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**Spectral characteristics
of forward–modelled global gravitational fields
for selected planetary bodies**

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1. Introduction:

- Gravitational fields may be characterised by spherical harmonic coefficients (SHCs),
- **Useful measures:** Root–Mean–Square (RMS) of SHCs per degree and its statistical fits,
- **Applications:** error propagation, optimisation, experimental designs, or least–squares constraints,
- **Problem:** vertical changes of spectral characteristics around topography.

2. Theoretical aspects:

A) Newton's integral:

$$V(r, \Omega) = G \int_{\Omega'} \int_{r'=0}^{r_s(\Omega')} \varrho(r', \Omega') \mathcal{L}^{-1}(r, \Omega, r', \Omega') r'^2 dr' d\Omega'$$

Notation:

$r, \Omega = (\varphi, \lambda)$ – spherical planetocentric coordinates
of the evaluation point P ,

$r', \Omega' = (\varphi', \lambda')$ – spherical planetocentric coordinates
of the volume integration element Q ,

r_s – radius of the topographic surface,

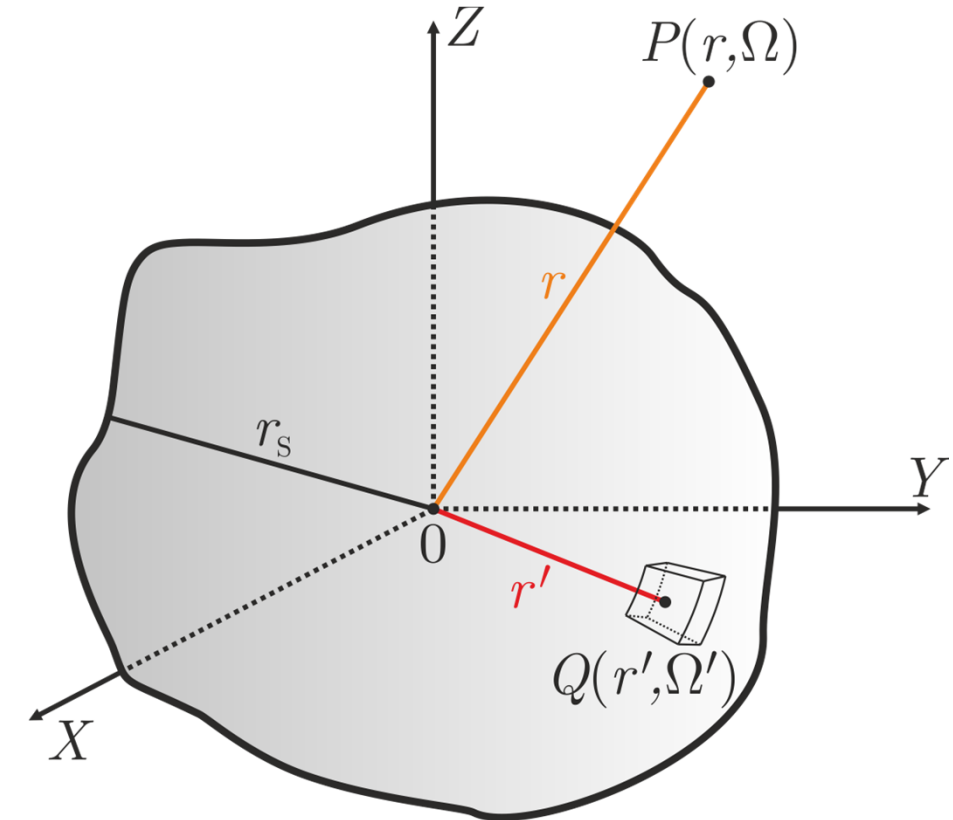
G – universal gravitational constant,

ϱ – volumetric density,

$\mathcal{L}$ – Euclidean distance,

$$\int_{\Omega'} (\cdot) d\Omega' = \int_{\lambda'=0}^{2\pi} \int_{\varphi'=-\frac{\pi}{2}}^{+\frac{\pi}{2}} (\cdot) \cos \varphi' d\varphi' d\lambda'.$$

Geometry:



B) Internal spherical harmonic series:

$$V(r, \Omega) = \frac{GM}{R_o} \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} \left(\frac{R_o}{r} \right)^{n+1} \bar{C}_{n,m}^o(r) \bar{Y}_{n,m}(\Omega) \\ + \frac{GM}{R_i} \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} \left(\frac{r}{R_i} \right)^n \bar{C}_{n,m}^i(r) \bar{Y}_{n,m}(\Omega)$$

Notation:

M – mass of the gravitating body,

R_o, R_i – scale factors,

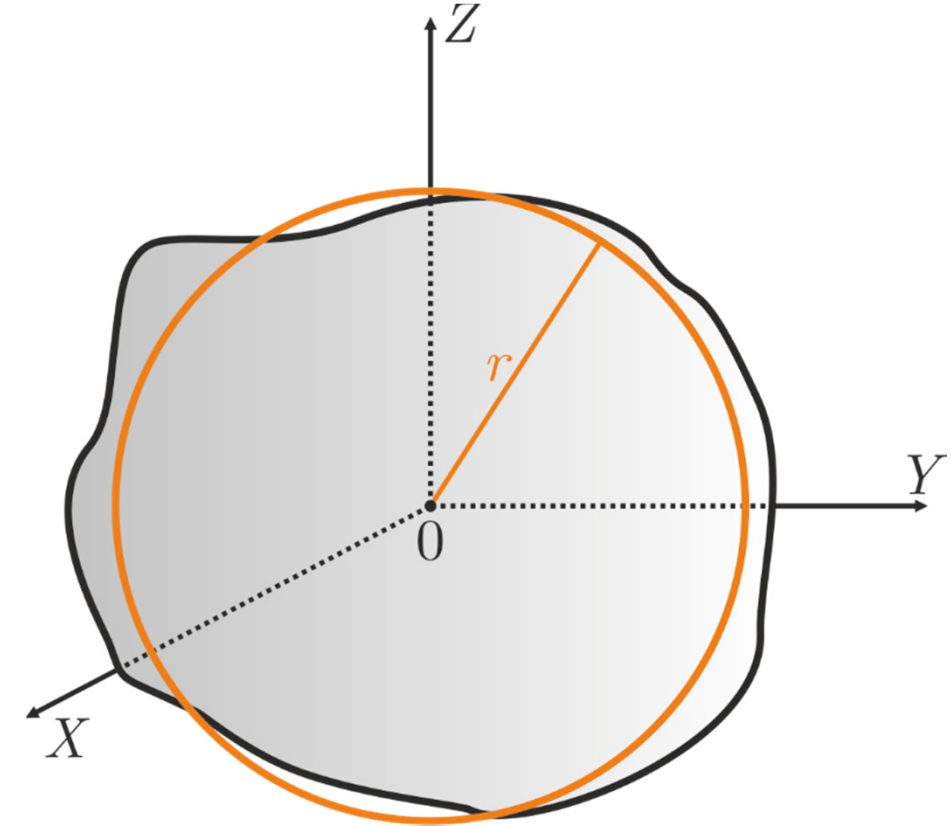
$\bar{Y}_{n,m}$ – spherical harmonic function,

Spherical harmonic coefficients:

$$\bar{C}_{n,m}^o(r) = \frac{1}{R_o^n M(2n+1)} \int_{\Omega'} \int_{r'=0}^r \varrho(r', \Omega') \bar{Y}_{n,m}(\Omega') r'^{n+2} dr' d\Omega',$$

$$\bar{C}_{n,m}^i(r) = \frac{R_i^{n+1}}{M(2n+1)} \int_{\Omega'} \int_{r'=r}^{r_s(\Omega')} \varrho(r', \Omega') \bar{Y}_{n,m}(\Omega') \left(\frac{1}{r'} \right)^{n-1} dr' d\Omega'.$$

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Notes on $\bar{C}_{n,m}^o$ and $\bar{C}_{n,m}^i$:

- Are not constant and vary with r ,
- Can only be determined by forward modelling, i.e. from ϱ and r_s .

C) External spherical harmonic series:

$$V(r, \Omega) = \frac{GM}{R} \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} \left(\frac{R}{r}\right)^{n+1} \bar{C}_{n,m} \bar{Y}_{n,m}(\Omega)$$

Notation:

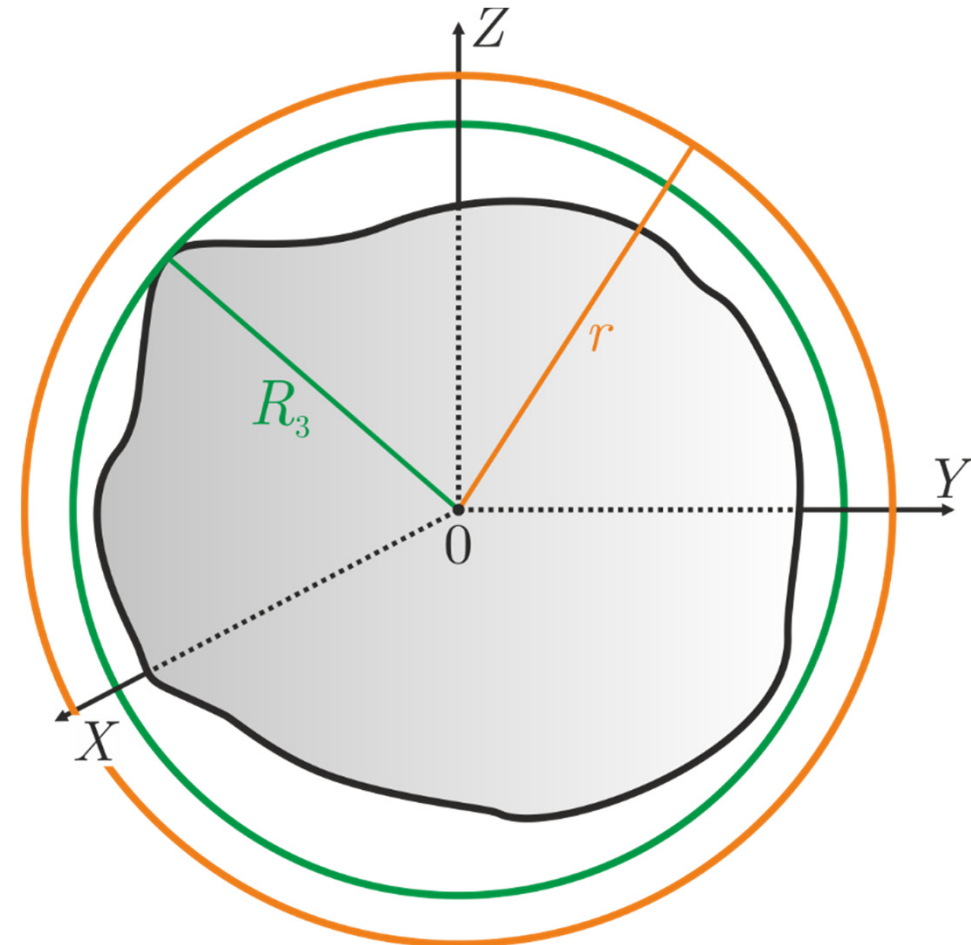
R – scale factor,

R_3 – radius of the minimum Brillouin sphere,

Spherical harmonic coefficient:

