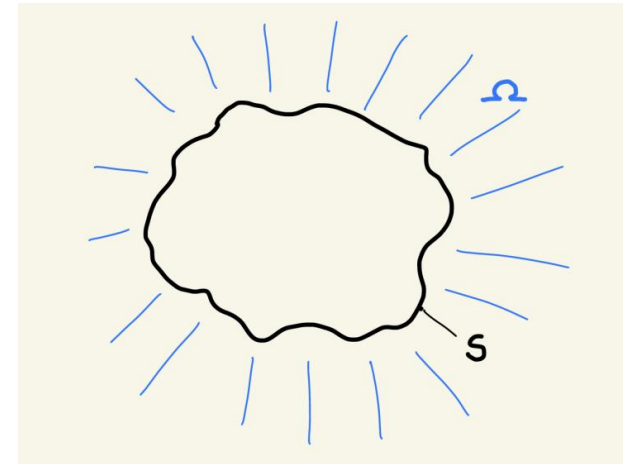
In more details, this problem can be formalized in the following way:

given $\sigma_P = (\lambda_P, \varphi_P)$, $W_S(\sigma_P)$, $g_S(\sigma_P)$ for any $P \in S$

find $S \equiv \{h(\sigma_P), P \in S\}$ and $W(\sigma, h)$

Thus this GBVP can be written as

$$\left\{ \begin{array}{l} \Delta \left[W - \frac{1}{2} \omega^2 (x^2 + y^2) \right] = 0 \text{ in } \Omega \equiv \{h > h(\sigma)\} \\ W|_{h=h(\sigma)} = W_S(\sigma_P) \text{ on } S \\ |\nabla W|_{h=h(\sigma)} = g_S(\sigma) \text{ on } S \\ W - \frac{1}{2} \omega^2 (x^2 + y^2) \rightarrow 0 \quad h \rightarrow \infty \end{array} \right.$$



This problem is then linearized considering for the linearization the (e.g. Marussi) telluroid $\tilde{S}$ that is described by the ellipsoidal height $\tilde{h}(\sigma)$ obtained from the equation

$$U(\sigma, \tilde{h}(\sigma)) = W_S(\sigma) = W(\sigma, h(\sigma))$$

The $\tilde{h}(\sigma)$ defines the normal height of P and U is the normal potential. Thus we can write

$$h(\sigma) = \tilde{h}(\sigma) + \zeta(\sigma)$$

where $\zeta(\sigma)$ is the height anomaly. So, at first order, considering the above equations, the linearized boundary equation on W becomes

$$W_S(\sigma) = U(\sigma, h(\sigma)) + T(\sigma, h(\sigma)) \cong U(\sigma, \tilde{h}(\sigma)) + \frac{\partial U}{\partial h} \zeta + T(\sigma, \tilde{h}(\sigma))$$

where $T(\sigma, \tilde{h}(\sigma))$ is the anomalous potential

Then, given the first equation in this slide, we have

$$0 = W_S(\sigma) - U(\sigma, \tilde{h}(\sigma)) \cong \frac{\partial U}{\partial h} \zeta + T(\sigma, \tilde{h}(\sigma)) = -\gamma \zeta + T(\sigma, \tilde{h}(\sigma))$$

that is, the Bruns equation (where γ is the normal gravity)

$$\zeta = \frac{T}{\gamma}$$

The linearized boundary equation on $g_s(\sigma)$ becomes the equation defining the free-air gravity anomaly

$$\Delta g_s(\sigma) = g_s(\sigma) - \gamma(\sigma, \tilde{h}(\sigma)) = -\frac{\partial T}{\partial h} + \frac{T}{\gamma} \frac{\partial \gamma}{\partial h} \Big|_{\tilde{h}}$$

The original problem can be then formulated as the following one for the harmonic anomalous potential T

$$\left\{ \begin{array}{l} \Delta T = 0 \quad \text{in } \tilde{\Omega} \equiv \{h \geq \tilde{h}(\sigma)\} \\ \Delta g_s(\sigma) = -\frac{\partial T}{\partial h} + \frac{T}{\gamma} \frac{\partial \gamma}{\partial h} \Big|_{\tilde{h}} \end{array} \right. \quad T = O\left(\frac{1}{r^3}\right)$$

This is the so-called *linearized Molodensky problem* (an oblique derivative BVP). Once we have T in $\tilde{\Omega}$, we can solve the original problem as

$$W(\sigma, h) = U(\sigma, h) + T(\sigma, h)$$

$$h(\sigma) = \tilde{h}(\sigma) + \zeta(\sigma) = \tilde{h}(\sigma) + \frac{T(\sigma, \tilde{h})}{\gamma(\sigma, \tilde{h})}$$

The Stokes-Helmert BVP problem

The key point in this approach is the Helmert reduction (*). Helmert's idea is to replace the actual Earth topography effect by a thin layer that can be placed on or beneath the geoid.

