

# Spheroidal Harmonics in Gravitational Field Modelling of Irregular Planetary Bodies

Jiří Belinger, Veronika Dohnalová, Martin Pitoňák, Pavel Novák, Michal Šprlák

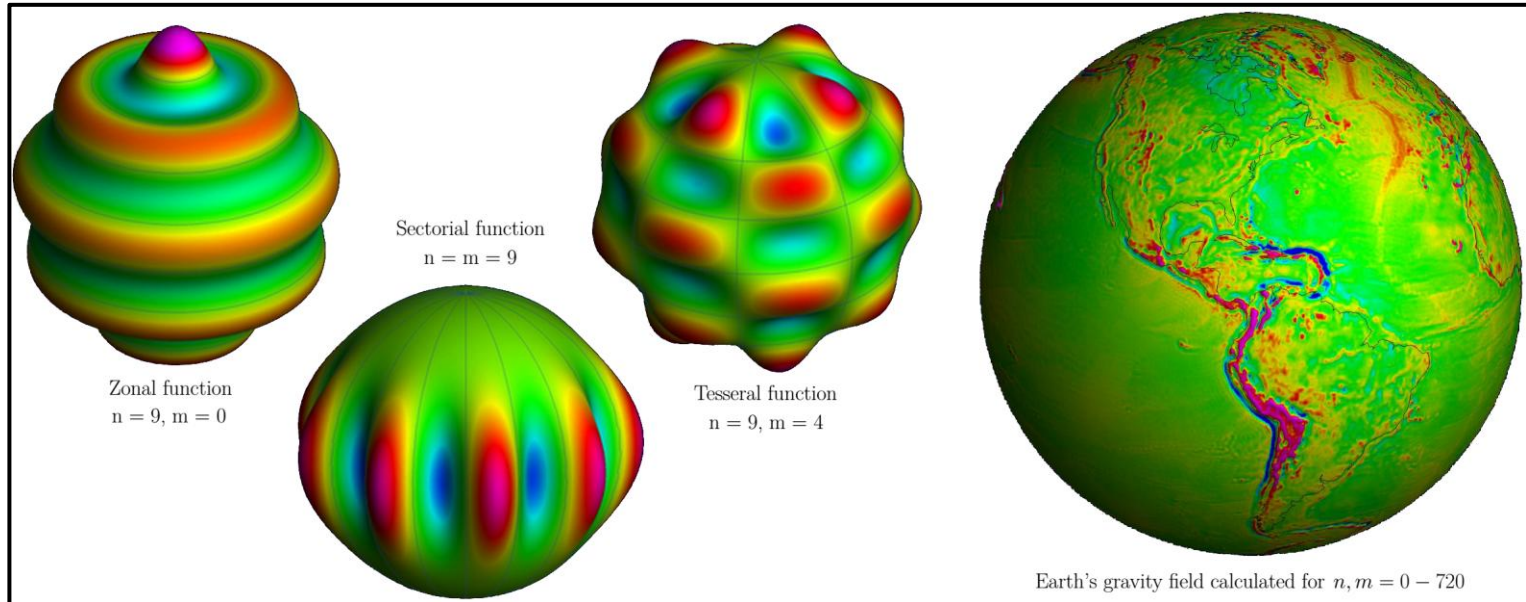
NTIS - New Technologies for the Information Society  
Faculty of Applied Sciences, University of West Bohemia in Pilsen

- **Introduction**
- **Theoretical background**
- **Software library**
- **Numerical experiments**
- **Conclusion**



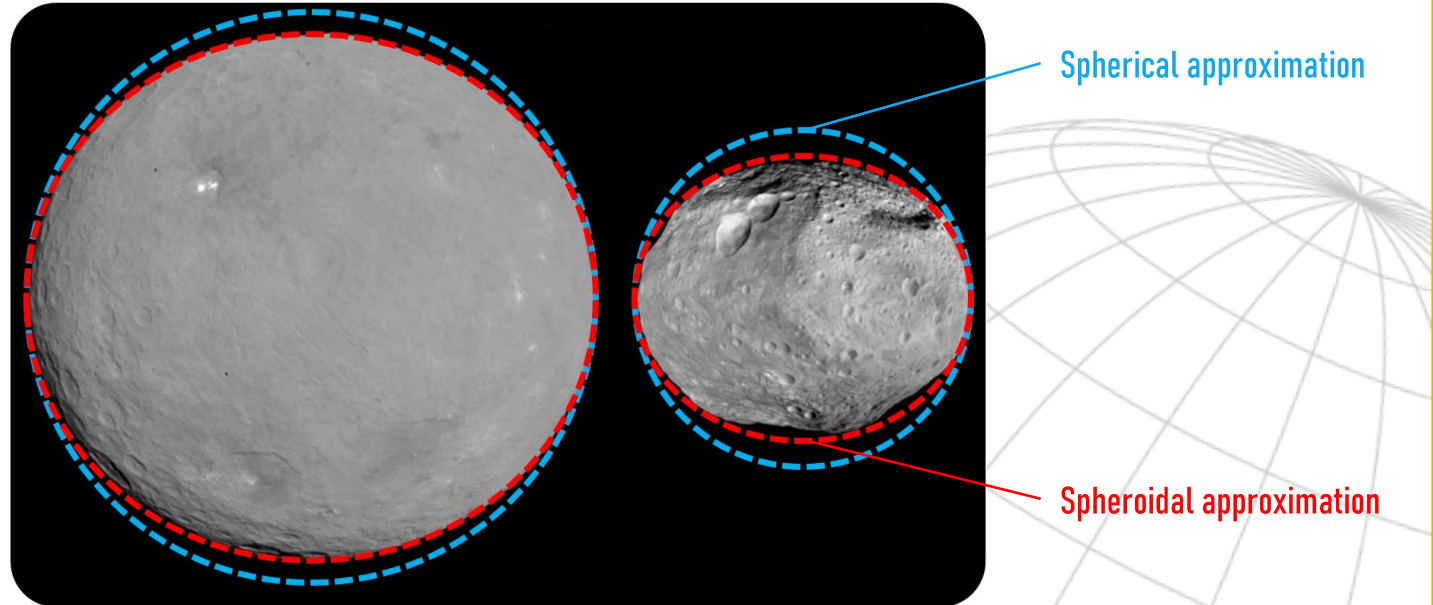
# Introduction

- Application of spectral spheroidal forward modelling



# Introduction

- Application of spectral spheroidal forward modelling
- Enhancement of gravitational field calculation possibilities



# Introduction

- Application of spectral spheroidal forward modelling
- Enhancement of gravitational field calculation possibilities
- Complication of theoretical relations

$$V_{\varphi\varphi\varphi}(r, \Omega) = -\frac{GM}{R^4} \sum_{n=0}^{\infty} \sum_{m=-n}^n \left(\frac{R}{r}\right)^{n+4} \bar{C}_{n,m} \left[ (3n+5) \frac{\partial}{\partial \varphi} - \frac{\partial^3}{\partial \varphi^3} \right] \bar{Y}_{n,m}(\Omega)$$

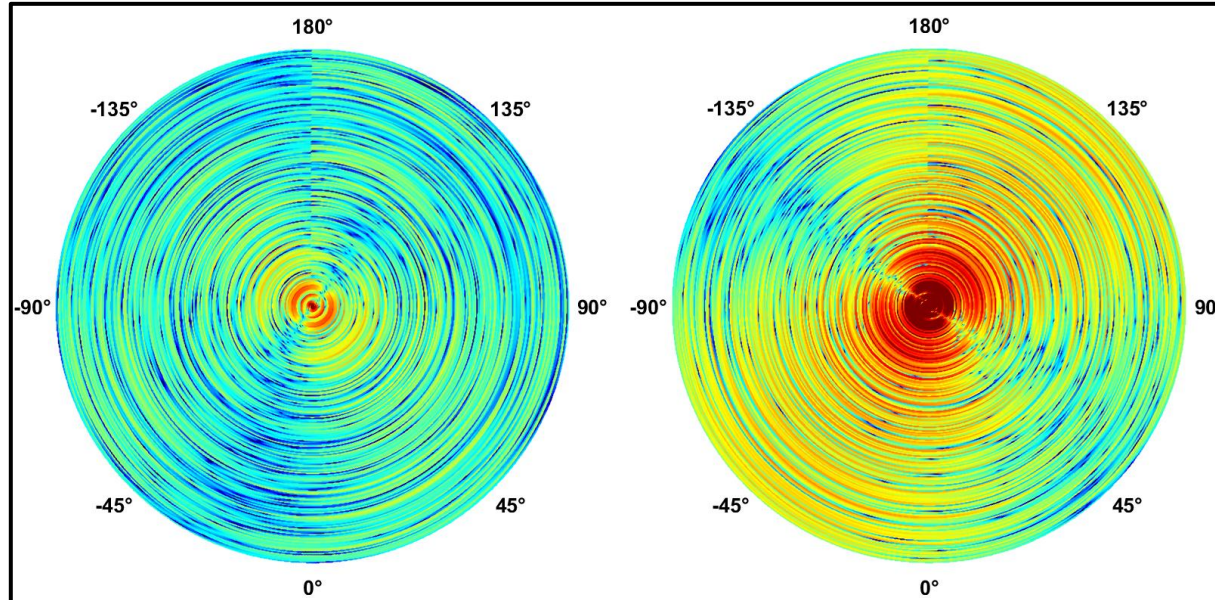
Spherical approximation

$$\begin{aligned} V_{\beta\beta\beta}(u, \Omega) = & \frac{GM}{R} \sum_{n=0}^{\infty} \sum_{m=-n}^n \left\{ -\frac{\frac{\partial}{\partial u} Q_{n,|m|}\left(i\frac{u}{\varepsilon}\right)}{Q_{n,|m|}\left(i\frac{b}{\varepsilon}\right)} \frac{6uv^2\varepsilon^2 \sin \beta \cos \beta}{L^7} \right. \\ & - \frac{Q_{n,|m|}\left(i\frac{u}{\varepsilon}\right)}{Q_{n,|m|}\left(i\frac{b}{\varepsilon}\right)} \frac{1}{L^3} \left[ 2 - \frac{3(u^2 + v^2)}{L^2} + \frac{6u^2v^2}{L^4} \right] \frac{\partial}{\partial \beta} + \frac{\frac{\partial}{\partial u} Q_{n,|m|}\left(i\frac{u}{\varepsilon}\right)}{Q_{n,|m|}\left(i\frac{b}{\varepsilon}\right)} \frac{3uv^2}{L^5} \frac{\partial}{\partial \beta} \\ & \left. - \frac{Q_{n,|m|}\left(i\frac{u}{\varepsilon}\right)}{Q_{n,|m|}\left(i\frac{b}{\varepsilon}\right)} \frac{3\varepsilon^2 \sin \beta \cos \beta}{L^5} \frac{\partial^2}{\partial \beta^2} + \frac{Q_{n,|m|}\left(i\frac{u}{\varepsilon}\right)}{Q_{n,|m|}\left(i\frac{b}{\varepsilon}\right)} \frac{1}{L^3} \frac{\partial^3}{\partial \beta^3} \right\} \bar{C}_{n,m}(u) \bar{Y}_{n,m}(\Omega) \end{aligned}$$