$$\bar{C}_{n,m} = \frac{1}{R^n M(2n+1)} \int_{\Omega'} \int_{r'=0}^{r_s(\Omega')} \varrho(r', \Omega') \bar{Y}_{n,m}(\Omega') r'^{n+2} dr' d\Omega'.$$

Geometry:



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Notes on $\bar{C}_{n,m}$:

- Are constant and define physical properties of the gravitating body,
- Can be determined by forward modelling or from gravitational field observations.

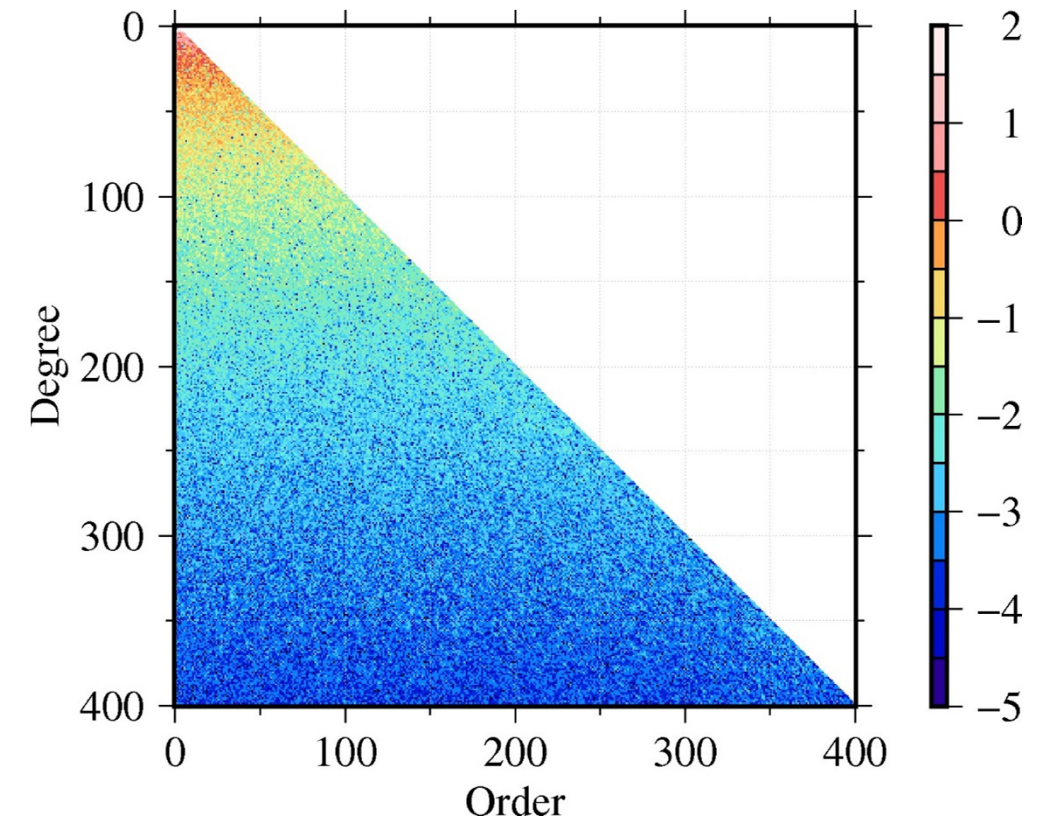
D) Spectral characteristics – 2D spectrum:

Definition:

$$\text{Mod}C_{n,m}(r) = \begin{cases} \left| \frac{GM}{R_o} \left(\frac{R_o}{r} \right)^{n+1} \bar{C}_{n,m}^o(r) \right. \\ \left. + \frac{GM}{R_i} \left(\frac{r}{R_i} \right)^n \bar{C}_{n,m}^i(r) \right|, \forall r < R_3 \\ \left| \frac{GM}{R} \left(\frac{R}{r} \right)^{n+1} \bar{C}_{n,m} \right|, \forall r \geq R_3 \end{cases}$$

Example:

(GL660B @ $R_3=1748.20$ km)



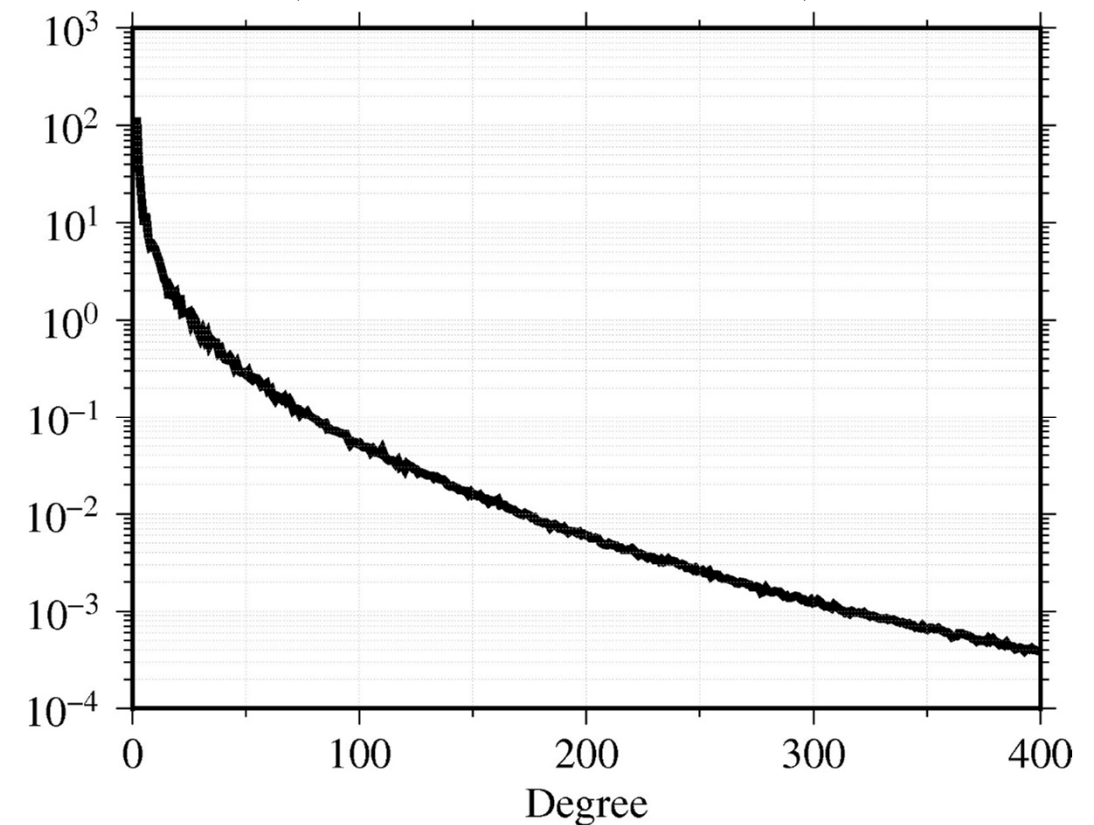
E) Spectral characteristics – 1D spectrum:

Definition:

$$\text{RMS}_n(r) = \begin{cases} \frac{GM}{\sqrt{2n+1}} \left\{ \sum_{m=-n}^{+n} \left[\frac{1}{R_o} \left(\frac{R_o}{r} \right)^{n+1} \bar{C}_{n,m}^o(r) + \frac{1}{R_i} \left(\frac{r}{R_i} \right)^n \bar{C}_{n,m}^i(r) \right]^2 \right\}^{1/2}, & \forall r < R_3 \\ \frac{GM}{R\sqrt{2n+1}} \left[\sum_{m=-n}^{+n} \left(\frac{R}{r} \right)^{2n+2} \bar{C}_{n,m}^2 \right]^{1/2}, & \forall r \geq R_3 \end{cases}$$

Example:

(GL660B @ $R_3=1748.20$ km)