This implies a change in the actual potential of the Earth as expressed in the following (also taking into account, as done previously in the Molodensky's BVPs, linearization)

$$W^H(P) = W(P) - \delta V^H(P) = U(P) + T(P) - \delta V^H(P) = U(P) + T^H(P)$$

$$T^H(P) = T(P) - \delta V^H(P)$$

where $W^H(P)$ is the Earth gravity potential corrected for the Helmert potential correction

$$\delta V^H(P) = V_t(P) - V_c(P)$$

with

$V_t(P)$ = gravitational potential effect of topographic masses

$V_c(P)$ = gravitational potential effect of the condensation layer

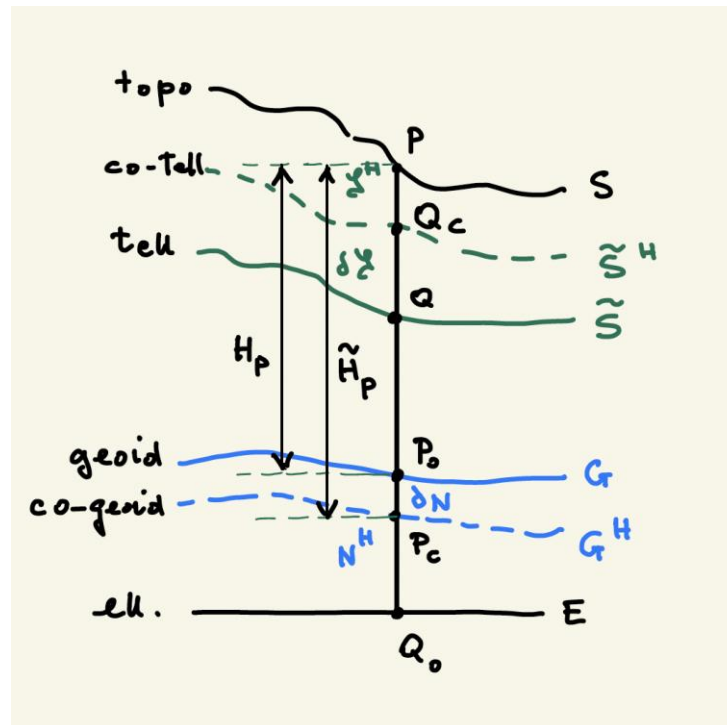
(*) Other reductions can be considered but here we focus on the Helmert one that has the smaller impact on the reduced values than the others

The geoid G is then changed in the co-geoid G^H that is taken as the boundary surface for the Stokes-Helmert BVP.

As done for the Molodensky problem, we need to express the boundary condition, i.e. the gravity anomalies, on the co-geoid

This boundary condition can be obtained in the following way

$$g^H(P_C) = -\frac{\partial W^H}{\partial h}\bigg|_{P_C} = -\frac{\partial U}{\partial h}\bigg|_{P_C} - \frac{\partial T^H}{\partial h}\bigg|_{P_C} = \gamma(P_C) - \frac{\partial T^H}{\partial h}\bigg|_{P_C}$$



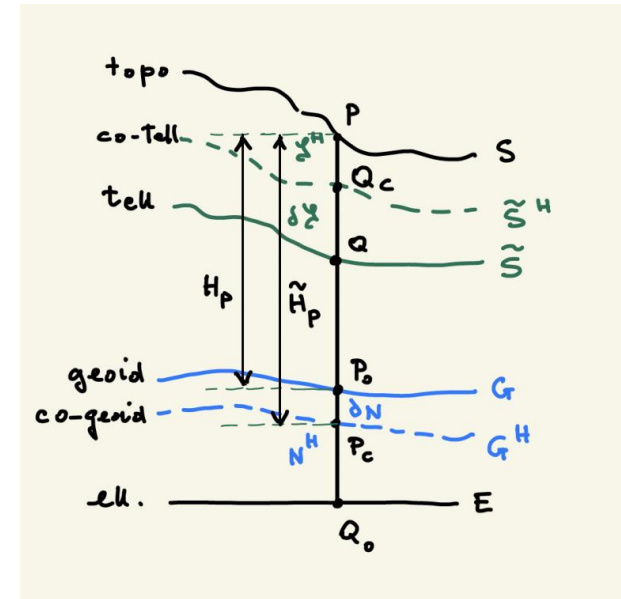
$$\gamma(P_c) \cong \gamma(Q_0) + \left. \frac{\partial \gamma}{\partial h} \right|_{Q_0} N^H = \gamma(Q_0) + \left. \frac{\partial \gamma}{\partial h} \right|_{Q_0} \frac{T^H(P_c)}{\gamma(Q_0)}$$

by substituting in the one of the previous page, we get

$$g^H(P_c) = \gamma(Q_0) - \left. \frac{\partial T^H}{\partial h} \right|_{P_c} + \left. \frac{\partial \gamma}{\partial h} \right|_{Q_0} \frac{T^H(P_c)}{\gamma(Q_0)}$$

which gives then the Helmert anomaly on the co-geoid

$$\Delta g^H = g^H(P_c) - \gamma(Q_0) = - \left. \frac{\partial T^H}{\partial h} \right|_{P_c} + \left. \frac{\partial \gamma}{\partial h} \right|_{Q_0} \frac{T^H(P_c)}{\gamma(Q_0)}$$



