

CONSIDERING REPRO-BLUUE FOR NETWORK ADJUSTMENTS IN THE MODERNIZED NATIONAL SPATIAL REFERENCE SYSTEM

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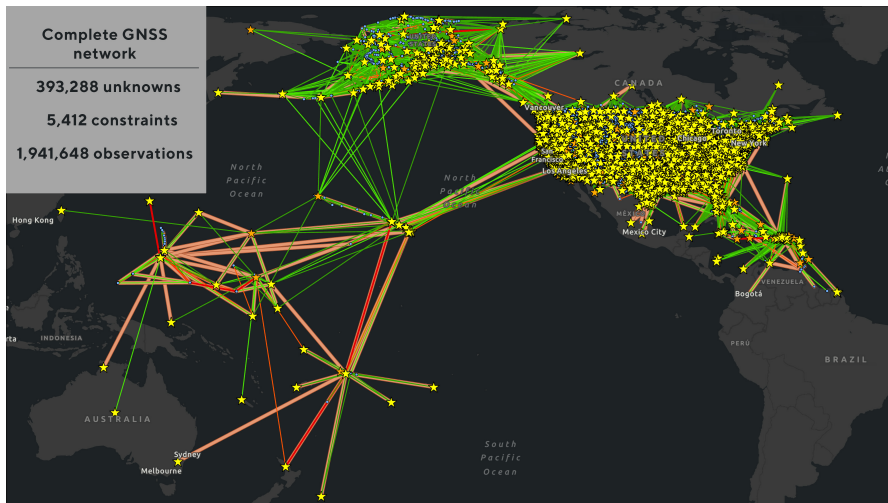
- LASER: A modern solver of least-squares adjustment problems for NGS
- Motivation for repro-BLUUE
 - Readjustment of national networks for the modernized NSRS
 - Keeping heights consistent at 53,000 points that are common to a 3D geodetic network and a leveling network
- repro-BLUUE: *reproducing Best Linear Uniformly Unbiased Estimator*
 - Review of Schaffrin's work: 1997–2012
 - A case for consideration by NGS
- Numerical example
- Conclusions and outlook
- References

Readjustment of national networks (1/5)

LASER: Least-squares Aadjustments: Statistics, Estimates, and Residuals

- LASER is a suite of libraries and programs for adjusting geodetic data.
- Adjustment types:
 1. 3D geodetic networks
 2. Leveling networks
 3. Gravity networks
 4. Calibration baselines
 5. Vertical gravity gradients
- Solution types:
 - Minimally constrained, aka free network adjustments
 - Overly constrained: fixed and stochastic constraints
- LASER is efficient.
 - Example network with 1.5 million observations and 300,000 unknowns:
 - Takes about 4 minutes when only the unknowns are solved for and about 4 hours when their covariance matrix is also computed.

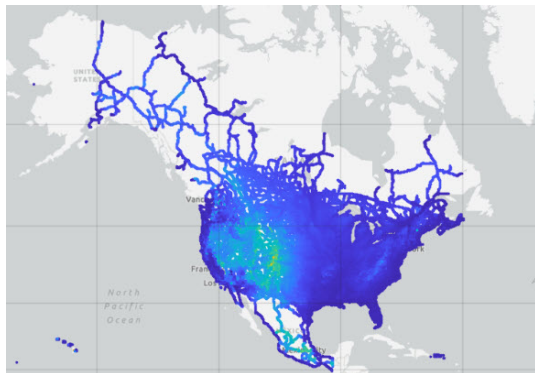
Readjustment of national networks (2/5)



A 3D geodetic network by NGS

Readjustment of national networks (3/5)

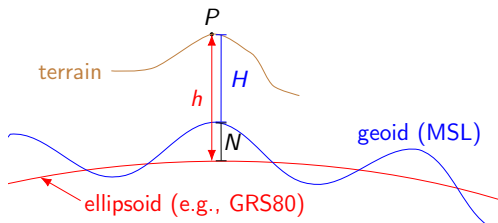
- The national networks span:
- CONUS (Contiguous United States)
- Alaska
- Hawaiian Islands
- Pacific Islands
- Puerto Rico and Virgin Islands
- Channel Islands



- The 3D network has approximately 132,000 points.
- The leveling network has over 1M points and over 2M observations.
- Around 53k points are common to both networks, which are adjusted independently.