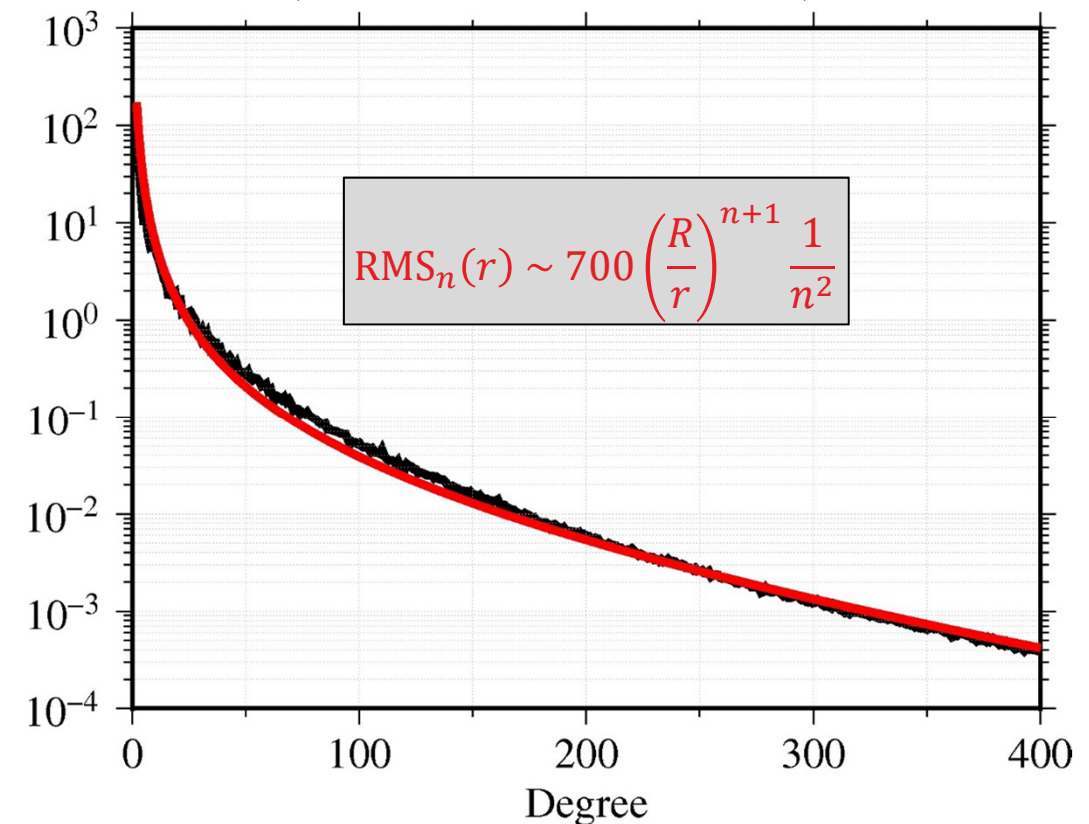
E) Spectral characteristics – 1D spectrum:

Definition:

$$\text{RMS}_n(r) = \begin{cases} \frac{GM}{\sqrt{2n+1}} \left\{ \sum_{m=-n}^{+n} \left[\frac{1}{R_o} \left(\frac{R_o}{r} \right)^{n+1} \bar{C}_{n,m}^o(r) + \frac{1}{R_i} \left(\frac{r}{R_i} \right)^n \bar{C}_{n,m}^i(r) \right]^2 \right\}^{1/2}, & \forall r < R_3 \\ \frac{GM}{R\sqrt{2n+1}} \left[\sum_{m=-n}^{+n} \left(\frac{R}{r} \right)^{2n+2} \bar{C}_{n,m}^2 \right]^{1/2}, & \forall r \geq R_3 \end{cases}$$

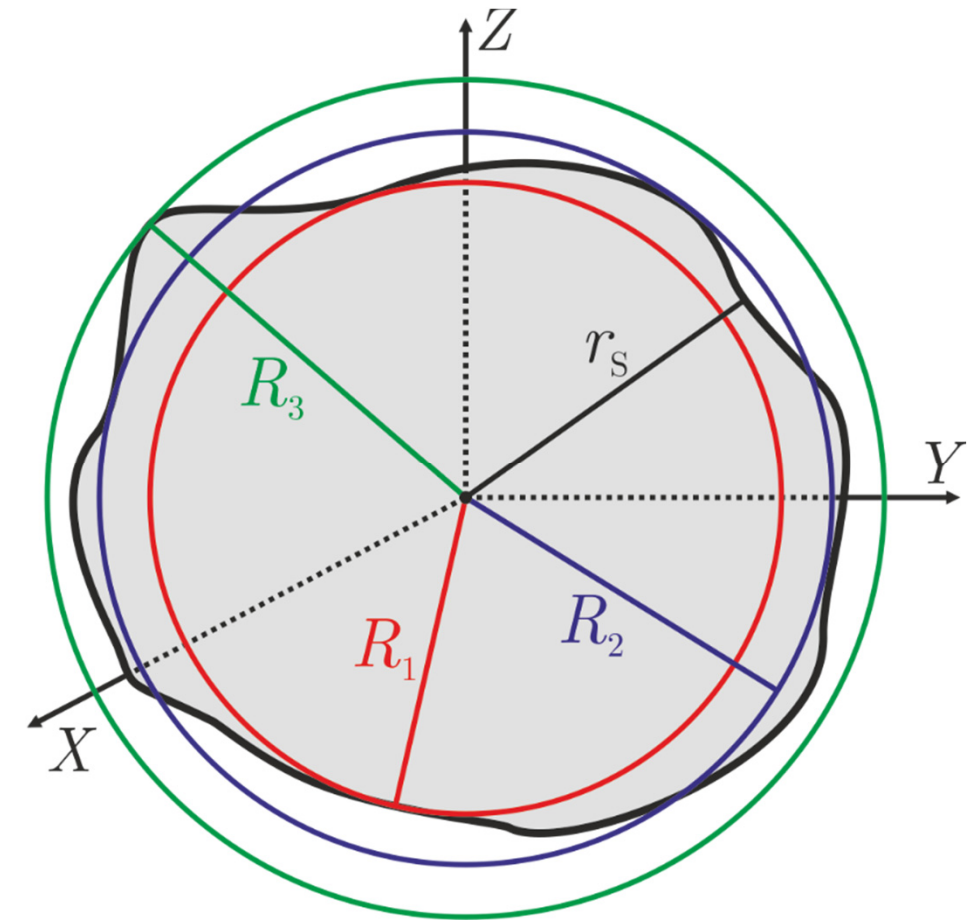
Example:

(GL660B @ $R_3=1748.20$ km)



3. Numerical experiments:

- Planetary bodies: Bennu, Ceres, Eros, Mars, Moon, Ryugu, and Vesta,
- Analysis of gravitational fields
@ R_1 (bottom), R_2 (best fit), and R_3 (top),
- Method: Rigorous Spectral Forward Modelling (Šprlák et al. 2018),
- Data: constant density and detailed topography.



r_s – topographic surface,
 $R_1 = \min(r_s)$,

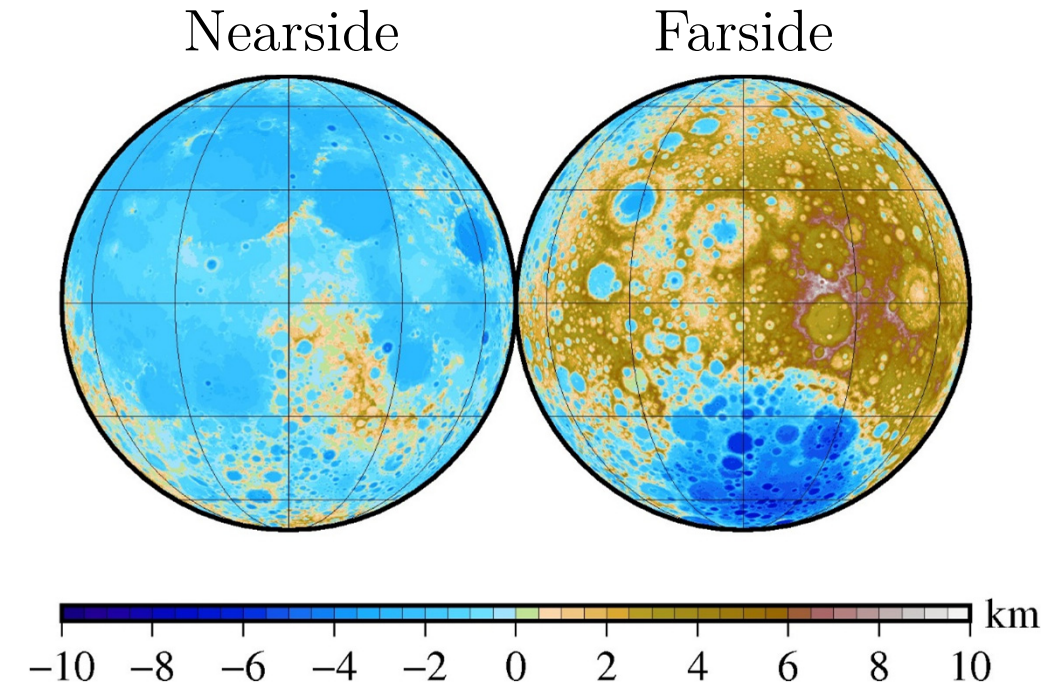
R_2 – best fitting sphere,
 $R_3 = \max(r_s)$.