Spheroidal approximation

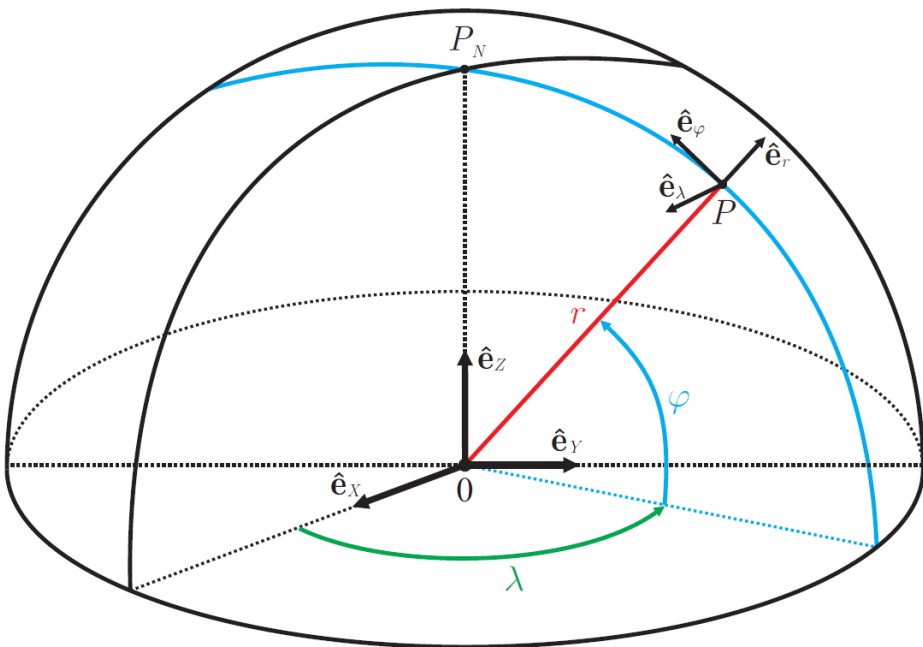
# Introduction

- Application of spectral spheroidal forward modelling
- Enhancement of gravitational field calculation possibilities
- Complication of theoretical relations
- Emergence of numerical problems



Singularities on poles

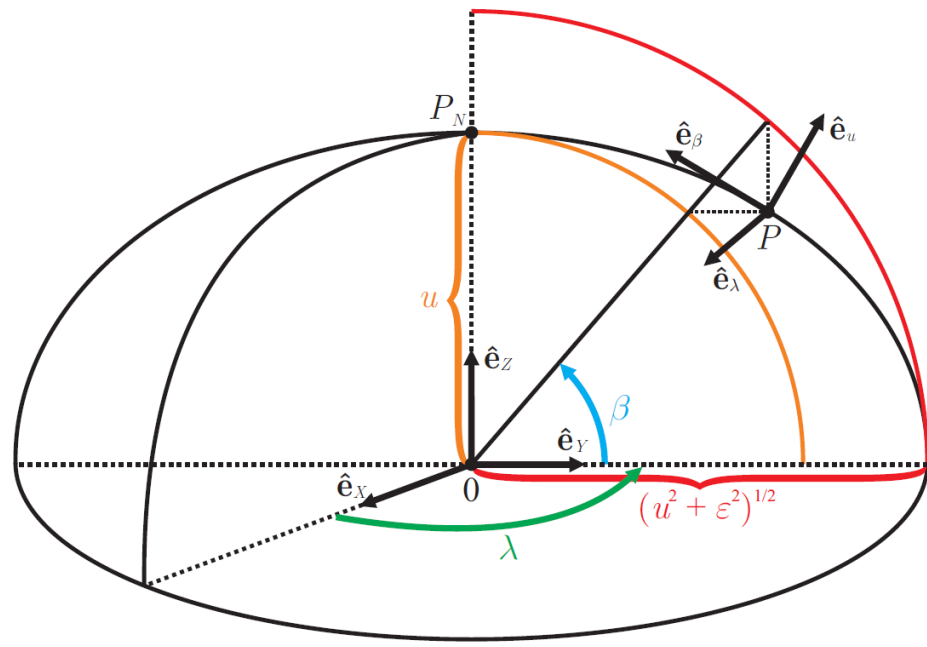
# Theoretical background (coordinate systems)



Spherical coordinate system

$r$  - radius  
 $\varphi$  - spherical latitude  
 $\lambda$  - spherical longitude

$\left. \begin{array}{l} \varphi \\ \lambda \end{array} \right\} \Omega$  - horizontal coordinates



Jacobi ellipsoidal coordinate system

$u$  - semi-minor axis of confocal spheroid  
 $\beta$  - reduced spheroidal latitude  
 $\lambda$  - spheroidal longitude  
 $\epsilon$  - linear eccentricity

$\left. \begin{array}{l} \beta \\ \lambda \end{array} \right\} \Omega$  - horizontal coordinates

# Theoretical background (analysis)

$$\bar{C}_{n,m}^o(u) = \frac{R_o}{Mb(2n+1)} q_{n,|m|} \left( \frac{b}{\varepsilon} \right) \int_{\lambda'=0}^{2\pi} \int_{\beta'=-\pi/2}^{\pi/2} \int_{u'=0}^u \rho(u', \beta', \lambda') \bar{P}_{n,m}(\sin \beta') \begin{Bmatrix} \cos m\lambda' \\ \sin |m|\lambda' \end{Bmatrix}$$

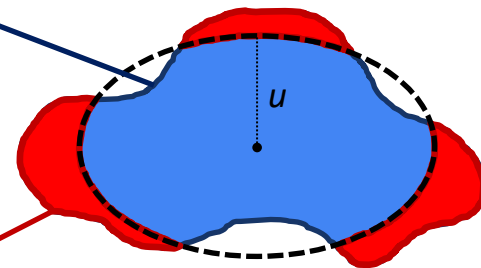
$$\times p_{n,|m|} \left( \frac{u'}{\varepsilon} \right) (u'^2 + \varepsilon^2 \sin^2 \beta') du' \cos \beta' d\beta' d\lambda'$$

Outer expansion

$$\bar{C}_{n,m}^i(u) = \frac{R_i}{Mb(2n+1)} p_{n,|m|} \left( \frac{b}{\varepsilon} \right) \int_{\lambda'=0}^{2\pi} \int_{\beta'=-\pi/2}^{\pi/2} \int_{u'=u}^{u'_S(\Omega')} \rho(u', \beta', \lambda') \bar{P}_{n,m}(\sin \beta') \begin{Bmatrix} \cos m\lambda' \\ \sin |m|\lambda' \end{Bmatrix}$$

$$\times q_{n,|m|} \left( \frac{u'}{\varepsilon} \right) (u'^2 + \varepsilon^2 \sin^2 \beta') du' \cos \beta' d\beta' d\lambda'$$

Inner expansion



Irregular body

$\bar{C}_{n,m}^o, \bar{C}_{n,m}^i$  - spheroidal harmonic coefficients of degree  $n$  and order  $m$

$R_o, R_i$  - scale factors

$b$  - semi-minor axis of reference spheroid

$M$  - body mass

$\rho$  - integration element density

$P_{n,m}, Q_{n,m}$  - associated Legendre functions of the 1st and 2nd kind

$p_{n,|m|}, q_{n,|m|}$  - modified associated Legendre functions of the 1st and 2nd kind

# Theoretical background (analysis)

$$\bar{C}_{n,m}^o(u) = \frac{R_o}{Mb(2n+1)} q_{n,|m|} \left( \frac{b}{\varepsilon} \right) \int_{\lambda'=0}^{2\pi} \int_{\beta'=-\pi/2}^{\pi/2} \left[ \int_{u'=0}^u \rho(u', \beta', \lambda') \bar{P}_{n,m}(\sin \beta') \begin{Bmatrix} \cos m\lambda' \\ \sin |m|\lambda' \end{Bmatrix} \right. \\ \left. \times p_{n,|m|} \left( \frac{u'}{\varepsilon} \right) (u'^2 + \varepsilon^2 \sin^2 \beta') du' \cos \beta' d\beta' d\lambda' \right]$$