Readjustment of national networks (4/5)

- The 53K common points are related by their respective heights.
- In the 3D geodetic network, computed heights h are geodetic.
- In the leveling network, estimated heights H are orthometric.



- Considering the geoid undulation N at the common points, the two types of heights at each point are related by the well-known Helmert projection $h = H + N$.
- Estimates for geoid undulations, and their variances, come from a separate process used to determine a new NGS geoid model called "GEOID2022."

Readjustment of national networks (5/5)

- Orthometric heights H at common points can be determined either by
 - $H = h - N$, with h from the adjustment of the geodetic network, or
 - directly from the leveling network adjustment.
- Suppose orthometric heights computed by $H = h - N$, with h from a geodetic network adjustment and N from a geoid model, will be used as control points in a leveling network adjustment.
- Moreover, suppose we want heights for the control points in the leveling network adjustment to be *reproduced* by the adjustment without losing or neglecting their variances that resulted from the geodetic network adjustment and the process used to determine the geoid.
- This is not unlike situations encountered in classical network densification problems, where repro-BLUE might be an appropriate estimator.

The partitioned GMM with stochastic constraints

1. GMM: The partitioned observation eqs. with their stochastic model read

$$\underset{n \times 1}{\mathbf{y}} = \underset{n \times m}{A} \underset{m \times 1}{\boldsymbol{\xi}} + \mathbf{e} = \begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{bmatrix} + \mathbf{e}, \quad \text{rk } A \leq m < n,$$
$$\mathbf{e} \sim (\mathbf{0}, \sigma_0^2 P_{n \times n}^{-1}).$$

2. GMM: The partitioned constraint eqs. with their stochastic model read

$$\underset{\ell \times 1}{\mathbf{z}_0} = \underset{\ell \times m}{K} \underset{m \times 1}{\boldsymbol{\xi}} + \mathbf{e}_0 = \begin{bmatrix} K_1 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{bmatrix} + \mathbf{e}_0, \quad \text{rk} \begin{bmatrix} A^\top & K^\top \end{bmatrix}_{m \times (n+\ell)} = m,$$
$$\mathbf{e}_0 \sim (\mathbf{0}, \underset{\ell \times \ell}{\sigma_0^2 Q_0} = \sigma_0^2 P_0^{-1}), \quad C\{\mathbf{e}, \mathbf{e}_0\} = 0.$$

- Define terms for the least-squares normal equations as

$$[N, \mathbf{c}] := \begin{bmatrix} A_1^\top \\ A_2^\top \end{bmatrix} P [A_1 \mid A_2, \mathbf{y}] \quad \text{and} \quad N_K := N + K^\top P_0 K.$$

The Least-Squares Solution (LESS) within the GMM with stochastic constraints $\hat{=}$ BLUE

- The LESS within the GMM with stoch. constraints is then provided by

$$\hat{\xi} = N_K^{-1}(\mathbf{c} + K^\top P_0 \mathbf{z}_0),$$

with dispersion matrix

$$D\{\hat{\xi}\} = \sigma_0^2 \cdot N_K^{-1} \quad \text{and its estimate} \quad \hat{D}\{\hat{\xi}\} = \hat{\sigma}_0^2 \cdot N_K^{-1}.$$

- The respective residual vectors are computed by

$$\tilde{\mathbf{e}} = \mathbf{y} - A\hat{\xi} \quad \text{and} \quad \tilde{\mathbf{e}}_0 = \mathbf{z}_0 - K\hat{\xi}.$$

- And the unknown variance component is estimated by

$$\hat{\sigma}_0^2 = \frac{\tilde{\mathbf{e}}^\top P \tilde{\mathbf{e}} + \tilde{\mathbf{e}}_0^\top P_0 \tilde{\mathbf{e}}_0}{n - m + \ell} =: \frac{\Omega}{r},$$

with model redundancy $r := n - m + \ell$, and Ω is called SSR.