A) Example 1 – the Moon:

Basic parameters:

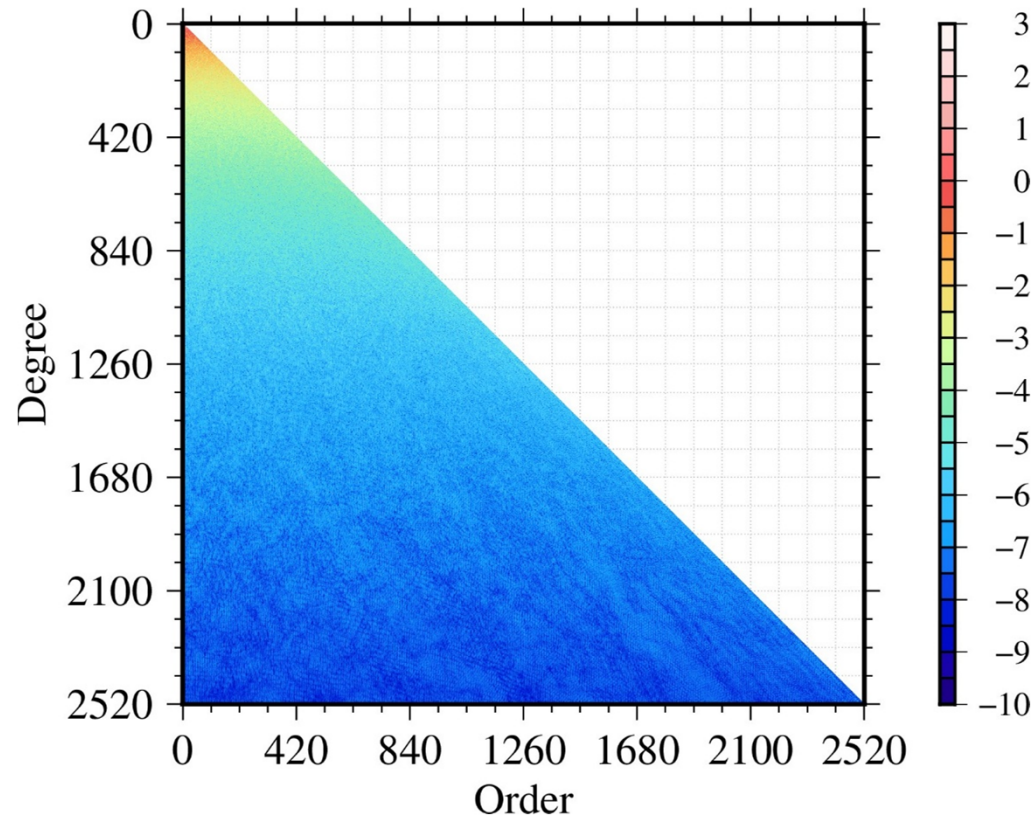
- Spherical shape, $f \approx 3.1 \times 10^{-4}$,
(Ardalan and Karimi 2014),
- Maximum spherical harmonic degree 2519,
- LOLA topography (Smith et al. 2010)
from LRO, resolution $1/14^\circ$,
- Volumetric density 2550 kg/m^3
(Wieczorek et al. 2013).

Topography: (wrt. $R=1737.4 \text{ km}$)

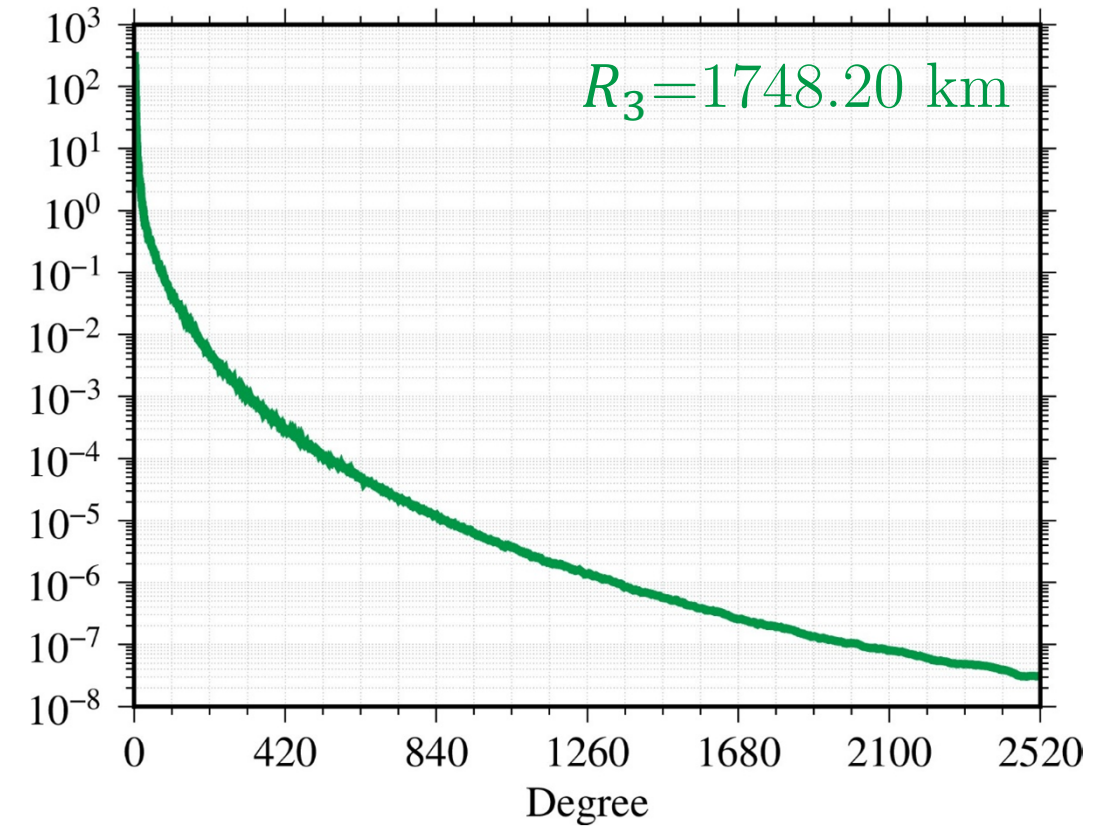


Spectral characteristics (@ $R_3=1748.20$ km, top):

$\text{Mod}C_{n,m}$:

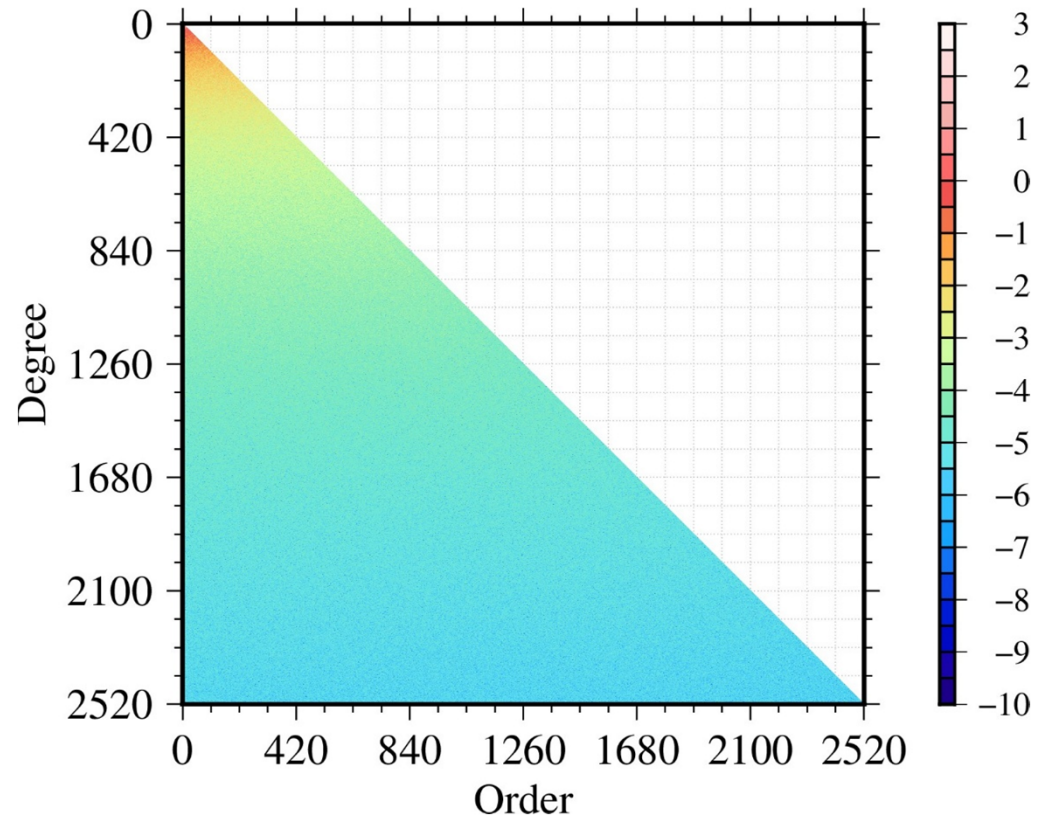


RMS_n :

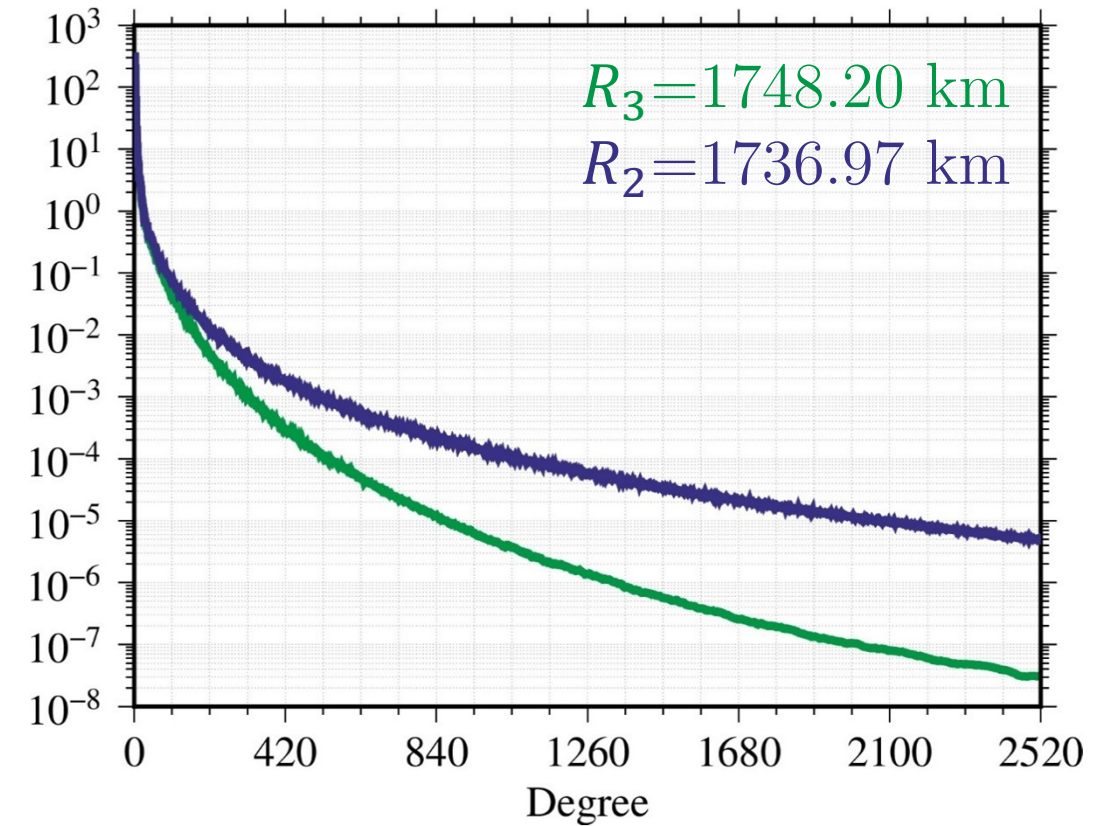


Spectral characteristics (@ $R_2=1736.97$ km, best fit):