Finally, we can write the linearized Helmert-Stokes problem as

$$\left\{ \begin{array}{l} \Delta T^H = 0 \quad \text{outside } G^H \\ -\frac{\partial T^H}{\partial h} + \frac{T^H}{\gamma} \frac{\partial \gamma}{\partial h} \Big|_{\tilde{s}} = \Delta g^H \text{ on } G^H \end{array} \right.$$

Once T^H is estimated by solving this BVP, one gets the geoid as

$$N(P_0) = \frac{T^H(P_c)}{\gamma(Q_0)} + \frac{\delta V^H(P_c)}{\gamma(Q_0)} = N^H + \delta N$$

From the observed $g(P)$ on the Earth surface to the $g^H(P_c)$ on the co-geoid

Since the observed gravity $g(P)$ is given on the Earth surface, we have to reduce it to the co-geoid in order to have $g^H(P_c)$ which is in the definition of Δg^H

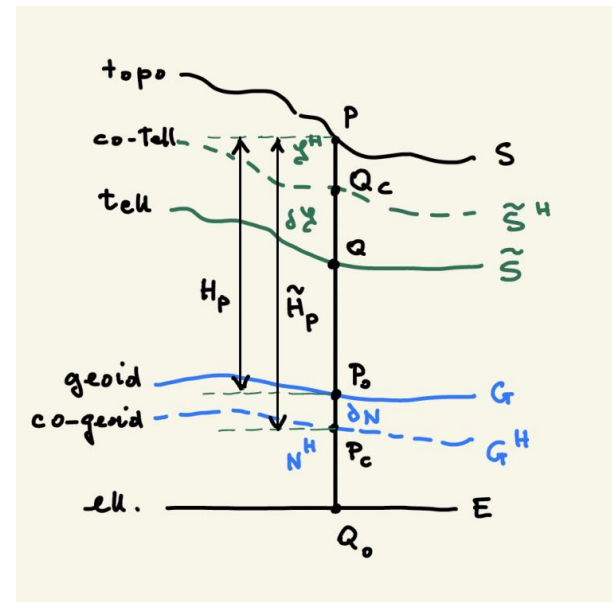
We can write

$$g^H(P_c) \cong g^H(P) - \left. \frac{\partial g^H}{\partial h} \right|_{P_c} (H_P + \delta N) = g^H(P) - \left. \frac{\partial g^H}{\partial h} \right|_{P_c} \tilde{H}_P$$

and set

$$g^H(P_c) = \gamma(P_c) - \frac{\partial T^H}{\partial h} \Big|_{P_c}$$

$$\tilde{H}_P = H_P + \delta N = H_P + \frac{\delta V^H(P_c)}{\gamma(Q_0)}$$



By substituting in the initial condition

$$g^H(P_c) \cong g^H(P) - \left. \frac{\partial g^H}{\partial h} \right|_{P_c} (H_P + \delta N) = g^H(P) - \left. \frac{\partial g^H}{\partial h} \right|_{P_c} \tilde{H}_P$$

and rearranging the terms we get

$$g^H(P_c) \cong g^H(P) - \left. \frac{\partial \gamma}{\partial h} \right|_{P_c} \tilde{H}_P + \left. \frac{\partial^2 T^H}{\partial h^2} \right|_{P_c} \tilde{H}_P = g(P) + \left. \frac{\partial \delta V^H}{\partial h} \right|_P - \left. \frac{\partial \gamma}{\partial h} \right|_{P_c} \tilde{H}_P + \left. \frac{\partial^2 T^H}{\partial h^2} \right|_{P_c} \tilde{H}_P$$

which finally gives the equation for the boundary values

$$\Delta g^H = g^H(P_c) - \gamma(Q_0) = g(P) + \left. \frac{\partial \delta V^H}{\partial h} \right|_P - \left. \frac{\partial \gamma}{\partial h} \right|_{P_c} \tilde{H}_P + \left. \frac{\partial^2 T^H}{\partial h^2} \right|_{P_c} \tilde{H}_P - \gamma(Q_0)$$

Remark 1: the term $\left. \frac{\partial^2 T^H}{\partial h^2} \right|_{P_c} \tilde{H}_P$ is unknown before T^H is computed. Thus iterations should be considered.

Also, after some manipulations, this boundary condition can be written as the standard Stokes boundary condition

$$-\left. \frac{\partial T}{\partial h} \right|_{P_c} + \left. \frac{\partial \gamma}{\partial h} \right|_{Q_0} \frac{T(P_c)}{\gamma(Q_0)} = g(P_c) - \gamma(Q_0) = \Delta g \quad \text{on } G^H$$

Other applied approximations of the linearized Helmert-Stokes's problem

To come to the solution of the Stokes's BVP, two other approximations are usually applied

- ✓ the spherical approximation: in expressing the boundary condition terms of the order of the squared eccentricity e^2 are neglected (boundary data are still evaluated at G^H)
- ✓ the constant radius approximation: the reference surface is considered the mean Earth sphere as an approximation of the ellipsoid

In this way we manage to write the solution of the linearized Helmert-Stokes problem as

$$N(P_0) = N^H + \delta N = + \frac{\delta V^H(P_c)}{\gamma(Q_0)} = \frac{R}{4\pi\gamma} \int_{\sigma} \Delta g^H(P) S(\psi_{PP_0}) d\sigma + \frac{\delta V^H(P_c)}{\gamma(Q_0)}$$

R = mean Earth sphere radius

σ = unit sphere

$$\text{Stokes function} = S(\psi) = \frac{1}{\sin(\psi/2)} - 6 \sin\left(\frac{\psi}{2}\right) + 1 - 5 \cos(\psi) - 3 \cos(\psi) \ln\left[\sin\left(\frac{\psi}{2}\right) + \sin^2\left(\frac{\psi}{2}\right)\right]$$

Remark: refinements for the so-called ellipsoidal corrections are then applied to the solution

The Least Squares Modification of Stokes's formula with Additive Corrections (LSMSA) (also known as the KTH method)

This is a method in which reducing and restoring components are combined and a LS modification of the Stokes's formula is also considered.