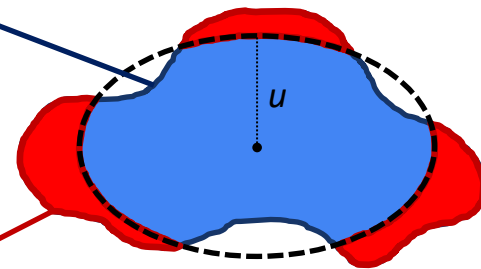
Outer expansion

$$\bar{C}_{n,m}^i(u) = \frac{R_i}{Mb(2n+1)} p_{n,|m|} \left( \frac{b}{\varepsilon} \right) \int_{\lambda'=0}^{2\pi} \int_{\beta'=-\pi/2}^{\pi/2} \left[ \int_{u'=u}^{u'_S(\Omega')} \rho(u', \beta', \lambda') \bar{P}_{n,m}(\sin \beta') \begin{Bmatrix} \cos m\lambda' \\ \sin |m|\lambda' \end{Bmatrix} \right. \\ \left. \times q_{n,|m|} \left( \frac{u'}{\varepsilon} \right) (u'^2 + \varepsilon^2 \sin^2 \beta') du' \cos \beta' d\beta' d\lambda' \right]$$

Inner expansion

$$\int_{u'=0}^u p_{n,|m|} \left( \frac{u'}{\varepsilon} \right) du'$$

$$\int_{u'=u}^{u'_S(\Omega')} q_{n,|m|} \left( \frac{u'}{\varepsilon} \right) du'$$



Irregular body

$\bar{C}_{n,m}^o, \bar{C}_{n,m}^i$  - spheroidal harmonic coefficients of degree n and order m

$R_o, R_i$  - scale factors

$b$  - semi-minor axis of reference spheroid

$M$  - body mass

$\rho$  - integration element density

$P_{n,m}, Q_{n,m}$  - associated Legendre functions of the 1st and 2nd kind

$p_{n,|m|}, q_{n,|m|}$  - modified associated Legendre functions of the 1st and 2nd kind

# Theoretical background (analysis)

$$\bar{C}_{n,m}^o(u) = \frac{R_o}{Mb(2n+1)} q_{n,|m|} \left( \frac{b}{\varepsilon} \right) \int_{\lambda'=0}^{2\pi} \int_{\beta'=-\pi/2}^{\pi/2} \int_{u'=0}^u \rho(u', \beta', \lambda') \bar{P}_{n,m}(\sin \beta') \begin{Bmatrix} \cos m\lambda' \\ \sin |m|\lambda' \end{Bmatrix}$$

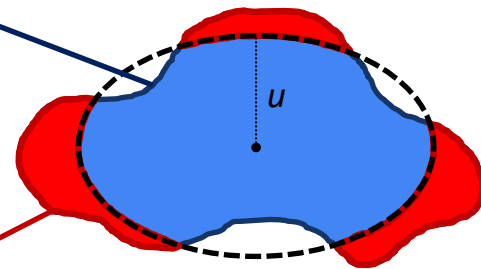
$$\times p_{n,|m|} \left( \frac{u'}{\varepsilon} \right) (u'^2 + \varepsilon^2 \sin^2 \beta') du' \cos \beta' d\beta' d\lambda'$$

Outer expansion

$$\bar{C}_{n,m}^i(u) = \frac{R_i}{Mb(2n+1)} p_{n,|m|} \left( \frac{b}{\varepsilon} \right) \int_{\lambda'=0}^{2\pi} \int_{\beta'=-\pi/2}^{\pi/2} \int_{u'=u}^{u'_S(\Omega')} \rho(u', \beta', \lambda') \bar{P}_{n,m}(\sin \beta') \begin{Bmatrix} \cos m\lambda' \\ \sin |m|\lambda' \end{Bmatrix}$$

$$\times q_{n,|m|} \left( \frac{u'}{\varepsilon} \right) (u'^2 + \varepsilon^2 \sin^2 \beta') du' \cos \beta' d\beta' d\lambda'$$

Inner expansion



Irregular body

$$\int_{u'=0}^u p_{n,|m|} \left( \frac{u'}{\varepsilon} \right) du'$$

$$p_{n,m} \left( \frac{u'}{\varepsilon} \right) = \left( \frac{u'}{b} \right)^n \left( \frac{u'}{\sqrt{u'^2 + \varepsilon^2}} \right)^m {}_2F_1 \left( -\frac{n+m}{2}, -\frac{n+m-1}{2}, -\frac{2n-1}{2}, -\frac{\varepsilon^2}{u'^2} \right)$$

Calculation using Gauss hypergeometric (HG) functions

$$\int_{u'=u}^{u'_S(\Omega')} q_{n,|m|} \left( \frac{u'}{\varepsilon} \right) du'$$

$$q_{n,|m|} \left( \frac{u'}{\varepsilon} \right) = \left( \frac{b}{\sqrt{u'^2 + \varepsilon^2}} \right)^{n+1} {}_2F_1 \left( \frac{n+|m|+1}{2}, \frac{n-|m|+1}{2}, \frac{2n+3}{2}, \frac{\varepsilon^2}{u'^2 + \varepsilon^2} \right)$$

# Theoretical background (synthesis)

$$V_{\beta}(u, \Omega) = \frac{GM}{R_o} \sum_{n=0}^{\infty} \sum_{m=-n}^n \bar{C}_{n,m}^o(u) \frac{Q_{n,|m|}\left(i\frac{u}{\varepsilon}\right)}{Q_{n,|m|}\left(i\frac{b}{\varepsilon}\right)} \frac{1}{L} \frac{\partial}{\partial \beta} \bar{P}_{n,|m|}(\sin \beta) K_m(\lambda)$$

$$+ \frac{GM}{R_i} \sum_{n=0}^{\infty} \sum_{m=-n}^n \bar{C}_{n,m}^i(u) \frac{P_{n,|m|}\left(i\frac{u}{\varepsilon}\right)}{P_{n,|m|}\left(i\frac{b}{\varepsilon}\right)} \frac{1}{L} \frac{\partial}{\partial \beta} \bar{P}_{n,|m|}(\sin \beta) K_m(\lambda)$$

$$L = \sqrt{u^2 + \varepsilon^2 \sin^2 \beta}$$

$$K_m(\lambda) = \begin{cases} \cos m\lambda, & m \geq 0, \\ \sin |m|\lambda, & m < 0. \end{cases}$$



# Theoretical background (synthesis)

$$\begin{aligned}
 V_\beta(u, \Omega) = & \frac{GM}{R_o} \sum_{n=0}^{\infty} \sum_{m=-n}^n \bar{C}_{n,m}^o(u) \frac{Q_{n,|m|}\left(i\frac{u}{\varepsilon}\right)}{Q_{n,|m|}\left(i\frac{b}{\varepsilon}\right)} \frac{1}{L} \frac{\partial}{\partial \beta} \bar{P}_{n,|m|}(\sin \beta) K_m(\lambda) \\
 & + \frac{GM}{R_i} \sum_{n=0}^{\infty} \sum_{m=-n}^n \bar{C}_{n,m}^i(u) \frac{P_{n,|m|}\left(i\frac{u}{\varepsilon}\right)}{P_{n,|m|}\left(i\frac{b}{\varepsilon}\right)} \frac{1}{L} \frac{\partial}{\partial \beta} \bar{P}_{n,|m|}(\sin \beta) K_m(\lambda)
 \end{aligned}$$

$$L = \sqrt{u^2 + \varepsilon^2 \sin^2 \beta}$$

$$K_m(\lambda) = \begin{cases} \cos m\lambda, & m \geq 0, \\ \sin |m|\lambda, & m < 0. \end{cases}$$



# Theoretical background (synthesis)

$$\left[ V_{\beta}(u, \Omega) = \frac{GM}{R_o} \sum_{n=0}^{\infty} \sum_{m=-n}^n \bar{C}_{n,m}^o(u) \frac{Q_{n,|m|}\left(\frac{i u}{\varepsilon}\right)}{Q_{n,|m|}\left(\frac{i b}{\varepsilon}\right)} \frac{1}{L} \frac{\partial}{\partial \beta} \bar{P}_{n,|m|}(\sin \beta) K_m(\lambda) \right. \\ \left. + \frac{GM}{R_i} \sum_{n=0}^{\infty} \sum_{m=-n}^n \bar{C}_{n,m}^i(u) \frac{P_{n,|m|}\left(\frac{i u}{\varepsilon}\right)}{P_{n,|m|}\left(\frac{i b}{\varepsilon}\right)} \frac{1}{L} \frac{\partial}{\partial \beta} \bar{P}_{n,|m|}(\sin \beta) K_m(\lambda) \right]$$

$$L = \sqrt{u^2 + \varepsilon^2 \sin^2 \beta}$$

$$K_m(\lambda) = \begin{cases} \cos m\lambda, & m \geq 0, \\ \sin |m|\lambda, & m < 0. \end{cases}$$

$\delta_{n,m}$  - Kronecker delta

$$V_{\beta}(u, \Omega) = \frac{GM}{2R_o L} \sum_{n=1}^{\infty} \bar{C}_{n,0}^o(u) \frac{Q_{n,0}\left(\frac{i u}{\varepsilon}\right)}{Q_{n,0}\left(\frac{i b}{\varepsilon}\right)} \sqrt{2n(n+1)} \bar{P}_{n,1}(\sin \beta) \\ + \frac{GM}{2R_o L} \sum_{n=1}^{\infty} \sum_{\substack{m=-n \\ m \neq 0}}^{+n} \bar{C}_{n,m}^o(u) \frac{Q_{n,|m|}\left(\frac{i u}{\varepsilon}\right)}{Q_{n,|m|}\left(\frac{i b}{\varepsilon}\right)} \left[ \sqrt{(n-|m|)(n+|m|+1)} \bar{P}_{n,|m|+1}(\sin \beta) \right. \\ \left. - \sqrt{(1+\delta_{|m|-1,0})(n+|m|)(n-|m|+1)} \bar{P}_{n,|m|-1}(\sin \beta) \right] K_m(\lambda) + \dots$$

