

Green's functions in geodesy

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- reviewing solutions to BVPs for the static gravity field
 - integral (classical) solutions based on surface integrals
 - integrating convolutions of input data with Green's functions ("GFs")
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- reviewing GFs in geodesy and decomposing their series representation
 - adopting GFs to imperfect observations of geopotential functionals
 - case study: GFs transforming airborne gravity vectors
 - summary

Green's functions ("GFs") in general

- problems governed by linear differential equations $\mathcal{L}u(\mathbf{y}) = f(\mathbf{x})$
- continuous geopotential in 3D space satisfies Poisson's DE
- because of linearity, PDE can be solved for a **source** f at discrete $\mathbf{x}'$
- total **effect** u at the point $\mathbf{x}$ compiled using integral convolution:

$$u(\mathbf{x}) = \int f(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') d\mathbf{x}'$$

- integral kernel G reflects physical properties of 3D space
- GFs have many interesting (and useful) properties

Examples of GFs applied in geodesy

GF	input ("source")	output ("effect")
Newton	density variations	gravitational potential
Stokes	gravity anomaly	geoid separation
Hotine	gravity disturbance	geoid separation
Abel-Poisson	surface potential	external potential
Farrell	surface loading mass	crustal deformation
Okada	fault dislocation	displacement/tilt/stress

GFs for transformation of potential gradients

- integral transformations of **rank- k** potential gradients ($t = R/r \leq 1$):

$$T(r, \Omega) = \int T_{\xi}^{(k)}(R, \Omega') G_{\xi}^{(k)}(t, \psi, \alpha') d\Omega'$$

- isotropic integral kernel for **radial gradients** ($\xi = r$):

$$G_r^{(k)}(t, \psi) = (-1)^k R^k \sum_{n=2}^{\infty} (2n+1) \frac{n!}{(n+k)!} t^{n+1} P_n(\psi)$$

- anisotropic integral kernel for **angular gradients** ($\xi = \Omega$):

$$G_{\Omega}^{(k)}(t, \psi, \alpha') = R^k \sum_{n=k}^{\infty} (2n+1) \frac{(n-k)!}{(n+k)!} t^{n+1} P_{n,k}(\psi) e^{-ik\alpha'}$$

- mixed gradients can be derived by differentiating angular gradients

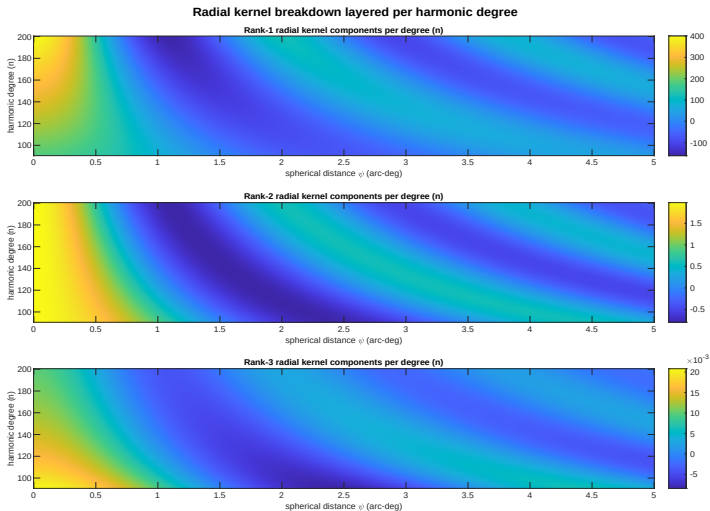
Generalized GF transforming gradients into potential

- generalized integral kernel for rank- k gradients:

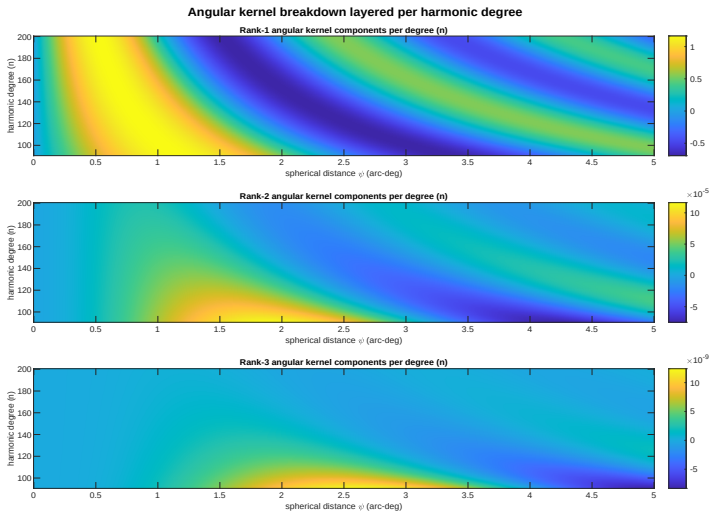
$$G^{(k)}(t, \psi, \alpha') = R^k \sum_{n=k}^{\infty} \frac{2n+1}{\lambda_n^{(k)}} t^{n+1} P_{n,k}(\psi) e^{-ik\alpha'}$$

term	mathematical role	meaning
R^k	integral normalization	dimensional consistency
$2n+1$	dim of harmonic space	geometric weighting
$\frac{1}{\lambda_n^{(k)}}$	reciprocal eigenvalue	low-pass filtering
t^{n+1}	radial power law	attenuation
$P_{n,k}$	basis function	angular dependency
$e^{-ik\alpha'}$	rotation vector	azimuthal dependency

Radial kernels per distance & harmonic degree

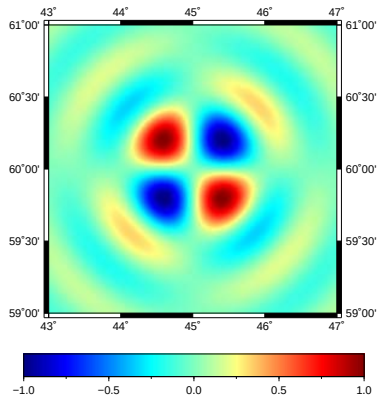


Angular kernels per distance & harmonic degree

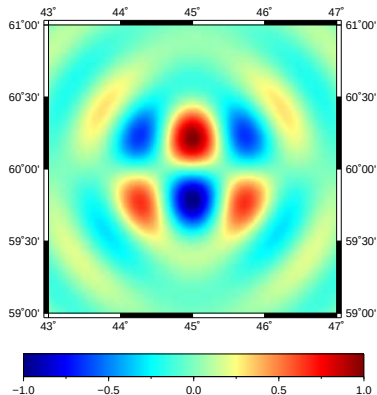


Azimuth-dependent (anisotropic) GFs

a) latitude-longitude



b) latitude-longitude-longitude



- kernel magnitudes are normalized as their shape is of interest

GFs in practical applications – modifications

- gradient data (signal) are noisy, spectrally and spatially limited
- GFs as integral kernels must be modified to get the best of data
- integrals are discretized to accommodate discrete observations
- integral kernels can/must be
 - spectrally limited to fit the spectrum of observations
 - stochastically modified to reduce observation noise
 - deterministically modified to reduce far-field effects

Stochastically-modified GFs as integral kernels

- integral transformation for rank-2 radial gradient T_{rr} :

$$T(R, \Omega) = \int T_{rr}(R, \Omega') \left[\sum_{n=2}^{\infty} (2n+1) A_n^{-1} P_n(\psi) \right] d\Omega'$$