- It can be shown that LESS w/in the GMM is the BLUE for this model.

Review of repro-BLUUE

Schaffrin 1997–2012

- In 1997, Burkhard Schaffrin derived the *best linear, uniformly unbiased estimator* with a “reproducing property,” such that

$$K\bar{\xi} = \mathbf{z}_0 \quad \text{and} \quad D\{K\bar{\xi}\} = \sigma_0^2 P_0^{-1} = \sigma_0^2 Q_0.$$

(i) *Linear*: $\bar{\xi} = L_1 \mathbf{y} + L_2 \mathbf{z}_0$, where L_1 and L_2 are TBD
 $m \times n$ $m \times \ell$

(ii) *Uniformly unbiased*:

$$\xi = E\{\bar{\xi}\} = (L_1 A) \cdot \xi + (L_2 K) \cdot \xi \quad \text{for all } \xi \in \mathbb{R}^m \Leftrightarrow L_1 A + L_2 K = I_m$$

(iii) *Reproducing property*:

$$\mathbf{0} = K\bar{\xi} - \mathbf{z}_0 = (KL_1) \cdot \mathbf{y} + (KL_2 - I_\ell) \cdot \mathbf{z}_0 \quad \text{for arbitrary } \mathbf{y} \in \mathbb{R}^n, \mathbf{z}_0 \in \mathbb{R}^\ell$$

$$\Leftrightarrow K \begin{bmatrix} L_1 & L_2 \end{bmatrix} = \begin{bmatrix} 0 & I_\ell \end{bmatrix} \Rightarrow A^T L_1^T K^T = 0 \quad \text{and} \quad K(I_m - L_2 K) = 0 \quad \text{due to (ii)}$$

(iv) *Best, i.e. minimum MSE/variance*:

$$\sigma_0^{-2} \cdot \text{tr MSE}\{\bar{\xi}\} = \sigma_0^{-2} \cdot \text{tr } D\{\bar{\xi}\} = \text{tr}(L_1 P^{-1} L_1^T) + \text{tr}(L_2 P_0^{-1} L_2^T) = \min$$

Review of repro-BLUE

Schaffrin 1997–2012

- The repro-BLUE turns out to be

$$\bar{\xi} = \hat{\xi} + K^{\top}(KK^{\top})^{-1}(\mathbf{z}_0 - K\hat{\xi}).$$

- Its dispersion matrix is given by

$$\begin{aligned} D\{\bar{\xi}\} &= \sigma_0^2 \cdot Q_{\bar{\xi}} = \\ &= D\{\hat{\xi}\} + \sigma_0^2 \cdot K^{\top}(KK^{\top})^{-1}(Q_0 - KN_K^{-1}K^{\top})(KK^{\top})^{-1}K, \end{aligned}$$

with its estimate as

$$\bar{D}\{\bar{\xi}\} = \bar{\sigma}_0^2 \cdot Q_{\bar{\xi}}.$$

- The estimated variance component is computed by

$$\begin{aligned} \bar{\sigma}_0^2 \cdot (n - 2m + 2\ell + \text{tr } NQ_{\bar{\xi}}) &= \\ &= (\tilde{\mathbf{e}}^{\top} P \tilde{\mathbf{e}} + \tilde{\mathbf{e}}_0^{\top} P_0 \tilde{\mathbf{e}}_0) + \tilde{\mathbf{e}}_0^{\top} (KK^{\top})^{-1} KN_K K^{\top} (KK^{\top})^{-1} \tilde{\mathbf{e}}_0. \end{aligned}$$