The geoid undulation is computed starting from free-air anomalies

$$\Delta g_{FA} = g(P) - \gamma(Q) \cong g(P) - \gamma(H_P)$$

The geoid undulation is given as

$$\hat{N} = \tilde{N} + \delta N_{COMB} + \delta N_{DWC} + \delta N_{ATM} + \delta N_{ELL}$$

where

- ✓ $\tilde{N} = \frac{R}{4\pi\gamma} \int_{\sigma} S^L(\psi) \Delta g_{FA} d\sigma + \frac{R}{2\gamma} \sum_{n=2}^L s_n \Delta g_n^{GGM}$
- ✓ Δg_n^{GGM} is the GGM derived gravity anomaly Laplace harmonic of degree n (to degree L)
- ✓ δN_{COMB} = combination of direct and indirect effect of topography on geoid
- ✓ δN_{DWC} = downward continuation effect on geoid
- ✓ δN_{ATM} = atmospheric effect on geoid
- ✓ δN_{ELL} = ellipsoidal corrections on geoid

Particularly, in

$$\tilde{N} = \frac{R}{4\pi\gamma} \int_{\sigma} S^L(\psi) \Delta g_{FA} d\sigma + \frac{R}{2\gamma} \sum_{n=2}^L s_n \Delta g_n^{GGM}$$

we have that

$$S^L(\psi) = S(\psi) - \sum_{K=2}^L \frac{2K+1}{2} s_K P_K(\cos\psi)$$

The $\{s_k\}$ can be found by applying the Least Squares Modification of Stokes' Formula (LSMSF). The equation for $\tilde{N}$ can be written in the following spectral form

$$\tilde{N} = \frac{R}{2\gamma} \sum_{n=2}^{\infty} (\lambda_n - s_n^*) (\Delta g_n + \varepsilon_n^T) + \frac{R}{2\gamma} \sum_{n=2}^L s_n (\Delta g_n + \varepsilon_n^{GGM})$$

with

$$\Delta g_n = \frac{2n+1}{4\pi} \int_{\sigma} \Delta g P_n(\cos\psi) d\sigma$$

$$\lambda_n = \frac{2}{n-1} \quad s_n^* = \begin{cases} s_n & n \leq L \\ 0 & \text{otherwise} \end{cases}$$

ε_n^T = random error in the gravity anomalies

ε_n^{GGM} = random error in the model gravity anomalies

Then, by considering the true geoid undulation

$$N = \frac{R}{2\gamma} \sum_{n=2}^{\infty} \lambda_n \Delta g_n$$

we can write

$$d\tilde{N} = \tilde{N} - N = \frac{R}{2\gamma} \sum_{n=2}^{\infty} (\lambda_n - s_n^*) \varepsilon_n^T + \frac{R}{2\gamma} \sum_{n=2}^L s_n \varepsilon_n^{GGM}$$

$$\sigma_{d\tilde{N}}^2 = E\{d\tilde{N}^2\} = \left(\frac{R}{2\gamma}\right)^2 \sum_{n=2}^{\infty} (\lambda_n - s_n^*)^2 \sigma_n^2 + \left(\frac{R}{2\gamma}\right)^2 \sum_{n=2}^L s_n^2 \sigma_n^2(GGM)$$