- estimator with spectral transfer functions $A_n = (n+1)(n+2)/R^2$:

$$\hat{T}_{nm} = W_n A_n^{-1} (T_{rr})_{nm}$$

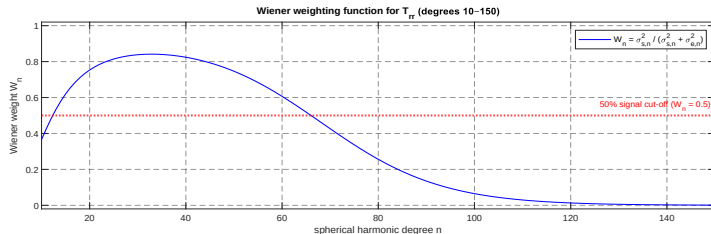
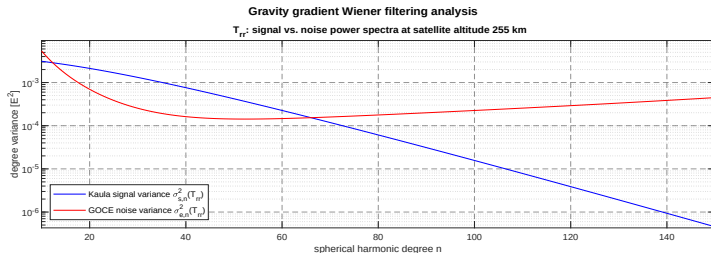
- minimizing $\mathbb{E} |\hat{T}_{nm} - T_{nm}|^2$ yields spectral weights $W_n \in [0, 1]$:

$$W_n = \frac{A_n^2 \sigma_{s,n}^2(T)}{A_n^2 \sigma_{s,n}^2(T) + \sigma_{e,n}^2}$$

- **stochastically-modified integral kernel** with $k_n = (2n+1)/A_n$:

$$G_{rr}^W(\psi) = \sum_{n=2}^{\infty} W_n k_n P_n(\psi)$$

Spectral filtering analysis for rank-2 radial gradients



Unified GFs for rank-1 radial gradients

- limited and noise-contaminated radial gradients:

$$\hat{T}(R, \Omega) = \int_{\Omega_0} G_r^*(\psi) \tilde{T}_r(R, \Omega') d\Omega'$$

- unified kernel function:

$$G_r^*(\psi) = \sum_{n=\min}^{\max} W_n k_n P_n(\psi) - \sum_{m=0}^N \frac{2m+1}{2} s_m P_m(\psi)$$

- modification coefficients s_n by minimizing global estimation error:

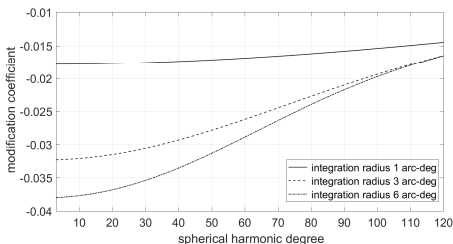
$$\sum_{n=\min}^{\max} \left\{ [W_n k_n - s_n]^2 \sigma_{e,n}^2(T_r) + [(1 - W_n) k_n + Q_n(\psi_0)]^2 \sigma_{s,n}^2(T_r) \right\}$$

- truncation coefficients Q_n model signal leakage through ψ_0 :

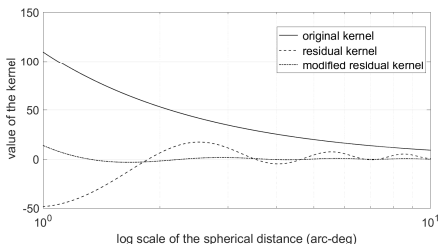
$$Q_n(\psi_0) = \sum_{k=\min}^{\max} (W_k k_k - s_k) e_{kn}(\psi_0)$$

Examples of modified GFs

a) modification coefficients for selected integrations domains



b) original, degree-limited and modified degree-limited integral kernel



Transformation of airborne rank-1 potential gradients

- vector integral transformation:

$$T(R, \Omega) = \int_{\Omega_0} \left[G_r(\psi, \tau) T_r(r, \Omega') + \mathbf{G}_\Omega(\psi, \tau, \alpha') \cdot \mathbf{T}_\Omega(r, \Omega') \right] d\Omega'$$