Repro-BLUUE when $K = [I_\ell, 0]$

e.g. in NGS leveling networks

- Repro-BLUUE for the unknown orthometric heights is computed by

$$\bar{\xi} = \begin{bmatrix} \mathbf{z}_0 \\ \hat{\xi}_2 \end{bmatrix} \quad \text{in contrast to} \quad \hat{\xi} = \begin{bmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \end{bmatrix}.$$

- Repro-BLUUE's estimated dispersion matrix is provided by

$$\bar{D}\{\bar{\xi}\} = \bar{\sigma}_0^2 \cdot \begin{bmatrix} Q_0 & (N_K^{-1})_{12} \\ (N_K^{-1})_{21} & (N_K^{-1})_{22} \end{bmatrix} = \bar{\sigma}_0^2 \cdot Q_{\bar{\xi}},$$

in contrast to

$$\hat{D}\{\hat{\xi}\} = \hat{\sigma}_0^2 \cdot N_K^{-1} = \hat{\sigma}_0^2 \cdot \begin{bmatrix} (N_K^{-1})_{11} & (N_K^{-1})_{12} \\ (N_K^{-1})_{21} & (N_K^{-1})_{22} \end{bmatrix} = \hat{\sigma}_0^2 \cdot Q_{\hat{\xi}}.$$

- Finally, Repro-BLUUE's estimated variance component is computed by

$$\begin{aligned} \bar{\sigma}_0^2 \cdot (n - 2m + 2\ell + \text{tr } NQ_{\bar{\xi}}) &= \\ &= \tilde{\mathbf{e}}^\top P \tilde{\mathbf{e}} + \tilde{\mathbf{e}}_0^\top P_0 \tilde{\mathbf{e}}_0 + \tilde{\mathbf{e}}_0^\top (A_1^\top P A_1 + P_0) \tilde{\mathbf{e}}_0 \end{aligned}$$

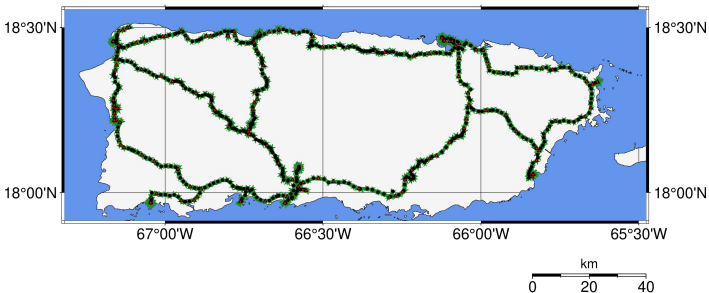
$$\Rightarrow \bar{\sigma}_0^2 = (r - m + \ell + \text{tr } NQ_{\bar{\xi}})^{-1} [r \cdot \hat{\sigma}_0^2 + \tilde{\mathbf{e}}_0^\top (A_1^\top P A_1 + P_0) \tilde{\mathbf{e}}_0].$$

Summarizing repro-BLUE when $K = [I_\ell, 0]$

- Steps to obtain repro-BLUE:
 1. Compute LESS within the GMM with stochastic constraints ($\hat{=}$ BLUE).
 2. Replace the resulting estimates $\hat{\xi}_1$ at the constrained points with their corresponding constrained values z_0 .
 3. Replace the upper $\ell \times \ell$ block of BLUE's cofactor matrix Q_ξ with the cofactor matrix for the stochastic constraints Q_0 to obtain the repro-BLUE cofactor matrix $Q_{\bar{\xi}}$.
 4. Compute $\text{tr } NQ_{\bar{\xi}}$ for use in the estimation of repro-BLUE's variance component $\bar{\sigma}_0^2$. Note that
 - Q_ξ , from which $Q_{\bar{\xi}}$ is derived, may be a very large ($m \times m$) and full matrix computed a few columns at a time with only its diagonal elements retained in computer memory.
 - However, N is typically very sparse, and the trace $\text{tr } NQ_{\bar{\xi}}$ only requires m dot products; thus the trace can be efficiently computed and accumulated when computing the columns of $Q_{\bar{\xi}}$ (with BLUE).