$\text{Mod}C_{n,m}$:

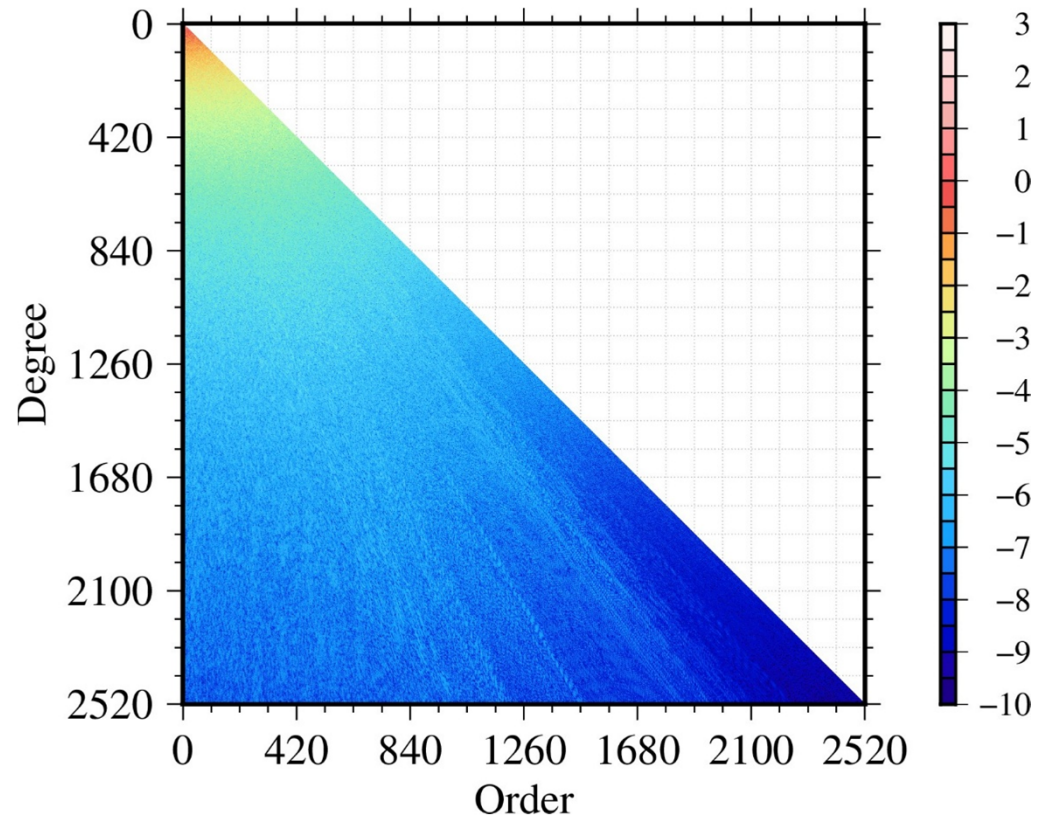


RMS_n :

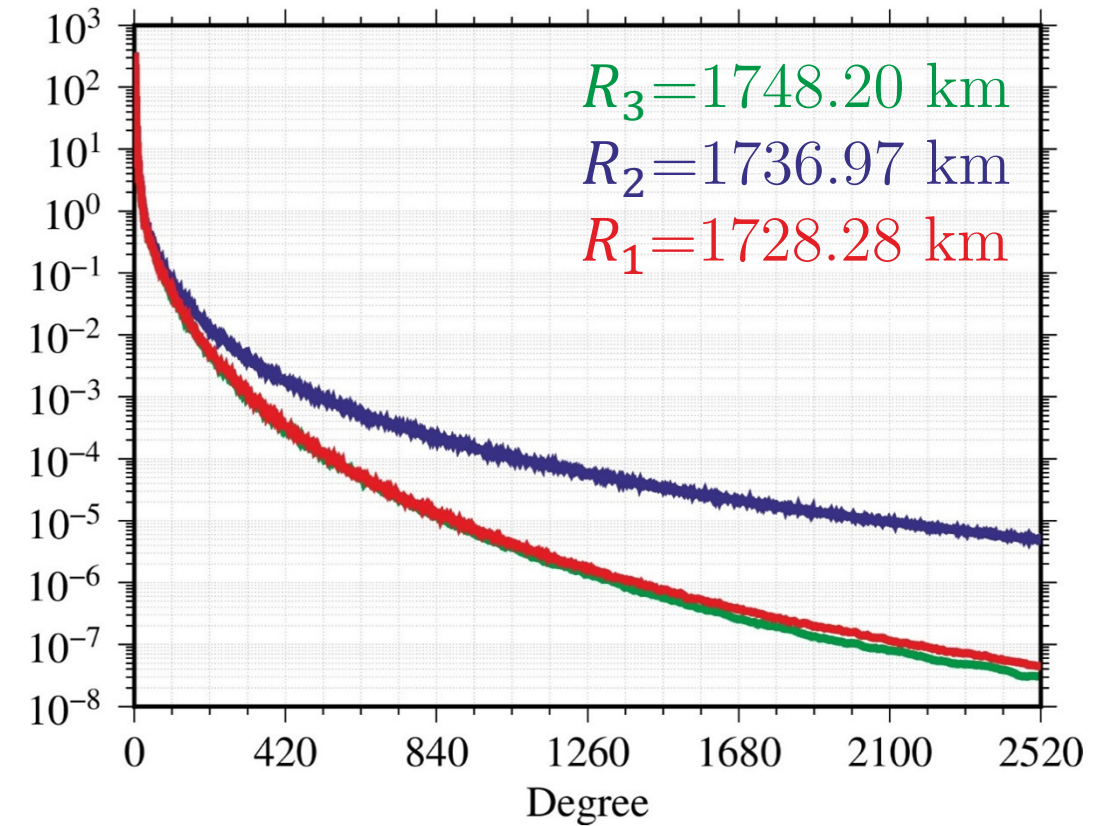


Spectral characteristics (@ $R_1=1728.28$ km, bottom):

$\text{Mod}C_{n,m}$:



RMS_n :

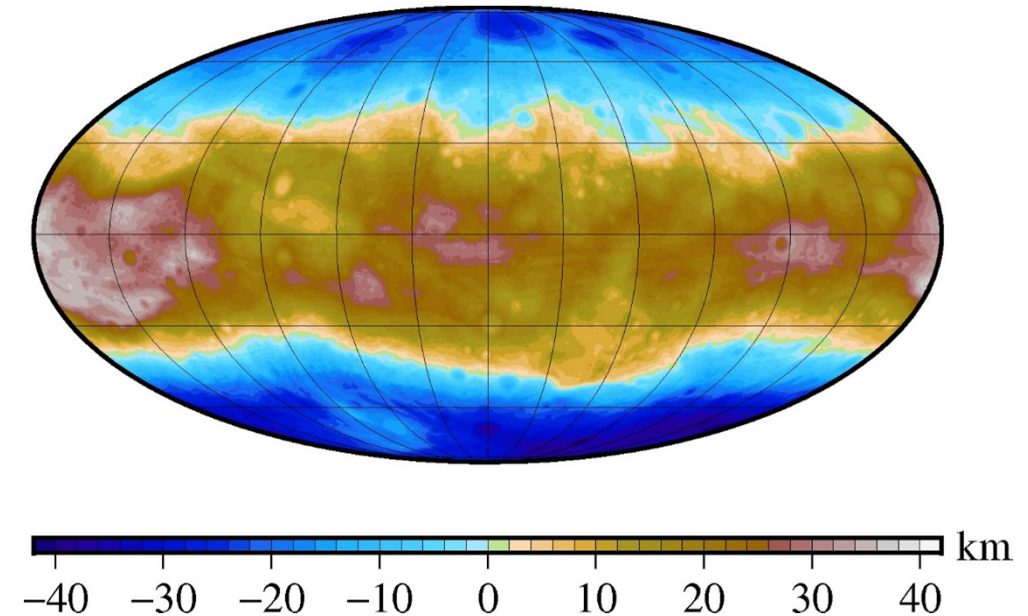


B) Example 2 – asteroid Vesta:

Basic parameters:

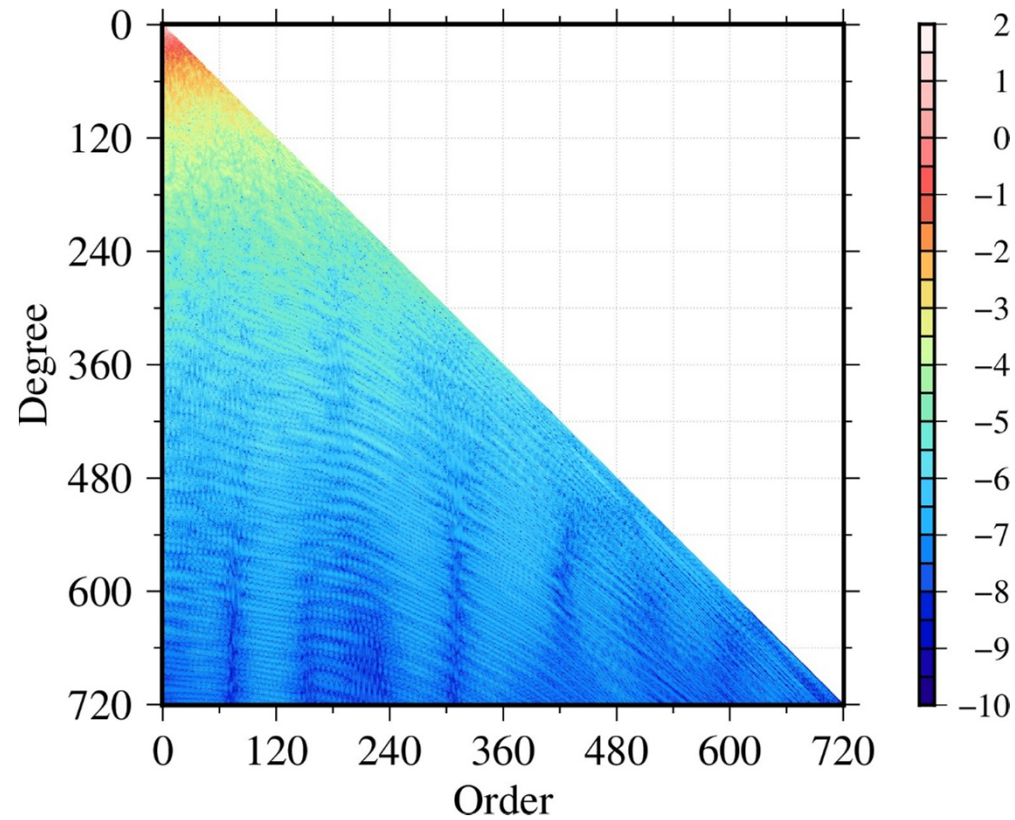
- Oblate shape, $f \approx 0.175$,
(Karimi et al. 2017),
- Maximum spherical harmonic
degree 719,
- DTM (Preusker et al. 2016) from Dawn,
resolution $1/4^\circ$,
- Volumetric density 3513 kg/m^3
(Zhang et al. 2024).

Topography: (wrt. $R=255.0 \text{ km}$)

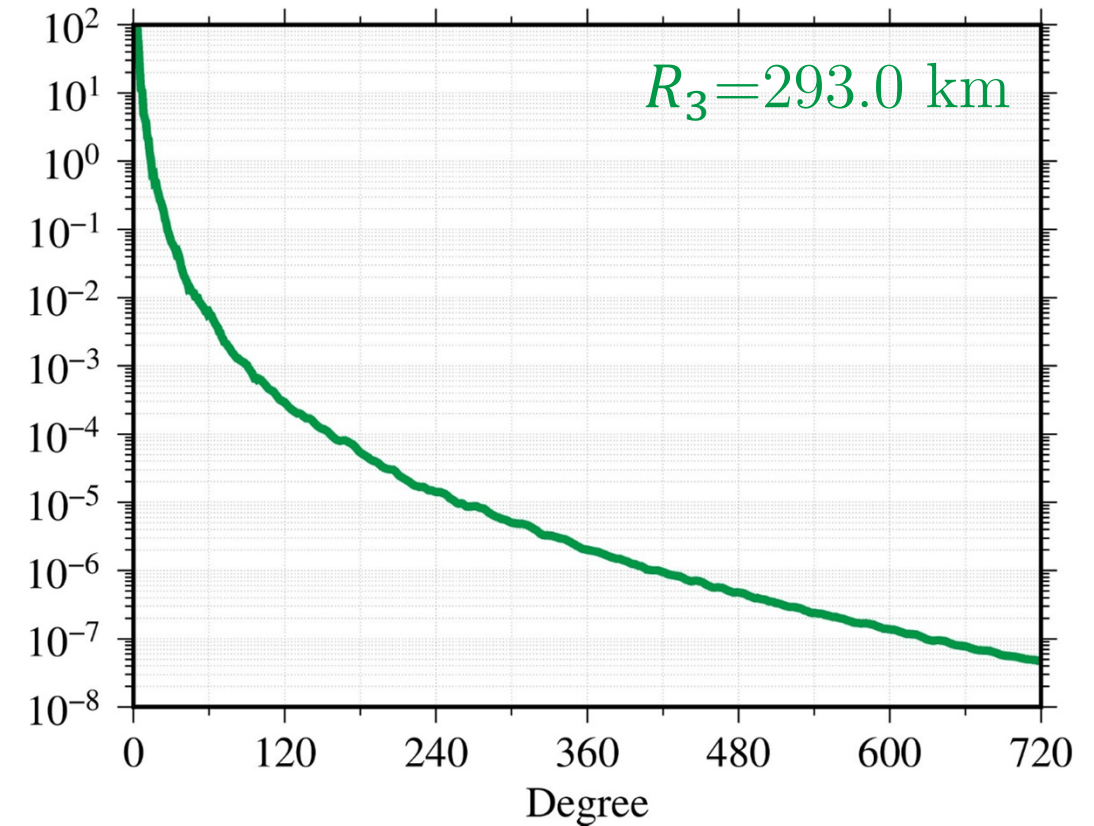


Spectral characteristics (@ $R_3=293.0$ km, top):

$\text{Mod}C_{n,m}$:

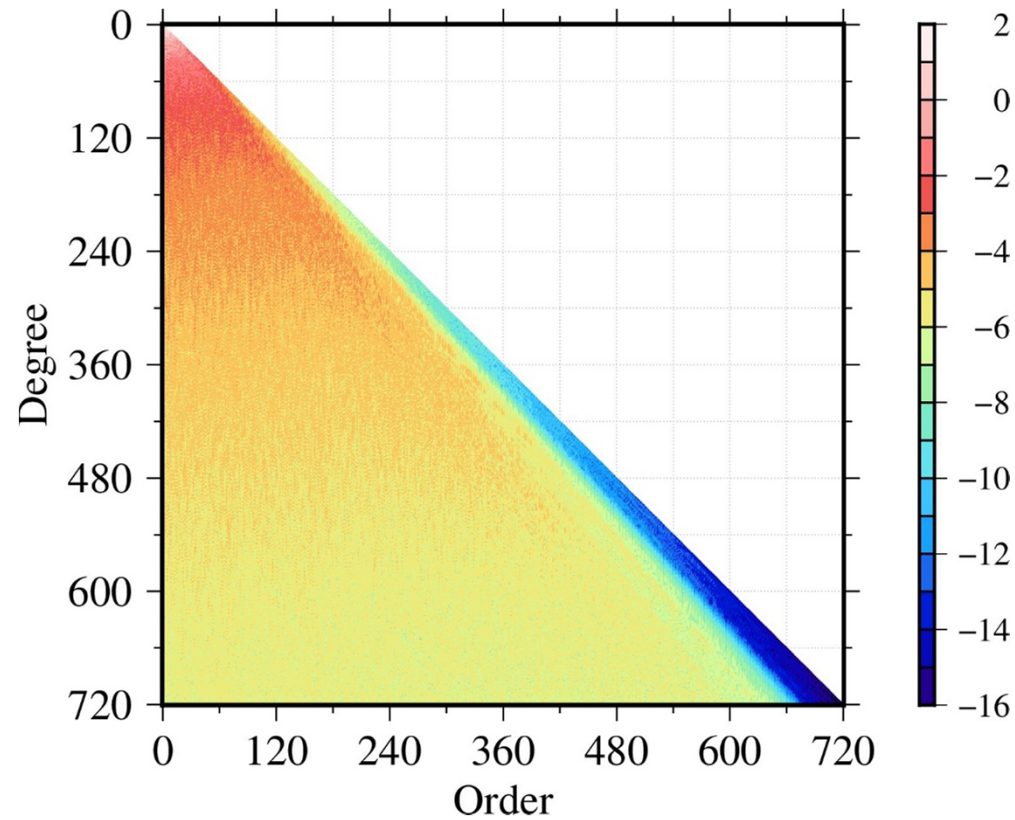


RMS_n :

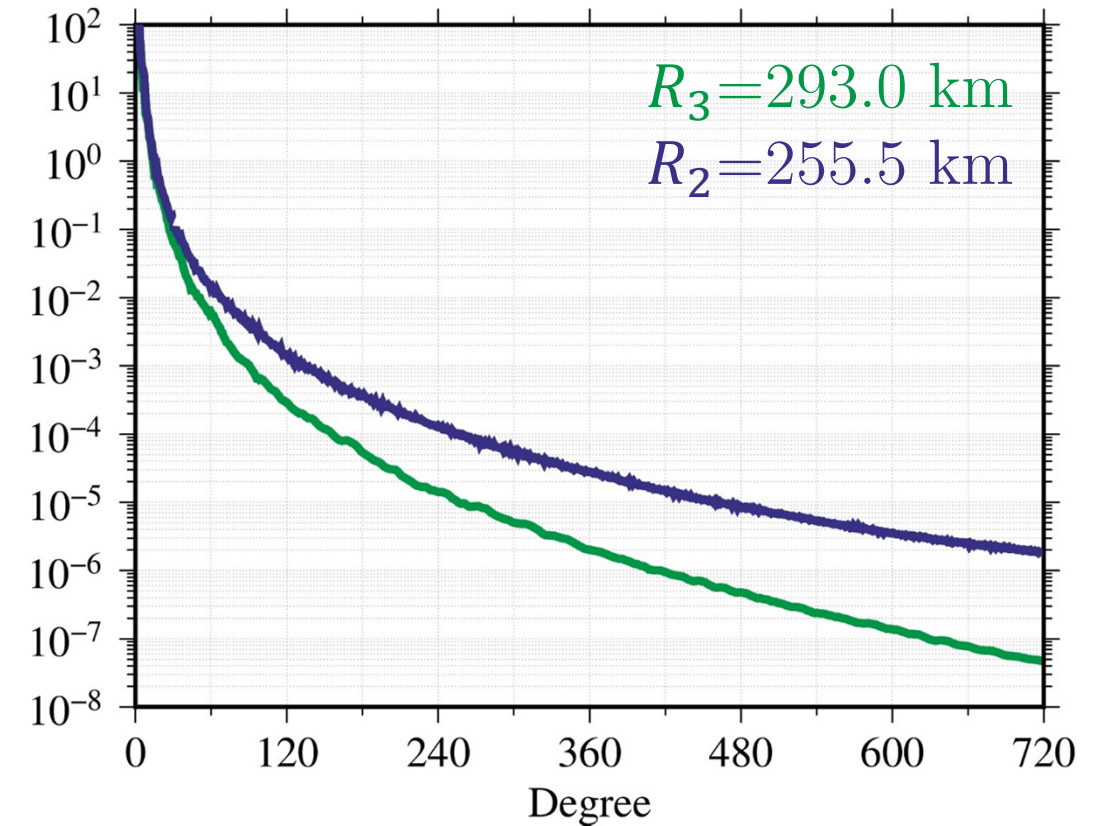


Spectral characteristics (@ $R_2=255.5$ km, best fit):