$V_{\beta}$

# Theoretical background (synthesis)

$$\left[ V_{\beta}(u, \Omega) = \frac{GM}{R_o} \sum_{n=0}^{\infty} \sum_{m=-n}^n \bar{C}_{n,m}^o(u) \frac{Q_{n,|m|}\left(\frac{i u}{\varepsilon}\right)}{Q_{n,|m|}\left(\frac{i b}{\varepsilon}\right)} \frac{1}{L} \frac{\partial}{\partial \beta} \bar{P}_{n,|m|}(\sin \beta) K_m(\lambda) \right. \\ \left. + \frac{GM}{R_i} \sum_{n=0}^{\infty} \sum_{m=-n}^n \bar{C}_{n,m}^i(u) \frac{P_{n,|m|}\left(\frac{i u}{\varepsilon}\right)}{P_{n,|m|}\left(\frac{i b}{\varepsilon}\right)} \frac{1}{L} \frac{\partial}{\partial \beta} \bar{P}_{n,|m|}(\sin \beta) K_m(\lambda) \right]$$

$$L = \sqrt{u^2 + \varepsilon^2 \sin^2 \beta}$$

$$K_m(\lambda) = \begin{cases} \cos m\lambda, & m \geq 0, \\ \sin |m|\lambda, & m < 0. \end{cases}$$

$\delta_{n,m}$  - Kronecker delta

$$V_{\beta}(u, \Omega) = \frac{GM}{2R_o L} \sum_{n=1}^{\infty} \bar{C}_{n,0}^o(u) \frac{Q_{n,0}\left(\frac{i u}{\varepsilon}\right)}{Q_{n,0}\left(\frac{i b}{\varepsilon}\right)} \sqrt{2n(n+1)} \bar{P}_{n,1}(\sin \beta) \\ + \frac{GM}{2R_o L} \sum_{n=1}^{\infty} \sum_{\substack{m=-n \\ m \neq 0}}^{+n} \bar{C}_{n,m}^o(u) \frac{Q_{n,|m|}\left(\frac{i u}{\varepsilon}\right)}{Q_{n,|m|}\left(\frac{i b}{\varepsilon}\right)} \left[ \sqrt{(n-|m|)(n+|m|+1)} \bar{P}_{n,|m|+1}(\sin \beta) \right. \\ \left. - \sqrt{(1+\delta_{|m|-1,0})(n+|m|)(n-|m|+1)} \bar{P}_{n,|m|-1}(\sin \beta) \right] K_m(\lambda) + \dots$$

$V_{\beta}, V_{\lambda}$

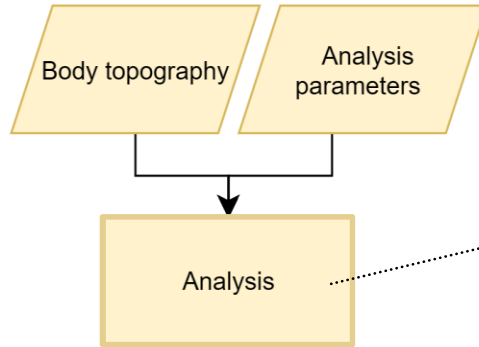
$V_{uu}, V_{u\beta}, V_{u\lambda}, V_{\beta\beta}, V_{\beta\lambda}, V_{\lambda\lambda}$

$V_{uuu}, V_{uu\beta}, V_{uu\lambda}, V_{u\beta\beta}, V_{u\beta\lambda}$

$V_{u\lambda\lambda}, V_{\beta\beta\beta}, V_{\beta\beta\lambda}, V_{\beta\lambda\lambda}, V_{\lambda\lambda\lambda}$

# Software library

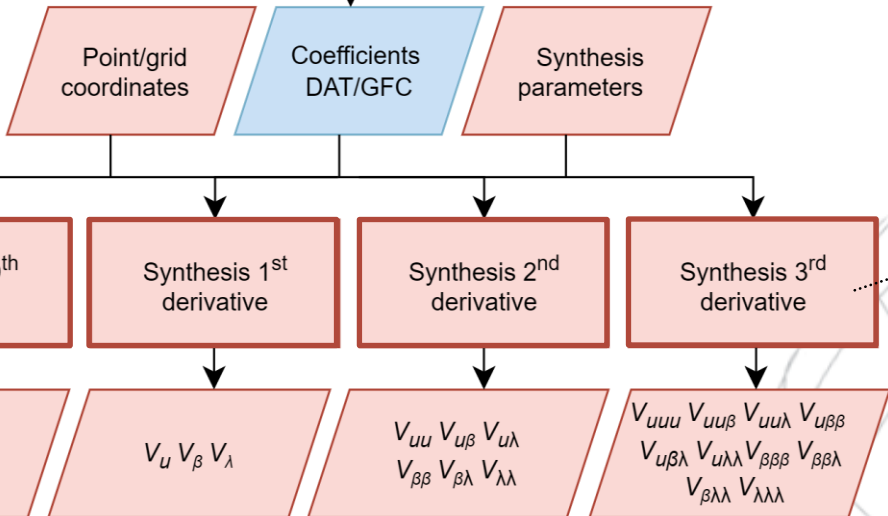
## Analysis



### Analysis files

- compute.c
- compute.h
- integ\_ui.c
- integ\_ui
- integ\_ui.h
- Makefile
- reading.c
- reading.h

## Synthesis



### Synthesis files

- compute.c
- compute.h
- grafim\_ui.c
- grafim\_ui
- grafim\_ui.h
- Makefile
- reading.c
- reading.h
- shs.c
- shs.h



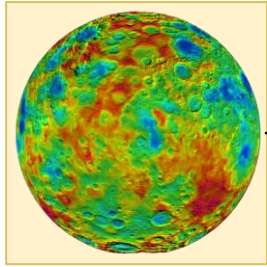
Data



Process

# Software library

3D Topography



Body topography

Analysis  
parameters

Analysis

Coefficients  
DAT/GFC

Analysis Inputs and Outputs

```
Input topography file: ceres_2520_5040_spheroid_446p0_bm_weight.g
n_max: 1000
GM G: 6.263015886E+10 6.6742E-11
u: 400000.0
R_o R_i: 439781.76406031208 439781.76406031208
a b: 482000.0 446000.0
b_B: 455479.0
b_C: 300000.0
p: 2162.0
Output coefficients file: shc_ceres_spheroid_1000.dat
```

Spheroidal coefficients

| L          | M | Co                       | So                       |
|------------|---|--------------------------|--------------------------|
| head ===== |   |                          |                          |
| 0          | 0 | 7.57962960300739175E-01  | 0.0000000000000000E+00   |
| 1          | 0 | 4.65817332432699196E-04  | 0.0000000000000000E+00   |
| 1          | 1 | -1.08205606204522342E-03 | -4.92868029245949234E-04 |
| 2          | 0 | 3.09500210939009455E-03  | 0.0000000000000000E+00   |
| 2          | 1 | 1.97295332932664484E-04  | -2.42515483791506481E-04 |
| 2          | 2 | -2.29088151251934391E-05 | 7.40656505343221931E-04  |

Data

Process

# Software library

## Computation point coordinates

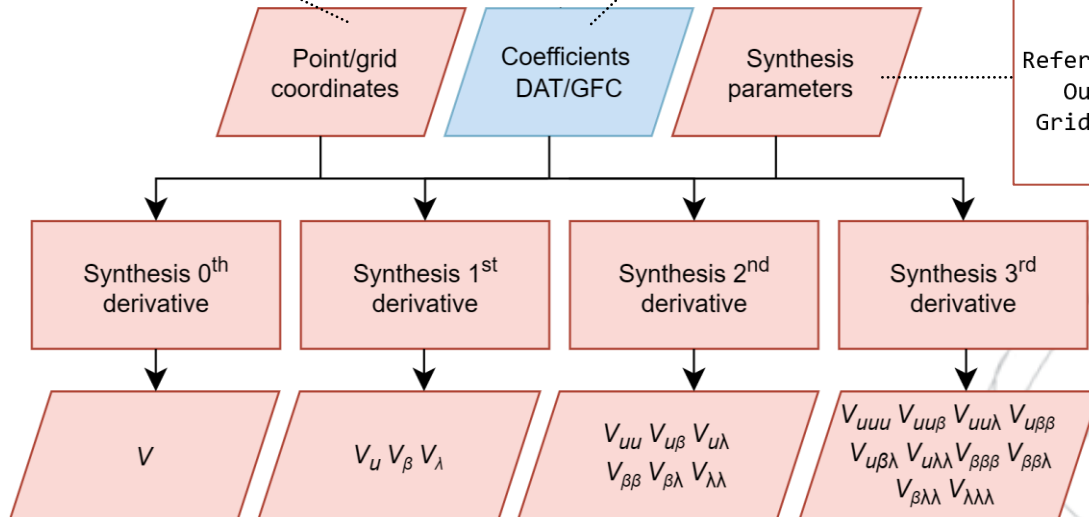
```
Grid(0) coor.: -89.875 89.875 0.25
                0.125 359.875 0.25
                400000.0
Point(1) coor.: 0.0 0.0 400000.0
                20.0 45.0 400000.0
                40.0 90.0 400000.0
```

## Spheroidal coefficients

| L          | M | Co                       | So                       |
|------------|---|--------------------------|--------------------------|
| head ===== |   |                          |                          |
| 0          | 0 | 7.57962960300739175E-01  | 0.000000000000000000E+00 |
| 1          | 0 | 4.65817332432699196E-04  | 0.000000000000000000E+00 |
| 1          | 1 | -1.08205606204522342E-03 | -4.92868029245949234E-04 |
| 2          | 0 | 3.09500210939009455E-03  | 0.000000000000000000E+00 |
| 2          | 1 | 1.97295332932664484E-04  | -2.42515483791506481E-04 |
| 2          | 2 | -2.29088151251934391E-05 | 7.40656505343221931E-04  |