Numerical example

Leveling network in Puerto Rico



LASER output:

Number of observations	1566	Number of closed loops/spurs	16
Number of parameters	-665	Density of norm. eq. matrix	0.5%
Number of constraints	+174	Density of cov. matrix	100%

Model redundancy	1075		

Numerical example

Repro-BLUUE computations for a leveling network in Puerto Rico

- The following results show that at the level of the *square root of the variance component*, repro-BLUUE is slightly over *twice as large* as that of BLUEU ($\hat{=}$ LESS within the GMM with stochastic constraints).

Quantity	Math. Expression/Symbol	Value
SSR BLUEU	$\Omega := \tilde{\mathbf{e}}^\top P \tilde{\mathbf{e}} + \tilde{\mathbf{e}}_0^\top P_0 \tilde{\mathbf{e}}_0$	131.27
Model redundancy	$r = n - m + \ell$	1075
Quad. term for repro-BLUUE	$\tilde{\mathbf{e}}_0^\top (A_1^\top P A_1 + P_0) \tilde{\mathbf{e}}_0$	2170.25
Trace term for repro-BLUUE	$\text{tr } N Q_{\tilde{\xi}}$	3686.15
Auxiliary term for repro-BLUUE	$n - 2m + 2\ell$	584
Est. VC BLUEU (sqrt)	$\hat{\sigma}_0^2 = \Omega / r$	0.12 (0.35)
Est. VC repro-BLUUE (sqrt)	$\bar{\sigma}_0^2$	0.54 (0.73)

$$\bar{\sigma}_0^2 = (n - 2m + 2\ell + \text{tr } N Q_{\tilde{\xi}})^{-1} [r \cdot \hat{\sigma}_0^2 + \tilde{\mathbf{e}}_0^\top (A_1^\top P A_1 + P_0) \tilde{\mathbf{e}}_0]$$

$$\hat{\sigma}_0^2 = \frac{131.27}{1075} = 0.12, \quad \bar{\sigma}_0^2 = \frac{131.27 + 2170.25}{584 + 3686.15} = 0.54$$

Additional examples

Variance component computations

Network	n	m	ℓ	r	$\sqrt{\hat{\sigma}_0^2}$	$\sqrt{\bar{\sigma}_0^2}$	Increase in sqrt VC
Culebra	56	17	5	44	0.56	0.77	38%
Guam	897	364	32	565	1.09	0.54	-50%
Johnston	57	26	1	32	0.15	0.15	0%
Maui	1296	572	5	729	0.39	0.45	15%
Puerto Rico	1566	665	174	1075	0.35	0.73	109%
Saipan	148	74	10	84	0.15	0.35	133%
St Croix	66	33	12	45	0.22	0.40	82%
St John	89	44	5	50	0.08	0.16	100%
St Thomas	36	19	12	29	0.14	0.19	36%
Tinian	95	46	29	78	0.22	0.27	23%
Tutulia	231	112	46	165	0.25	0.22	-12%
Vieques	110	34	9	85	0.18	0.48	167%
n = num. observations, m = num. unknowns						Min	-50%
ℓ = num. constraints						Max	167%
r = model redundancy						Median	37%

Conclusions and outlook

- Is updating the NGS least-squares adjustment program LASER to handle repro-BLUUE for the national leveling network feasible? Apparently so. . .
 - Additional computations are not complicated.
 - But updates to the code would not be trivial due to LASER's special logic for handling large systems of equations.
- Ideally, LASER would also be extended to allow for both unknown 3D and 1D points to be estimated in a *single* network adjustment.
 - It currently allows for leveling data to be included in 3D geodetic networks, but only among 3D points.
 - This would not guarantee that NGS computers could simultaneously adjust the national geodetic and leveling networks in a single adjustment, as we do not know at this time if they are powerful enough for that.
 - But at least for smaller networks it would allow the results of such combined adjustments to be directly compared to leveling network adjustments based on repro-BLUUE as described here.

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