$\text{ModC}_{n,m}$:

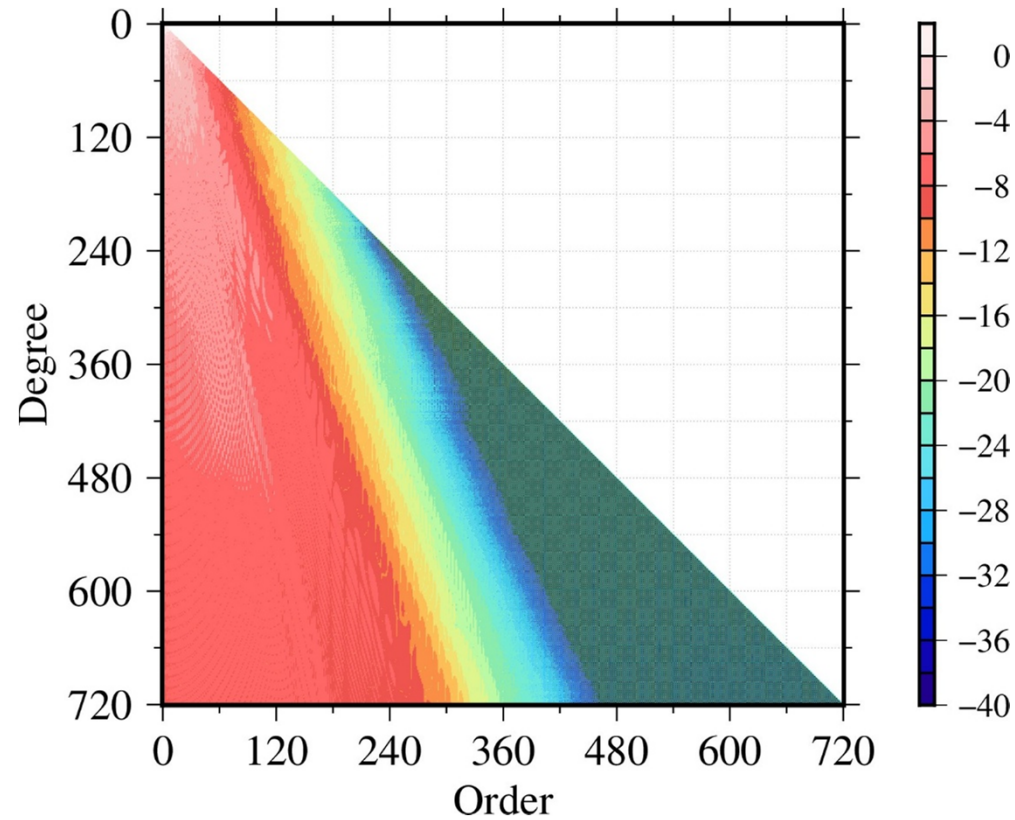


RMS_n :

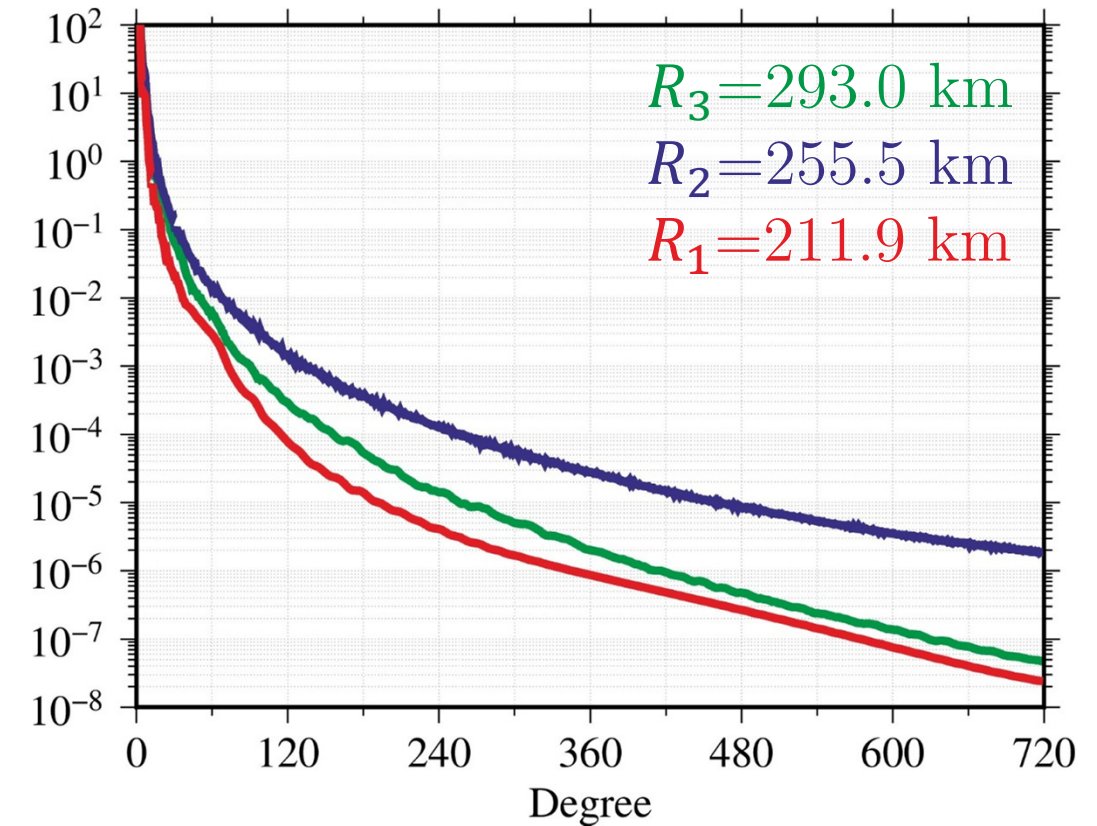


Spectral characteristics (@ $R_1=211.9$ km, bottom):

$\text{Mod}C_{n,m}$:



RMS_n :



4. Conclusions:

- Selected gravitational fields analysed near topography using forward modelling,
- Significant vertical changes of spectral characteristics demonstrated,
- Strongest power observed at the best–fitting spheres (i.e., nearest to topography),
- Gravitational 2D spectra may be anisotropic, more distinctively for flattened planetary bodies.

Open questions (selected):

- Are 2D gravitational spectra isotropic in general?
- Do degree–dependent power laws make any sense for anisotropic gravitational 2D spectra?
- Does the character of 1D and 2D gravitational spectra change when using a spheroidal harmonic representation?



Google Sites:

<https://www.sites.google.com/view/michal-sprlak>



ResearchGate:

@Michal Šprlák



LinkedIn:

@Michal Šprlák



YouTube:

@PhysicalGeodesybyMS

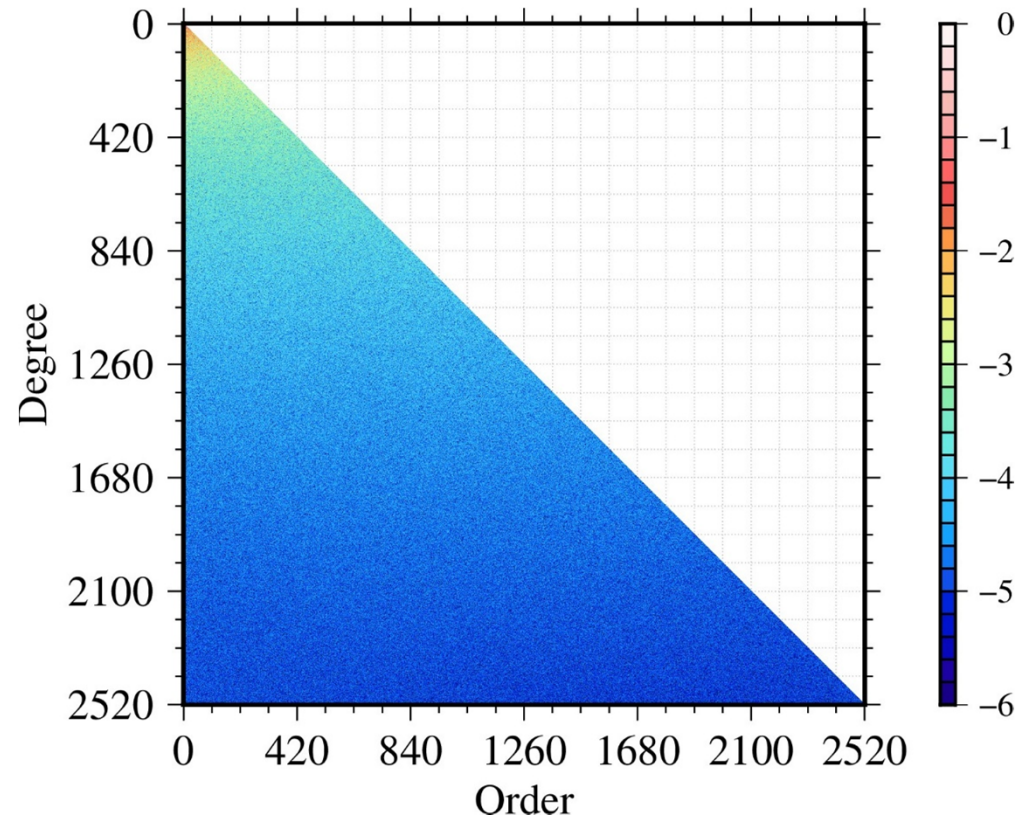
Thank you for your attention!