with

σ_n^2 = error degree variances of the terrestrial data

$\sigma_n^2(GGM)$ = error degree variances of the GGM gravity anomalies

Then, the minimum variance related to the LS solution of $\hat{N}$ is given by

$$\hat{s}_n = \lambda_n \sigma_n^2 / (\sigma_n^2 + \sigma_n^2(GGM))$$

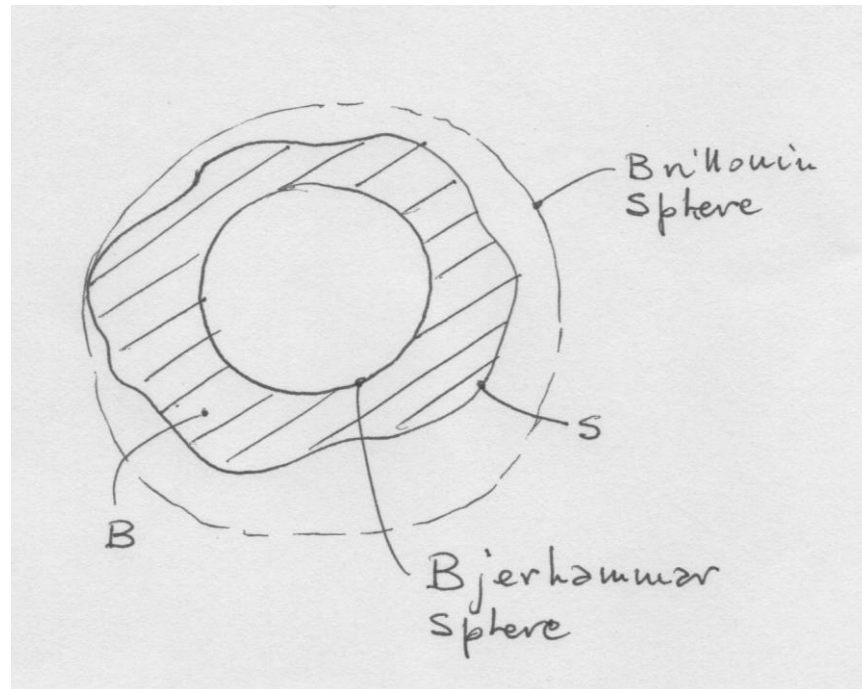
Remark: a similar principle can be applied to data over a cap ψ_0

The deterministic collocation (Hilbert space approach)

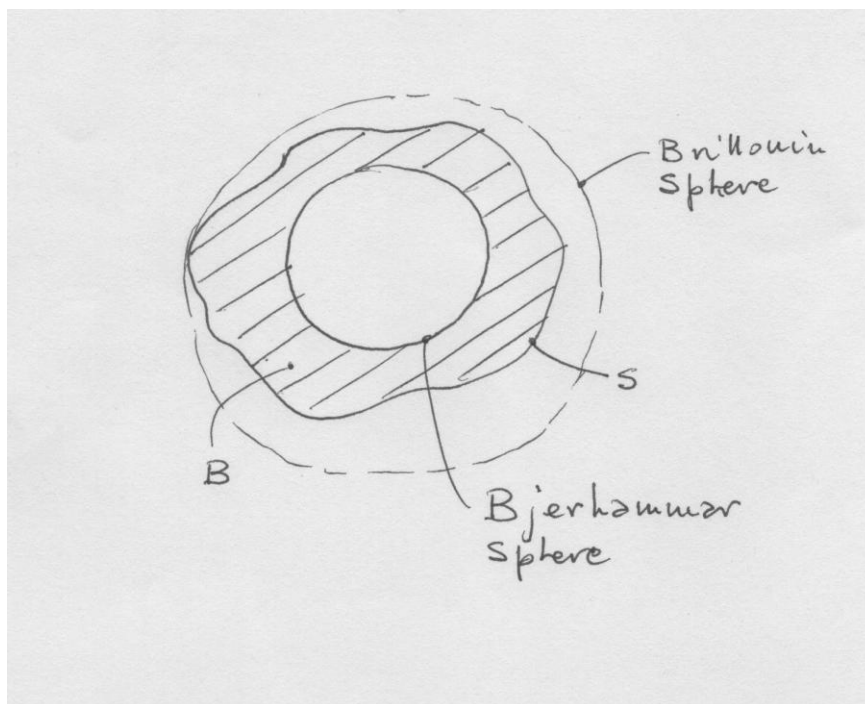
The collocation formula can be derived in the context of a proper Hilbert space (H) of harmonic function.

We assume to consider harmonic functions down to the Bjerhammar sphere, a sphere that is totally inside the Earth body B

These harmonic functions can correctly approximate the harmonic functions outside S by virtue of the Runge-Krarup theorem



The reference sphere can be either the Brillouin sphere or the Bjerhammar sphere



Using the Runge-Krarup theorem, one can prove that $T(P)$ harmonic outside S , can be properly approximated by a $\tilde{T}(P)$ that is harmonic down to the Bjerhammar sphere.

REMARK: $\tilde{T}(P)$ is an approximation of $T(P)$ only in its domain of harmonicity, i.e. outside the Earth surface S

Since we consider harmonic functions down to the Bjerhammar sphere, they can be represented as

$$f = \sum_{n=0}^{+\infty} \sum_{m=-n}^n f_{nm} \frac{R_B^n}{r^{n+1}} Y_{nm}(\vartheta, \lambda)$$

R_B =radius of the Bjerhammar sphere $Y_{nm}(\theta, \lambda)$ = spherical harmonics

We define then an Hilbert space H of harmonic function having:

- The scalar product

$$\langle f, g \rangle = \sum_{n=0}^{+\infty} p_n \sum_{m=-n}^n f_{nm} g_{nm}$$

- The (rotationally invariant) norm

$$\|f\|_H^2 = \sum_{n=0}^{+\infty} p_n \sum_{m=-n}^n f_{nm}^2$$

The spherical harmonics are functions of the H space and define an orthonormal complete (ON) base in the H space

$$s_{nm} = \frac{R_B}{r} Y_{nm}(\vartheta, \lambda)$$

$$\|s_{nm}\|_H^2 = p_n$$

$$\langle s_{nm}, s_{ik} \rangle_H = \delta_{ni} \delta_{mk}$$

Thus $\frac{s_{nm}}{\sqrt{p_n}}$ is an ON complete base in H

In the H space, observations are defined as linear(ized) functional L_i of T

$$M_i^k = obs_i(P_k) = L_i^k(T)$$

As an example, one can consider the free-air gravity anomaly

$$\Delta g(P_k) \cong -\frac{\partial T}{\partial r} \Big|_{P_k} - 2 \frac{T(P_k)}{r(P_k)} = L_{\Delta g}^k(T)$$