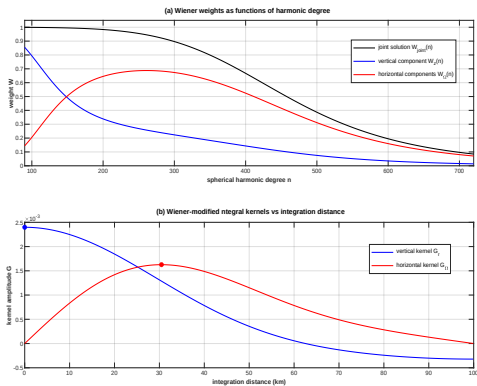
- weighted integral kernels with $\tau = 1/t > 1$:

$$G_r(\psi, \tau) = \sum_{n=\min}^{\max} k_n^{(r)} W_n^{(r)} \tau^{n+1} P_n(\psi)$$
$$\mathbf{G}_\Omega(\psi, \tau, \alpha') = \sum_{n=\min}^{\max} k_n^{(\Omega)} W_n^{(\Omega)} \tau^{n+1} P_{n,1}(\psi) \mathbf{f}(\alpha')$$

- spectral weights for the combined solution:

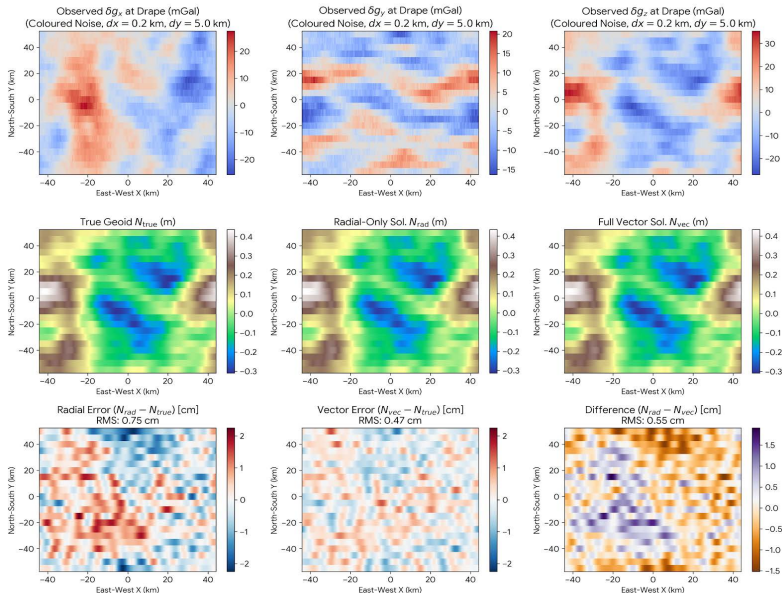
$$W_n^{(r)} = \frac{\sigma_{e,n}^2(\Omega)}{\sigma_{e,n}^2(\Omega) + n^2 \sigma_{e,n}^2(r)} , \quad W_n^{(\Omega)} = \frac{n^2 \sigma_{e,n}^2(r)}{\sigma_{e,n}^2(\Omega) + n^2 \sigma_{e,n}^2(r)}$$