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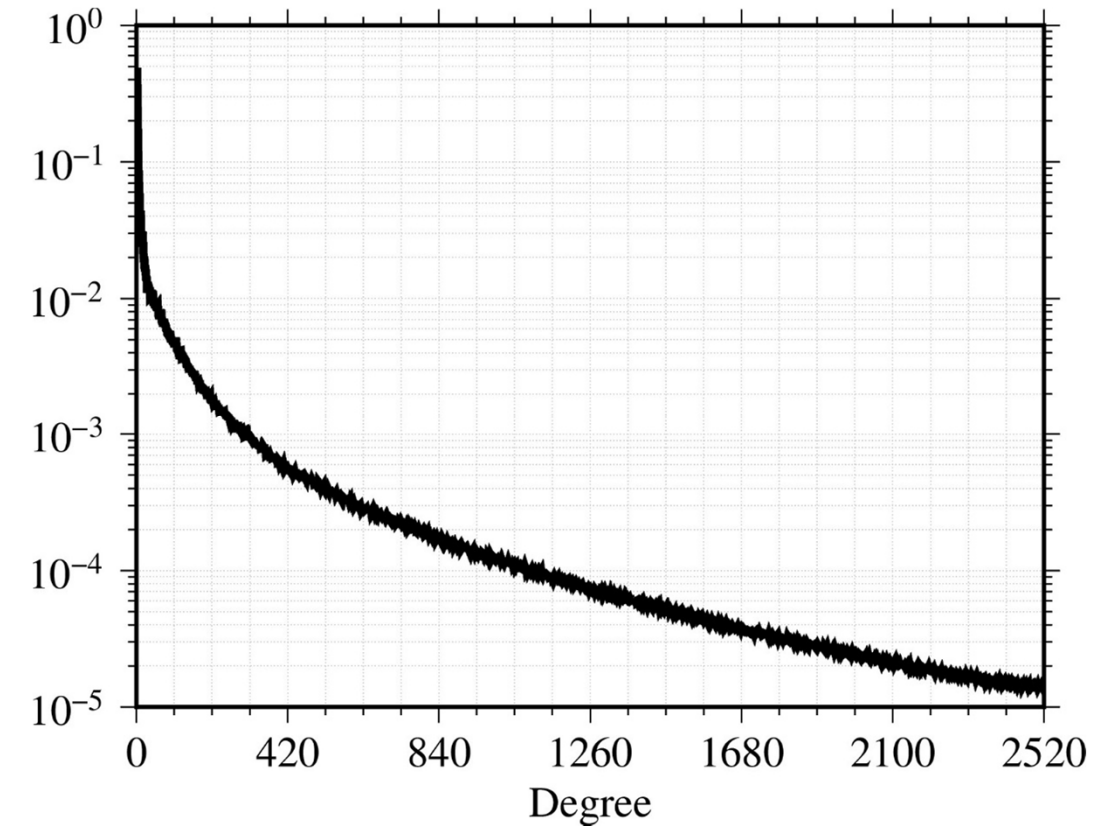
Backup slides

Topography of the Moon – spectral characteristics:

$\text{Mod}C_{n,m}$:

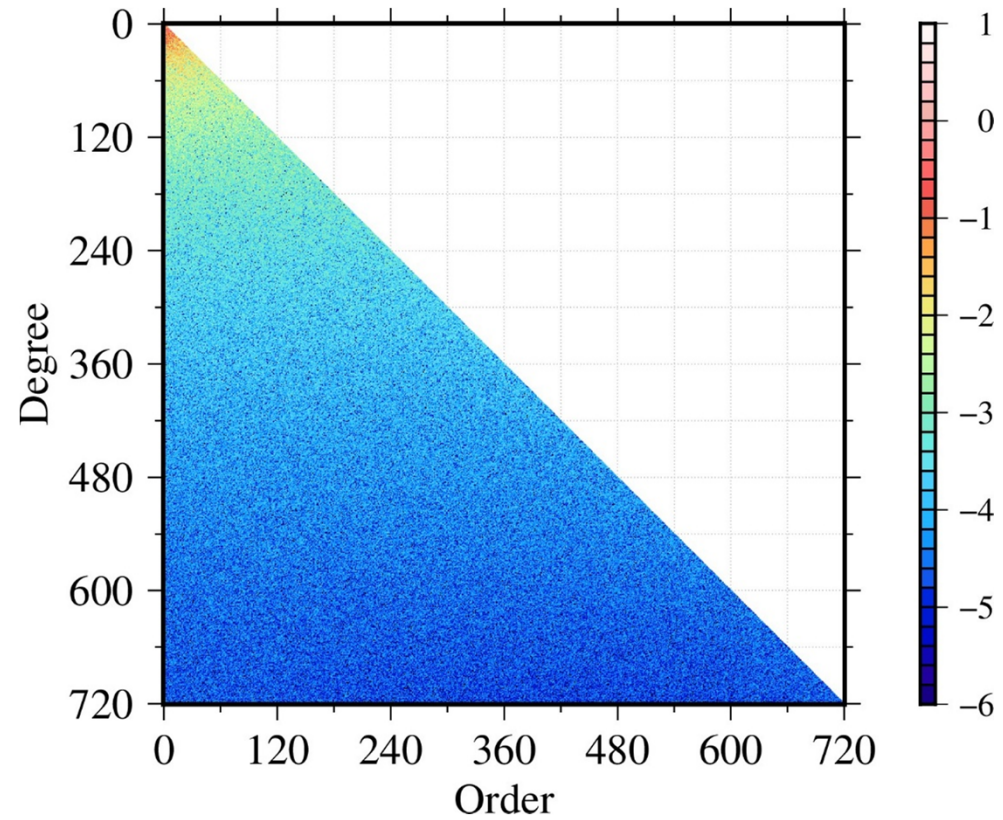


RMS_n :

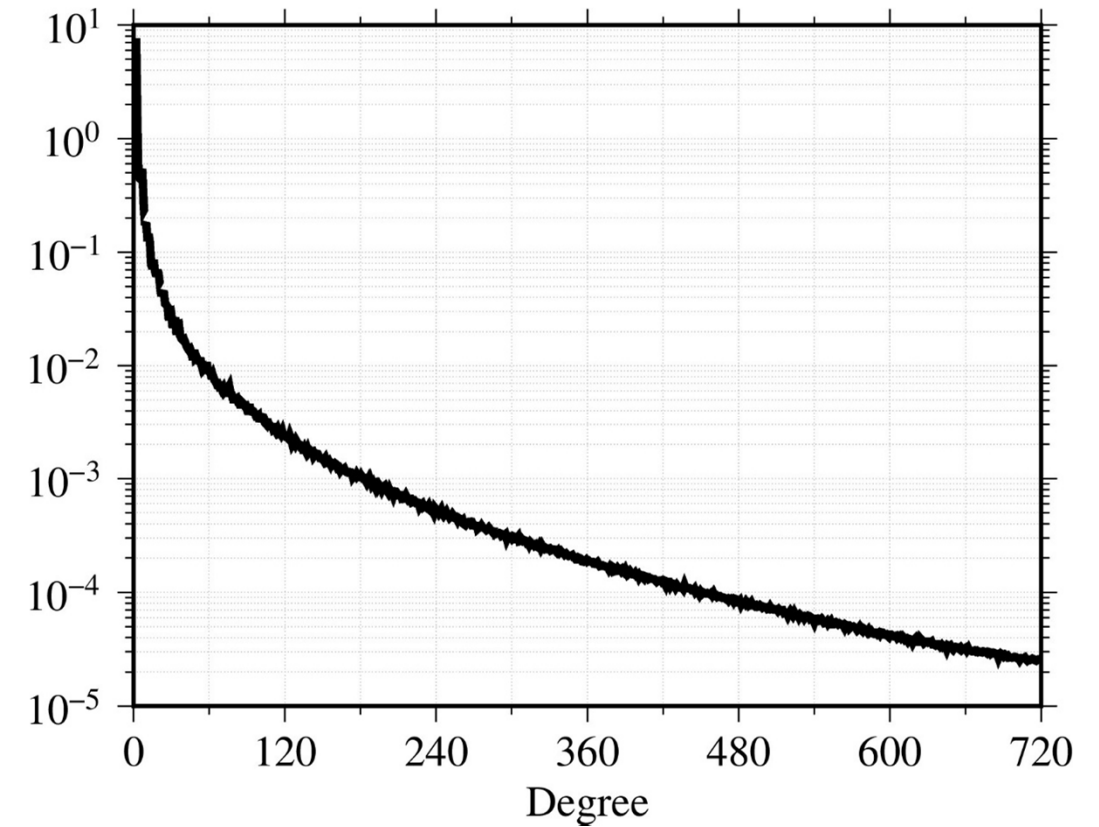


Topography of Vesta – spectral characteristics:

$\text{Mod}C_{n,m}$:



RMS_n :



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