Based on the Riesz theorem, one can find an element F_i in the H space so that

$$L_i(T) = \langle F_i, T \rangle \Big|_H$$

By assuming that

$$p_n = O(n^3) \quad n \longrightarrow \infty$$

we can define a function which uniformly converges to a continuous function down to the Bjerhammar sphere

$$\begin{aligned} K(P, Q) &= \sum_{n,m=2}^{+\infty} \frac{s_{nm}(P)s_{nm}(Q)}{p_n} = \sum_{n,m=2}^{+\infty} \frac{1}{p_n} \frac{R_B^{n+1}}{r_P^n} Y_{nm}(\mathcal{G}_P, \lambda_P) \frac{R_B^{n+1}}{r_Q^n} Y_{nm}(\mathcal{G}_Q, \lambda_Q) = \\ &= \sum_{n=2}^{+\infty} \frac{1}{p_n} \frac{R_B^2}{r_P r_Q} \sum_{m=-n}^n Y_{nm}(\mathcal{G}_P, \lambda_P) Y_{nm}(\mathcal{G}_Q, \lambda_Q) = \sum_{n=2}^{+\infty} \frac{1}{p_n} \frac{R_B^2}{r_P r_Q} (2n+1) P_n(\cos \psi_{PQ}) \end{aligned}$$

$$K(P, Q) = \sum_{n=2}^{+\infty} \frac{(2n+1)}{p_n} \frac{R_B^2}{r_P r_Q} P_n(\cos \psi_{PQ})$$

$P_n(\bullet)$ = Legendre polynomials

The $K(P,Q)$ function has the following relevant property

$$\begin{aligned} \langle K(P,Q), T(Q) \rangle_H &= \left\langle \sum_{n,m=2}^{+\infty} \frac{s_{nm}(P)s_{nm}(Q)}{p_n}, \sum_{i,k=2}^{+\infty} T_{ik}s_{ik}(Q) \right\rangle_H = \sum_{n,m}^{+\infty} \sum_{i,k}^{+\infty} \frac{s_{nm}(P)}{p_n} T_{ik} \langle s_{nm}(Q), s_{ik}(Q) \rangle_H = \\ &= \sum_{n,m}^{+\infty} \sum_{i,k}^{+\infty} \frac{s_{nm}(P)}{p_n} T_{ik} p_n \delta_{in} \delta_{mk} = \sum_{n,m}^{+\infty} T_{nm} s_{nm}(P) = \sum_{n,m}^{+\infty} T_{nm} \begin{bmatrix} R_B \\ r_P \end{bmatrix}^{n+1} Y_{nm}(P) \square T(P) \end{aligned}$$

Due to that $K(P,Q)$ is the **Reproducing Kernel of H**.

Furthermore, the representer of any bounded linear functional in H can be found using $K(P,Q)$

$$L_i^k(T) = L_i \left(\langle K(P_k, Q), T(Q) \rangle \Big|_H \right) = \langle L_i[K(P_k, Q)], T(Q) \rangle \Big|_H$$

Since it holds that (Riesz theorem) $L_i(T) = \langle F_i, T \rangle \Big|_H$, one can write

$$F_i(Q) = L_i K(P_k, Q) \square K(L_i^k, Q)$$

The minimum norm principle and the corresponding solutions are:

$$T_{nm} = \left\langle \frac{s_{nm}}{p_n}, T \right\rangle_H = 0 \quad n < N_{\max}$$

$$M_i^k = L_i^k(T) = \left\langle K(L_i^k, Q), T(Q) \right\rangle_H$$

$$K^N(P,Q)=\sum_{n,m=N_{\max}+1}^{+\square}\frac{s_{nm}(P)s_{nm}(Q)}{p_n}$$

$$K^N(L_r^n, L_s^m) = L_r L_s K^N(P_n, P_m)$$

$$K^N(L_i^k, P) = L_i K^N(P_K, P)$$

The equivalence between the stochastic and the deterministic collocation:
the example $\Delta g_r \rightarrow T_r$

$$\begin{aligned}\hat{T}_r &= C_{T_r \Delta g_r} \left(C_{\Delta g_r \Delta g_r} + \sigma_{n_{\Delta g}}^2 I \right)^{-1} (\Delta g_r + n_{\Delta g}) = \\ &= L_{\Delta g} C_{T_r T_r} \left[L_{\Delta g} L_{\Delta g} C_{T_r T_r} + \sigma_{n_{\Delta g}}^2 I \right]^{-1} (\Delta g_r + n_{\Delta g})\end{aligned}$$

Stochastic Collocation

$$\hat{T}_r = L_{\Delta g} K^N \left[L_{\Delta g} L_{\Delta g} K^N \right]^{-1} \Delta g_r$$