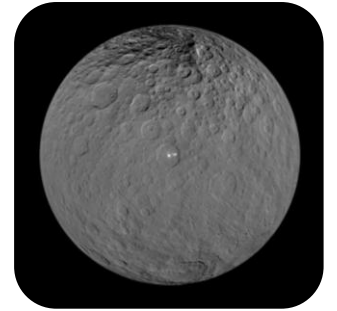
## Synthesis Inputs and Outputs

```
Input coefficients file: shc_ceres_spheroid_1000.gfc
                        n_min n_max: 0 500
Reference spheroid selection: 1
Output quantities results: res_ell_500.dat
Grid(0)/Point(1) input file: 1
Input coordinates file:  coor_ceres.dat
```

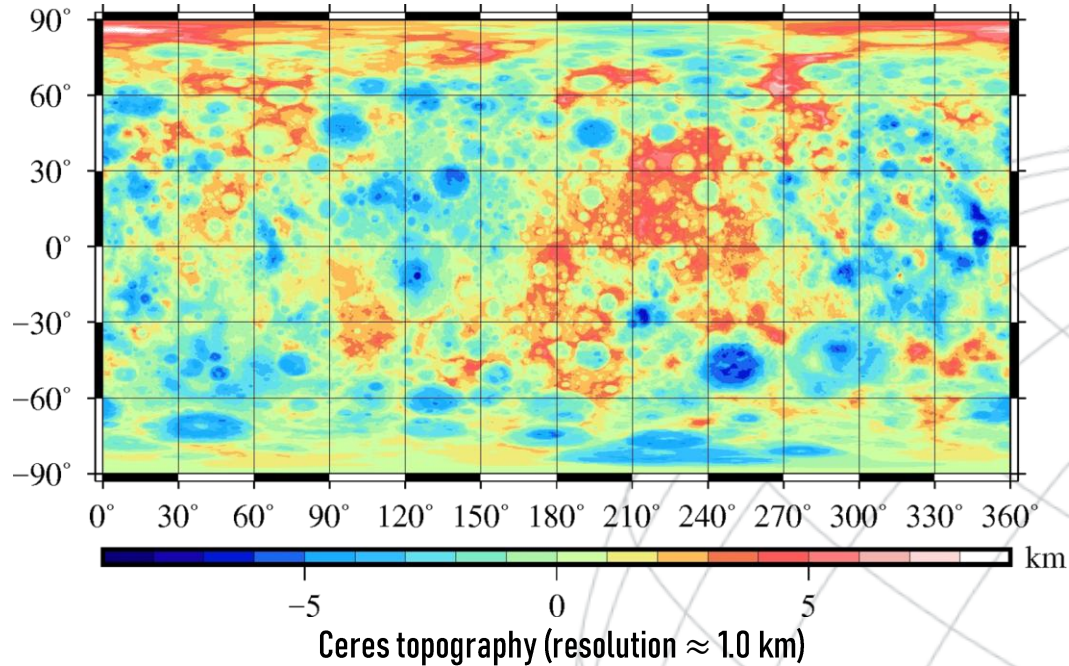


# Numerical experiments

- Comparison between spherical and spheroidal approximation (analysis/synthesis)
- Planetary body – dwarf planet Ceres (reference spheroid  $a = 482.0$  km,  $b = 446.0$  km)
- Topography models in resolutions from 360 ( $\approx 2.1$  km) to 2520 ( $\approx 0.3$  km)
- Outer expansion and combination of outer/inner expansion

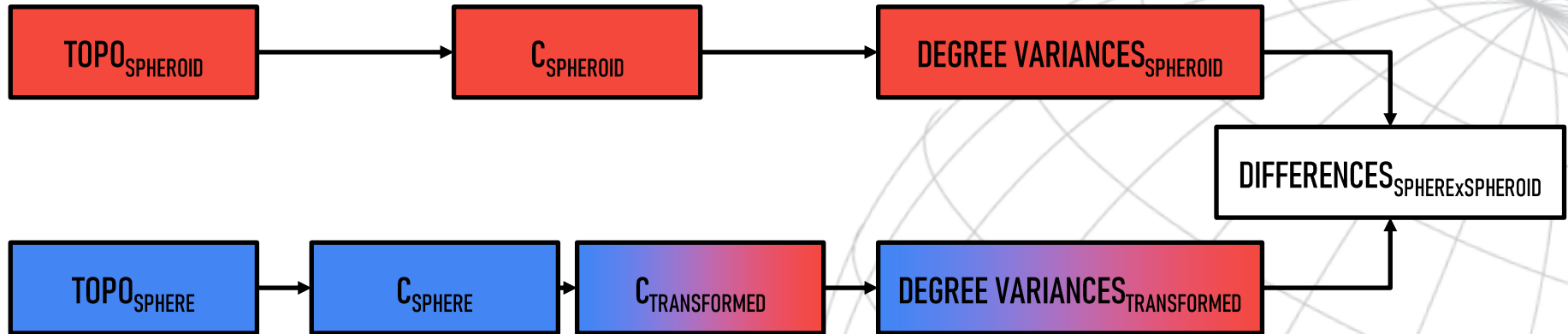


Ceres



# Numerical experiments (analysis)

- Spherical/spheroidal harmonic coefficients calculated (only outer expansion)
  - Equivalent reference surfaces (sphere/spheroid)
  - Equivalent topographies adjusted for surfaces
  - Maximal degree of harmonic expansion  $n = 360$  (computational demands)
  - Coefficients calculated for all topography resolutions ( $0.5^\circ$ ,  $0.25^\circ$ ,  $0.125^\circ$ ,  $0.0714^\circ$ )
  - Spherical coefficients transformed to spheroidal counterparts
- Degree variances of original spheroidal and transformed spherical coefficients compared



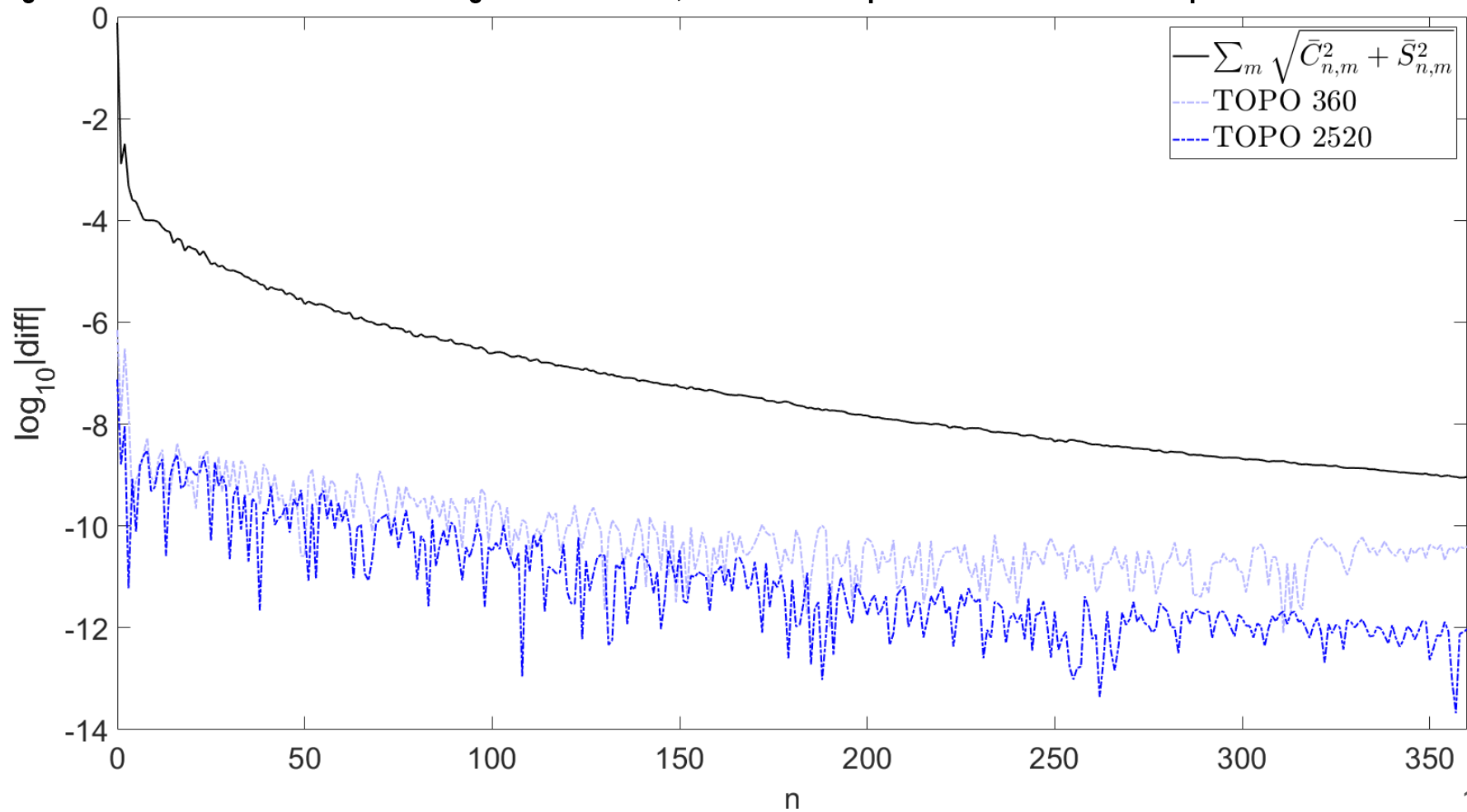
# Numerical experiments (analysis)