Spectral weights and modified kernels for rank-1 gradients

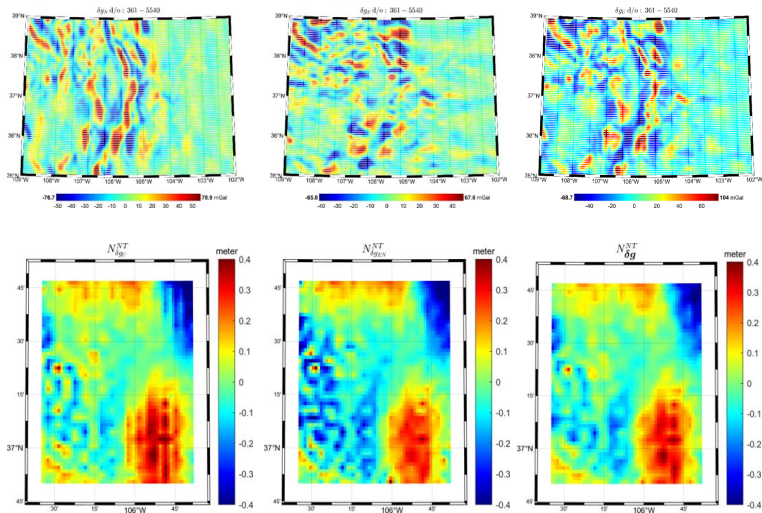


- vertical weight dominates $n < 150$, joint weight < 0.5 at $n = 460$
- vertical kernel $< 10\%$ beyond 60 km, horizontal kernel peaks at 30 km
- integration beyond $\approx 40\text{--}50$ km yields negligible contribution

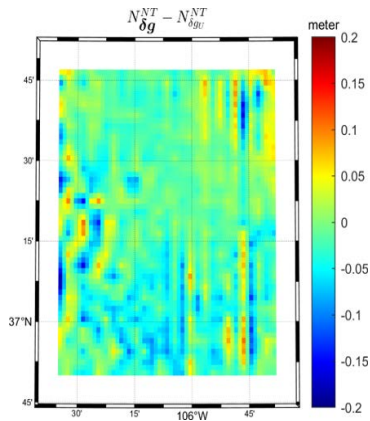
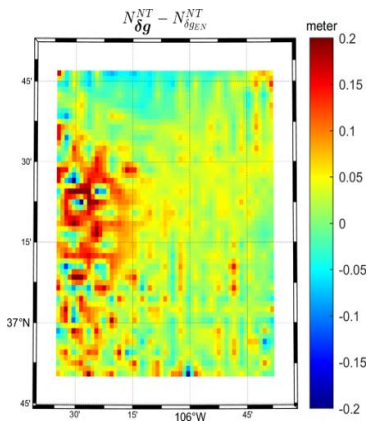
Transforming **synthetic** airborne rank-1 gradient vectors



Transforming **real** airborne rank-1 gradient vectors

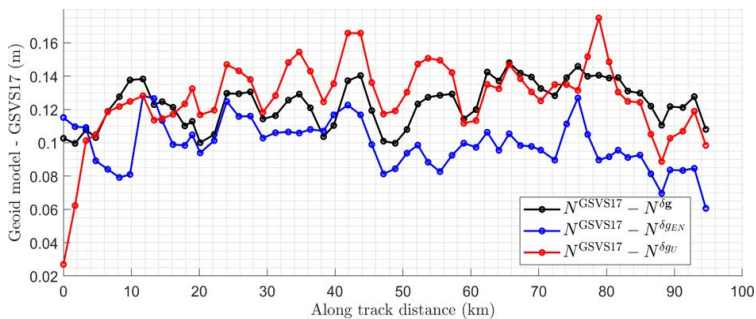


Differences of the solutions



- mean differences are 5.6 (full – horizontal) and 4.1 cm (full – radial)

Solutions relatively to ground reference (GSVS2017)



gradient	min	max	mean	std
radial	0.027	0.175	0.127	0.023
angular	0.061	0.128	0.099	0.014
full vector	0.100	0.148	0.124	0.013

Summary

- GFs transforming geodetic signals have interesting properties
- when applied with real data, following issues must be considered:
 - data noise amplification
 - near-field integration, truncation errors and edge effects
 - irregular data sampling (along- vs cross-track)
 - topographic mass violations
 - computational complexity, e.g., from $\mathcal{O}(N^2)$ to $\mathcal{O}(N^3)$
- new GFs (with new data) improve residual gravity field recovery
- synthetic and real vector data report up to 40% improvement