Deterministic Collocation

They are equivalent if

$$C_{T_r T_r}(P, Q) = \prod_{n=N_{MAX}+1}^{+\infty} \sigma_n^2 \frac{R_B^2}{r_P r_Q} P_n(\cos \psi_{PQ}) = \prod_{n=N_{MAX}+1}^{+\infty} \frac{(2n+1)}{p_n} \frac{R_B^2}{r_P r_Q} P_n(\cos \psi_{PQ}) = K^N(P, Q)$$

$$\sigma_n^2 = \frac{2n+1}{p_n}$$

REMARK: The equivalence holds if we also assume that $n_{\Delta g}=0$ and $\sigma_n^{err} = 0$

Estimating the geoid and the gravity field: the Spherical Radial Basis Functions (SRBF) approach

Consider a Bjerhammar sphere of radius R_B and two points in space $\underline{r}_P$ and $\underline{r}_Q$.

The function $\Psi(\underline{r}_P, \underline{r}_Q)$

$$\Psi(\underline{r}_P, \underline{r}_Q) = \sum_{l=0}^{\infty} \omega_l(\underline{r}_Q) \frac{2l+1}{4\pi R_B^2} \left(\frac{R_B}{r_P}\right)^{l+1} P_l(\cos\psi_{PQ})$$

$r_P \geq R_B \qquad r_Q < R_B$

$P_l(\bullet)$ = Legendre polynomials

is defined as the **exterior SRBF at pole position $\underline{r}_Q$ evaluated at position $\underline{r}_P$**

By definition, this SRBF is:

- i) harmonic in $r_P \geq R_B$
- ii) rotational symmetric around the axis $\frac{\underline{r}_Q}{r_Q}$

Different choices of $\omega_l(r_Q)$ give different types of SRBFs.

As an example, one can use point masses SRBFs. By considering that:

$$\frac{1}{r_{PQ}} = \frac{1}{r_P \sqrt{1 + \frac{r_Q^2}{r_P^2} - 2 \frac{r_Q}{r_P} \cos \psi_{PQ}}} = \sum_{n=0}^{+\infty} \frac{r_Q^n}{r_P^{n+1}} P_n(\cos \psi_{PQ}) \quad \frac{r_Q}{r_P} < 1$$

we have

$$\sum_{l=0}^{\infty} \frac{r_Q^l}{r_P^{l+1}} P_l(\cos \psi_{PQ}) = \sum_{l=0}^{\infty} \omega_l(r_Q) \frac{2l+1}{4\pi R_B^2} \left(\frac{R_B}{r_P}\right)^{l+1} P_l(\cos \psi_{PQ})$$

$$\frac{r_Q^l}{r_P^{l+1}} = r_Q^l \left(\frac{R_B}{r_P}\right)^{l+1} \frac{1}{R_B^{l+1}} = \omega_l(r_Q) \frac{2l+1}{4\pi R_B^2} \left(\frac{R_B}{r_P}\right)^{l+1}$$

$$\frac{r_Q^l}{R_B^{l+1}} = \omega_l(r_Q) \frac{2l+1}{4\pi R_B^2}$$

$$\omega_l(r_Q) = \frac{4\pi R_B^2}{2l+1} \left(\frac{r_Q}{R_B}\right)^l \frac{1}{R_B} = \frac{4\pi R_B}{2l+1} \left(\frac{r_Q}{R_B}\right)^l$$

The anomalous potential is then expressed as

$$T(\underline{r}_P) = \sum_{n=1}^N \beta_n \Psi(\underline{r}_P, \underline{r}_{Q_n}) \quad \beta_n = \text{real constant values}$$

Remark: as for the collocation approach that was described previously, we are in the framework of the Runge-Krarup theorem. $T(P)$ is considered a function in a space of regular harmonic function down to the Bjerhammar sphere. As this theorem states $T(P)$ is an approximation of the true anomalous potential outside the Earth surface.

The coefficient β_n are estimated by Least Squares starting from observed values that are considered, as in collocation, linear(ized) functional $L_i(\cdot)$ of $T(P)$, like e.g.

$$\Delta g(P) = -\frac{\partial T}{\partial r} - \frac{2}{r}T \Big|_P = L_{\Delta g}^P(T)$$

$$\delta g(P) = -\frac{\partial T}{\partial r} \Big|_P = L_{\delta g}^P(T)$$

The following Least Squares (LS) problem is then set up as

$$l_i = \sum_{n=1}^N \beta_n L_i[\Psi(\underline{r}_P, \underline{r}_{Q_n})] + \epsilon_i$$

$$l_i = L_i(T) + \epsilon_i \quad i \geq N$$

that is, the LS system

$$\underline{l} = A\underline{x} + \underline{\epsilon}$$

$$A = [a_{i,n}] = \{L_i[\Psi(\underline{r}_P, \underline{r}_{Q_n})]\} \quad \underline{l} = [L_i(T)] \quad i = 1, \dots, I$$

$$\underline{x} = [\beta_n] \quad n = 1, \dots, N \quad I \geq N$$

The covariance structure of the observations is usually taken as

$$C_{ll} = \begin{bmatrix} C_{l_1 l_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & C_{l_p l_p} \end{bmatrix}$$

i.e. of block-diagonal form

In applying SRBF the following problems has to be considered:

- ✓ the choice of the type of SRBF
- ✓ the horizontal positions of the SRBF
- ✓ the bandwidth of the SRBF
- ✓ the proper number of SRBF
- ✓ **usually, regularized LS are applied**

Summarizing, the SRBF approach to the estimation of $T(P)$:

- ✓ is based on simple functional models expressing all the main gravity field functional
- ✓ is suitable for global or local gravity field modelling (when observations are reduced by the low- and high-frequency components of the gravity field)
- ✓ is suitable for local refinements of a spherical harmonic representation of the gravity field
- ✓ can be adapted to different data distributions and gravity signal features
- ✓ collocation can be seen as a particular case of SRBF. As an example when estimating $T(P)$ from gravity anomaly values $\Delta g(P_k) = \Delta g_k$, we have

$$T(P) = \sum_{i=1}^N C_{T\Delta g}(P, P_i) \sum_{k=1}^N [C_{\Delta g\Delta g} + \sigma_n^2]_{ik}^{-1} (\Delta g_k + n_k) = \sum_{i=1}^N \lambda_i C_{T\Delta g}(P, P_i)$$

$$C_{\Delta gT}(P, Q) = \frac{1}{r_P} \sum_{n=2}^{+\infty} \sigma_n^2 (n-1) \left(\frac{R^2}{r_P r_Q} \right)^{n+1} P_n(\cos \psi_{PQ})$$

Some technical issues in geoid/height anomaly estimation

a) The knowledge of the Earth crust density

To estimate the geoid, computations involving the density of the crust are needed.

This is the case when considering the gravity reduction to the co-geoid or when estimating the geoid/height anomaly difference.

Models of the Earth crust density are available (mainly from geology) but no estimate of their reliability are usually given

For example, this has a critical impact on the the equation that is commonly used for checking the precision of geoid estimate, i.e. the well-known equation

$$h = H + N$$

If we consider a levelling line starting from P_0 on the geoid to the point Q on the Earth surface we have that the orthometric height of Q can be written as

$$H_Q = \Delta L_{P_0Q} + OC_{P_0Q} = \Delta L_{P_0Q} + \int_{P_0}^Q \frac{\Delta g}{\gamma} - H_Q \frac{\Delta g_Q}{\gamma} + \int_{Q_0}^Q 4\pi G \rho \frac{h}{\gamma} dh$$

where ΔL_{P_0Q} is the levelling increment, Δg is the gravity anomaly and Q_0 is on the geoid (G =gravity constant).

If geoid in Q is computed from height anomaly in the same point, we have that

$$\begin{aligned} N &= \zeta_Q + \int_{Q_0}^Q \frac{\Delta g}{\gamma} dh \cong \zeta_Q + H_Q \frac{\Delta g_Q}{\gamma} - \int_{Q_0}^Q 4\pi G \rho \frac{h}{\gamma} dh = \\ &= \frac{1}{\gamma} (T_Q + \Delta g_Q H_Q) - \int_{Q_0}^Q 4\pi G \rho \frac{h}{\gamma} dh \end{aligned}$$

The term in the green box is then the same but with opposite sign in the two equations. Thus an error in this term that depends on the density cannot be found in the equation $h = H + N$.

As one can prove that the error in the geoid due to this term amounts to

$$|\varepsilon(N)| \cong \frac{2\pi G \varepsilon(\rho) H^2}{\gamma_0}$$

an error $\varepsilon(\rho) = 0.2 \frac{g}{cm^3}$ gives $\varepsilon(N) \cong 0.9cm$ when $H = 1km$. So the equation $h=H+N$ cannot assess the geoid precision at cm level with such an error in the density (see also Sjöberg, Journal of Geodetic Science 8:30-33, 2018).

Also, density errors can directly affect the reduction of gravity data that are needed in the scheme that has been presented when discussing the Helmert-Stokes method.

b) Fitting geoid estimate to GNSS/Levelling (the hybrid *geoid*)

To fit gravimetric geoid estimates to GNSS/Levelling geoid derived values different interpolating surfaces are used. Among them, one can list:

- ✓ a simple bias
- ✓ a bias and a tilt
- ✓ higher order polynomials
- ✓ LSC on the residuals $\Delta N = N_{grav} - N_{GNSS/Lev}$

This is indeed a required procedure to get geoid undulation values to be used in transforming ellipsoidal heights h into orthometric heights H in e.g. engineering applications.

However, frequently, it is implicitly assumed that what is obtained after these transformation is still geoid. This is questionable and somehow misleading as not all these transformation have a clear physical interpretation.

Also, this can give an over-estimate of the geoid precision.

c) The tide system

As it well known, different tide systems are used in defining the geoid undulation. They are:

- ✓ the tide free
- ✓ the zero tide
- ✓ the mean tide

Transformations among the different tide systems are well defined. However, in the computations, this is frequently a hidden topic that is not explicitly mentioned in the solution metadata.

This is particularly important when dealing with IHRS/IHRF computations that are one-to-one with geoid/height anomaly estimations (one possible way of getting H_{IHRF} in one point is to use its existing geoid/height anomaly value).

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Thank you for your attention

and

my apologies

*as I'm pretty sure that in my presentation I forgot to
mention some important issues on geoid determination*