- Spherical/spheroidal harmonic coefficients calculated (only outer expansion)
  - Equivalent reference surfaces (sphere/spheroid)
  - Equivalent topographies adjusted for surfaces
  - Maximal degree of harmonic expansion  $n = 360$  (computational demands)
  - Coefficients calculated for all topography resolutions ( $0.5^\circ$ ,  $0.25^\circ$ ,  $0.125^\circ$ ,  $0.0714^\circ$ )
  - Spherical coefficients transformed to spheroidal counterparts
- Degree variances of original spheroidal and transformed spherical coefficients compared

Relative differences between spheroidal and transformed spherical coefficient degree variances

| Grid spacing | $0.5^\circ$ | $0.25^\circ$ | $0.125^\circ$ | $0.0714^\circ$ |
|--------------|-------------|--------------|---------------|----------------|
| Mean         | 7.6772e-03  | 2.3656e-03   | 1.0584e-03    | 0.3024e-03     |
| Std          | 1.3976e-02  | 0.3569e-02   | 0.1313e-02    | 0.0303e-02     |
| Min          | 9.3098e-07  | 1.9897e-07   | 3.1218e-07    | 0.1253e-07     |
| Max          | 6.2351e-02  | 1.4275e-02   | 0.6514e-02    | 0.1367e-02     |

Magnitude of differences between degree variances (transformed spherical coefficients  $\times$  spheroidal coefficients)

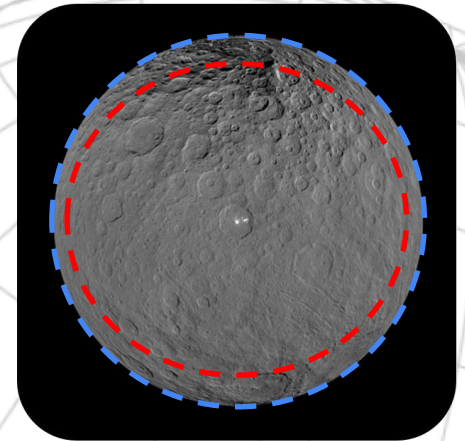


# Numerical experiments (synthesis)

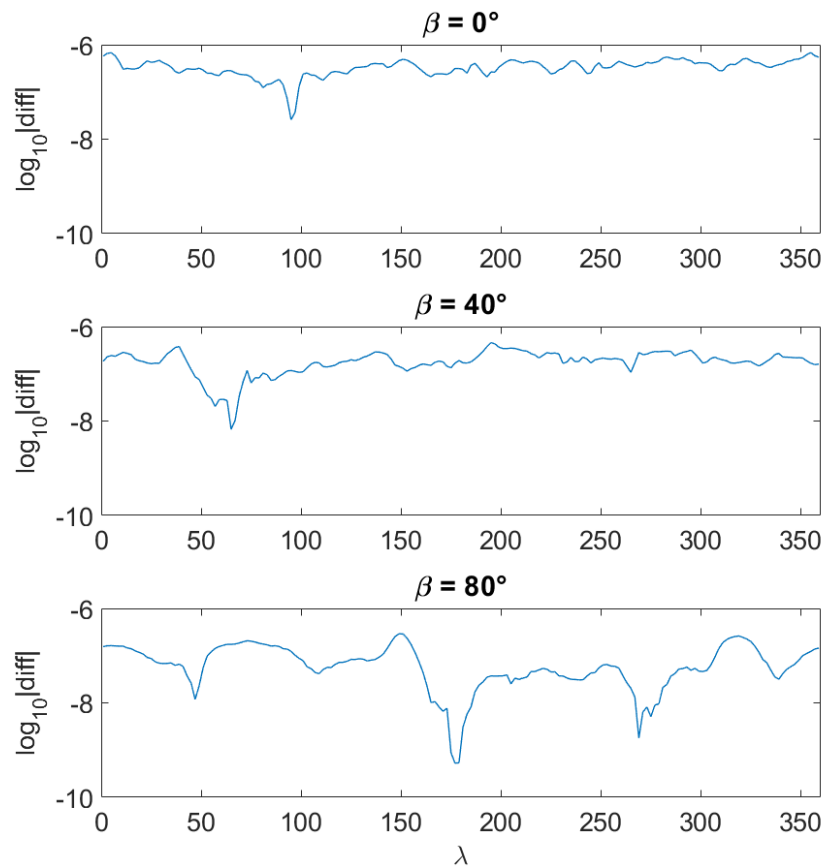
- Synthesis based on sets of spherical and spheroidal coefficients
  - Two calculation surfaces (Brillouin spheroid – outer expansion, spheroid inside the body masses – outer/inner expansion)
  - Maximal degree of expansion  $n = 720$  (computational demands)
  - Results for gravitational potential and its 1<sup>st</sup> and 2<sup>nd</sup> order spatial derivatives
- Computation points layout:
  - 1,620 computation points on each surface
  - 9 parallels ( $u_{\text{Brillouin}} = 455,479.0 \text{ m}$  /  $u_{\text{inside}} = 400,000.0 \text{ m}$ ,  $\beta = -80^\circ : 20^\circ : 80^\circ$ ,  $\lambda = 1^\circ : 2^\circ : 359^\circ$ )

Relative differences (spheroidal  $\times$  spherical) - synthesis of potential and vertical components

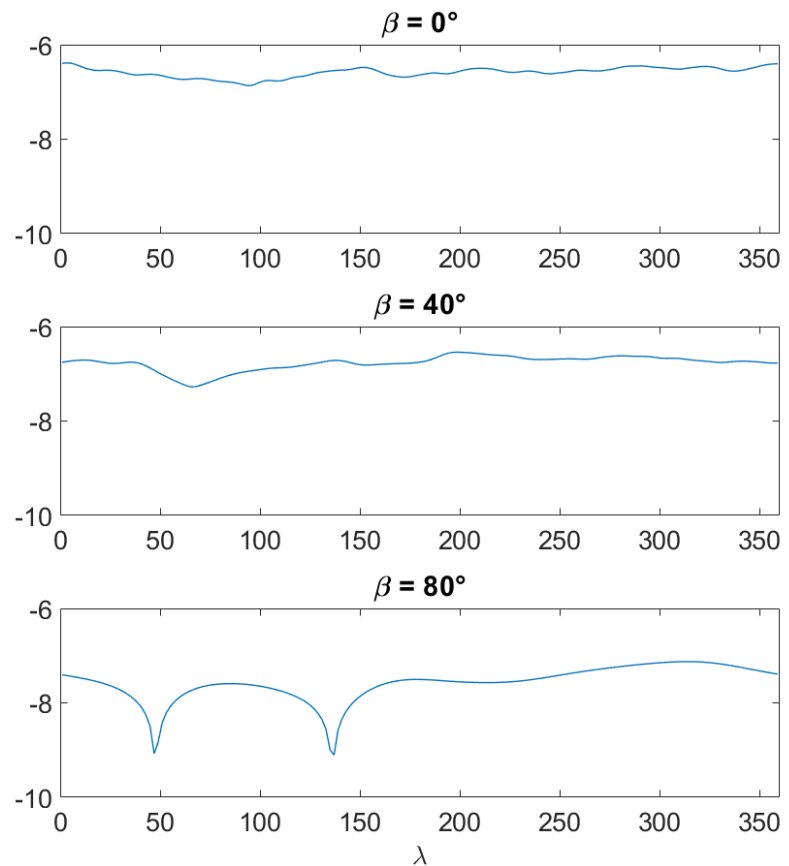
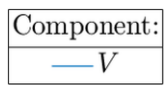
|      | Outer expansion              |                       |                          | Outer and inner expansion    |                       |                          |
|------|------------------------------|-----------------------|--------------------------|------------------------------|-----------------------|--------------------------|
|      | $V \text{ [m}^2/\text{s}^2]$ | $V_u \text{ [m/s}^2]$ | $V_{uu} \text{ [1/s}^2]$ | $V \text{ [m}^2/\text{s}^2]$ | $V_u \text{ [m/s}^2]$ | $V_{uu} \text{ [1/s}^2]$ |
| Mean | 1.7351e-07                   | 2.0843e-06            | 2.7861e-04               | 1.5093e-07                   | 4.8352e-07            | 1.5927e-06               |
| Std  | 1.2460e-07                   | 1.9131e-06            | 3.0820e-04               | 9.1237e-08                   | 3.7729e-07            | 1.5423e-06               |
| Min  | 5.8352e-11                   | 6.5788e-10            | 5.2453e-08               | 1.1104e-10                   | 2.5164e-10            | 2.9013e-10               |
| Max  | 6.8180e-07                   | 2.0829e-05            | 2.3422e-03               | 4.1168e-07                   | 2.4594e-06            | 1.5766e-05               |



Relative differences along parallels - potential

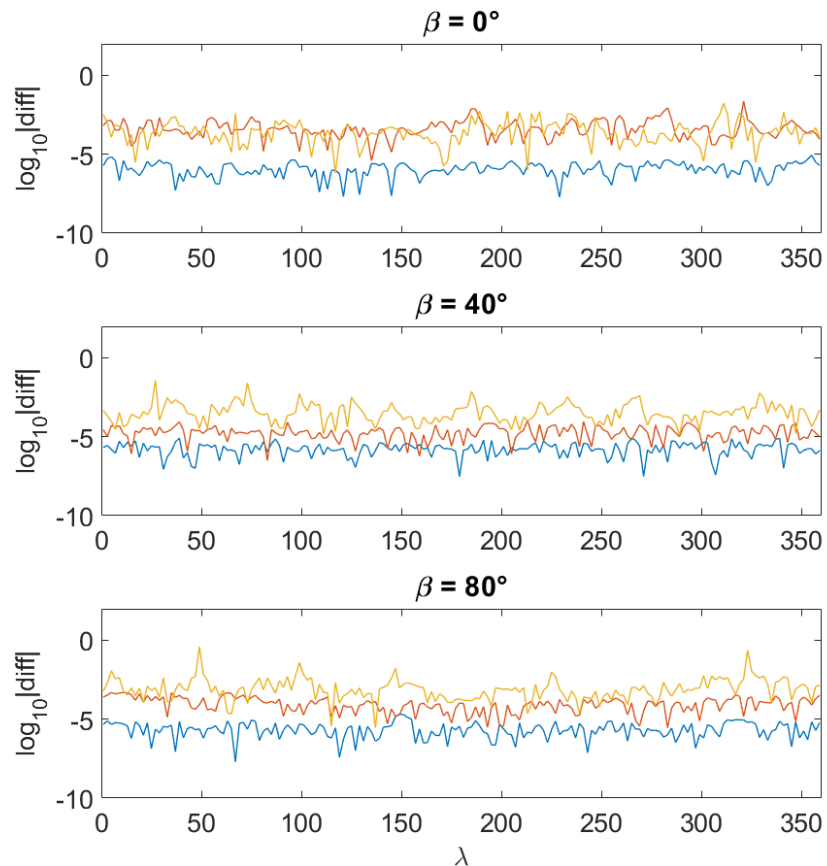


Outer expansion

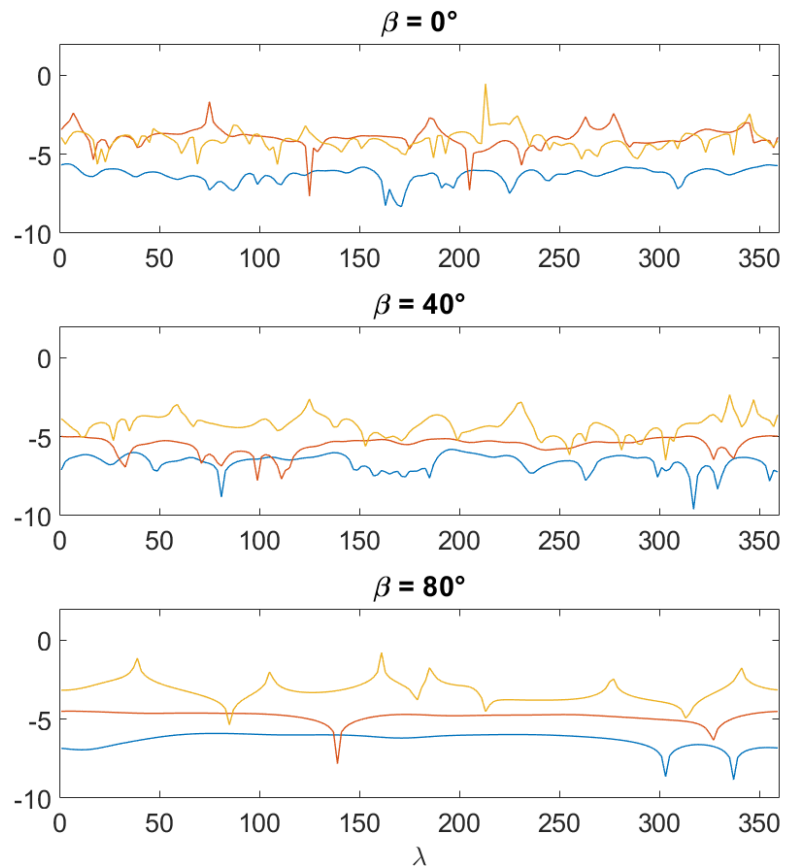
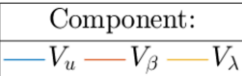


Outer and Inner expansion

# Relative differences along parallels – 1<sup>st</sup> derivatives

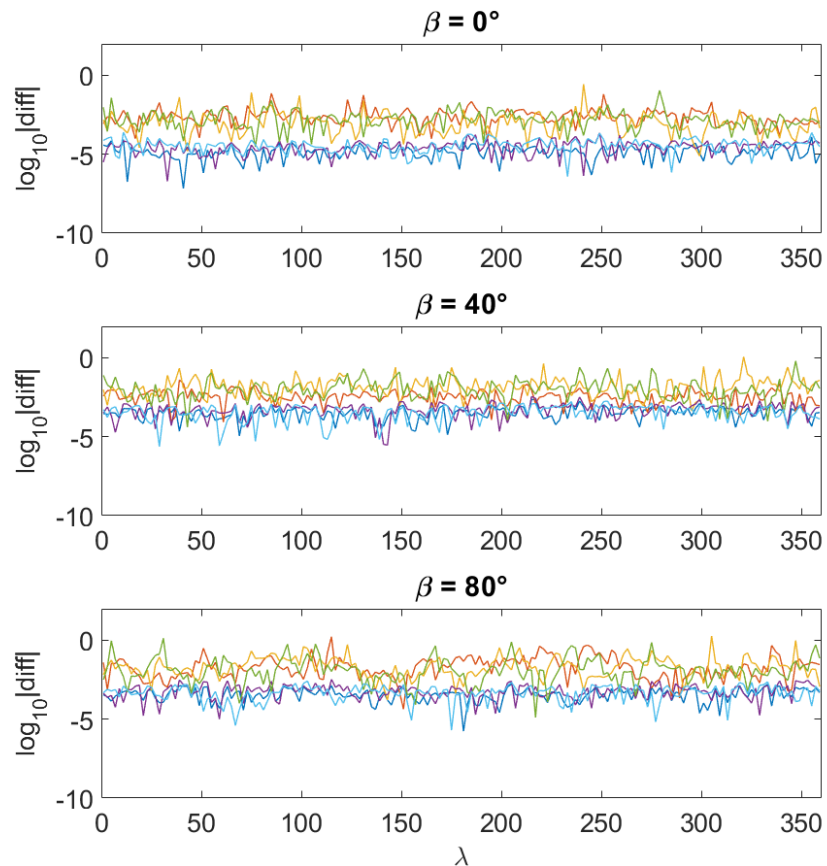


Outer expansion



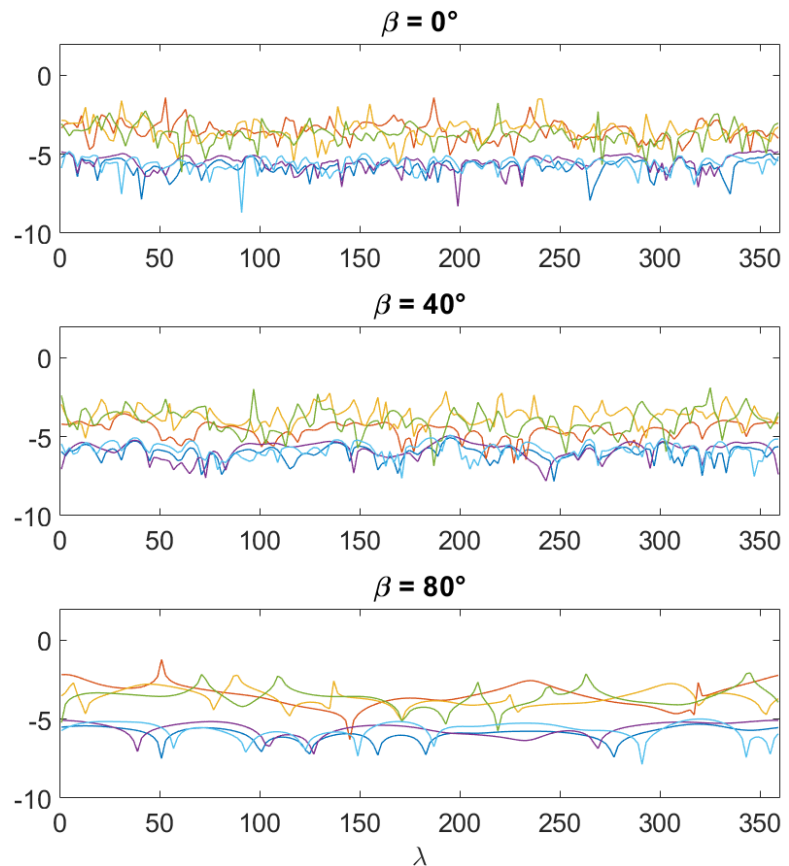
Outer and Inner expansion

Relative differences along parallels – 2<sup>nd</sup> derivatives



Outer expansion

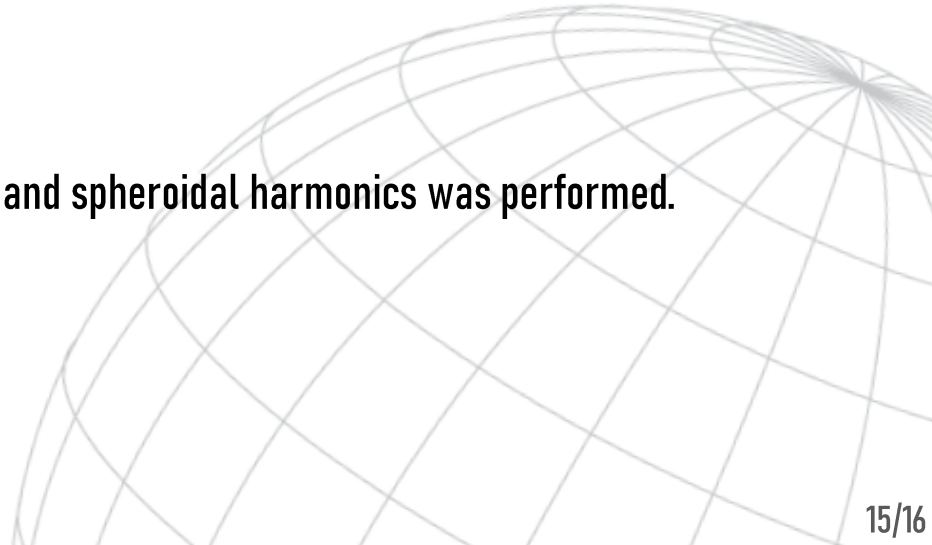
| Component: |                |                    |              |                  |                      |
|------------|----------------|--------------------|--------------|------------------|----------------------|
| $V_{uu}$   | $V_{u\lambda}$ | $V_{\beta\lambda}$ | $V_{u\beta}$ | $V_{\beta\beta}$ | $V_{\lambda\lambda}$ |



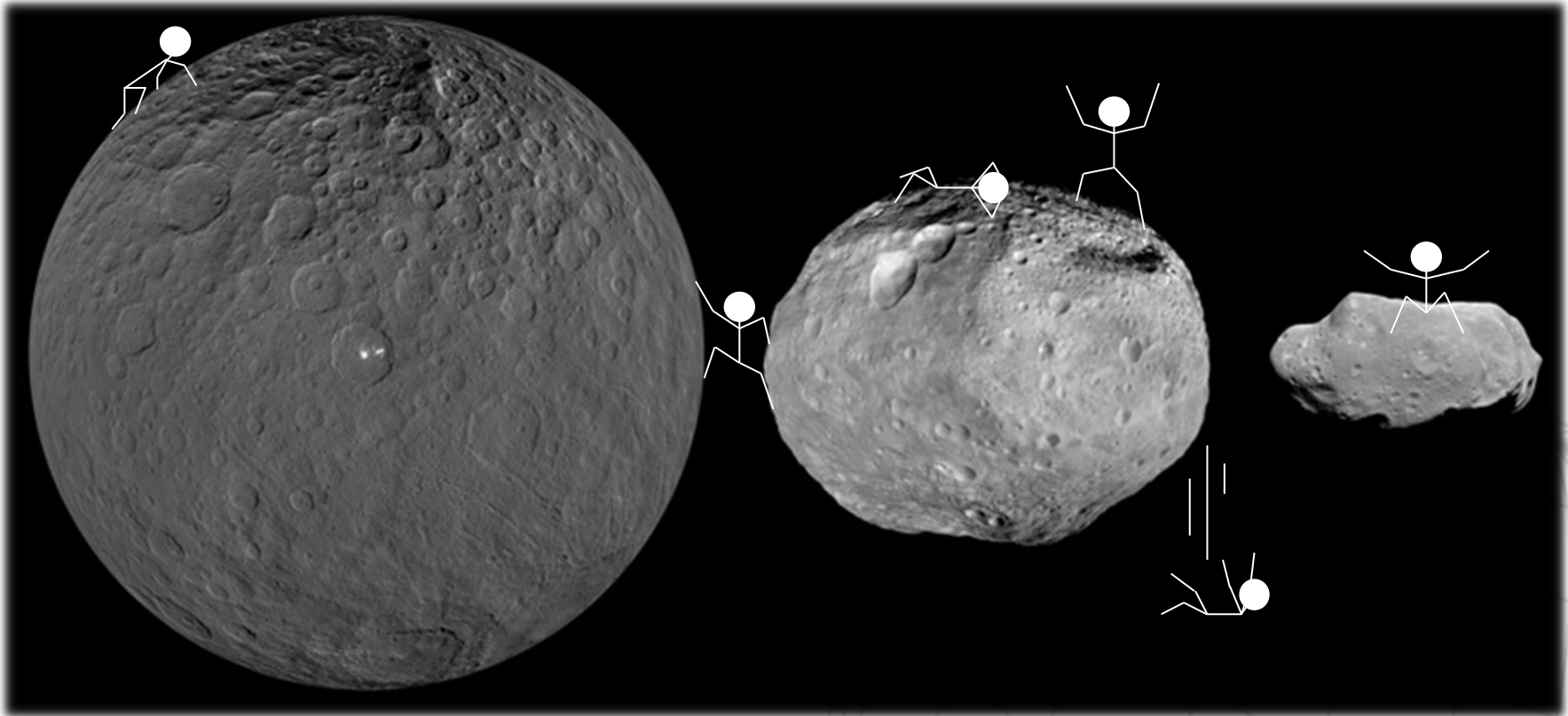
Outer and Inner expansion

# Conclusion

- **New theory for calculation using spheroidal harmonic analysis and synthesis was created.**
  - Relations for 20 functionals of gravitational field
  - Relations for oblate and prolate spheroids
  - Combination of outer and inner expansion
- **Theoretical equations were implemented into software library (C language).**
  - Implementation of all 20 functionals
  - Calculation for oblate spheroids
  - Combination of outer and inner expansion
- **Real data (Ceres) comparison between spherical and spheroidal harmonics was performed.**
- **Future work:**
  - High-degree testing of the 3<sup>rd</sup> spatial derivatives
  - Software library publication



# Thank you for your attention



Author was supported by SGS-2025-008 (Application of Mathematics and Informatics in Geomatics VI).

Computational resources provided by the e-INFRA CZ project (ID:90254), supported by the Ministry of Education, Youth and Sports of the Czech Republic.

Jiří Belinger (belinger@fav.zcu.cz)

Relative differences between spheroidal and spherical synthesis results for individual components

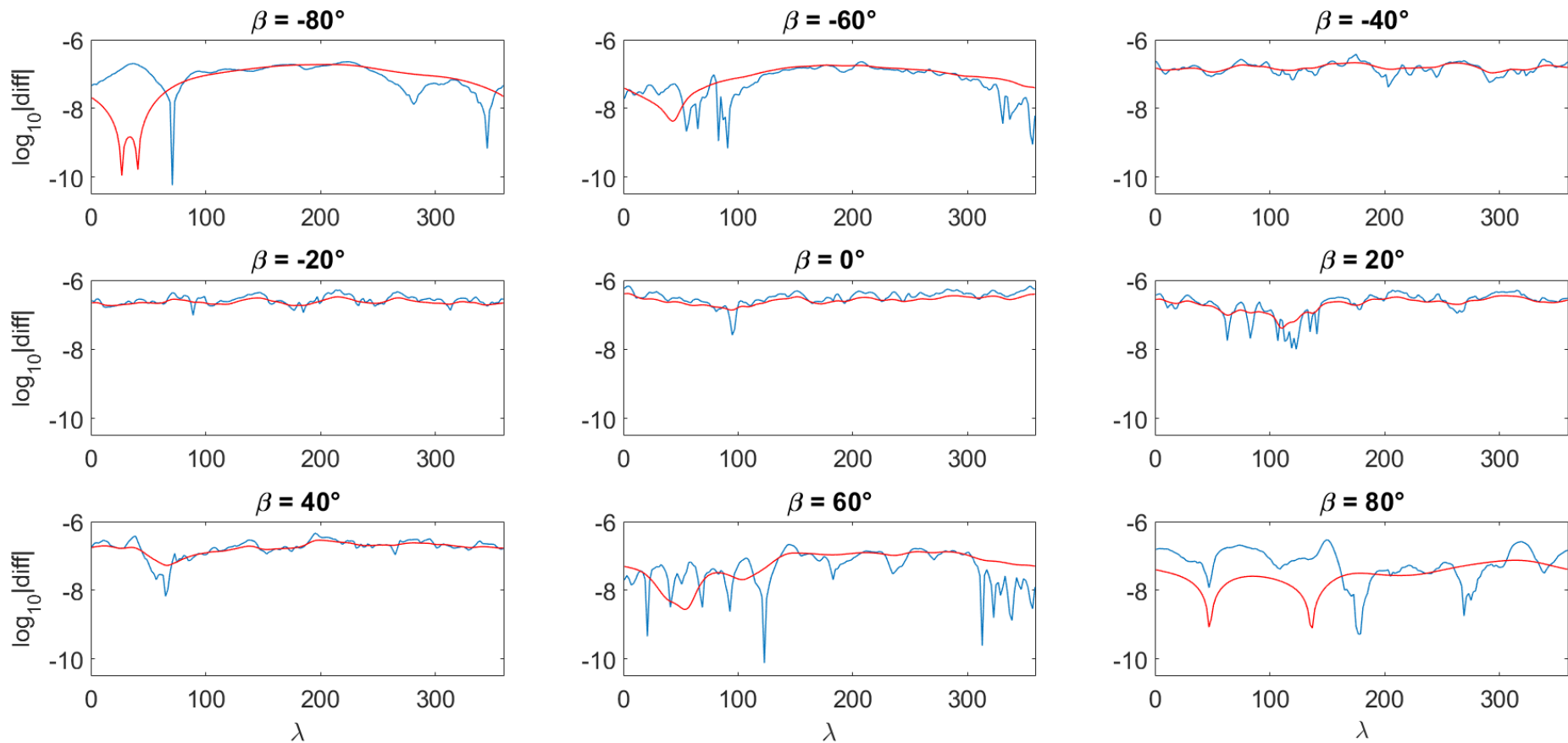
Outer expansion

|      | Pot. [m <sup>2</sup> /s <sup>2</sup> ] | 1 <sup>st</sup> derivatives [m/s <sup>2</sup> ] |                |                | 2 <sup>nd</sup> derivatives [1/s <sup>2</sup> ] |                 |                 |                 |                 |                 |
|------|----------------------------------------|-------------------------------------------------|----------------|----------------|-------------------------------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
|      | V                                      | V <sub>u</sub>                                  | V <sub>β</sub> | V <sub>λ</sub> | V <sub>uu</sub>                                 | V <sub>uβ</sub> | V <sub>uλ</sub> | V <sub>ββ</sub> | V <sub>βλ</sub> | V <sub>λλ</sub> |
| Mean | 1.7351e-07                             | 2.0843e-06                                      | 1.3780e-04     | 1.6395e-03     | 2.7861e-04                                      | 2.1462e-02      | 3.5785e-02      | 4.9035e-04      | 3.6233e-02      | 3.7882e-04      |
| Std  | 1.2460e-07                             | 1.9131e-06                                      | 7.4145e-04     | 0.0132e-02     | 3.0820e-04                                      | 3.2335e-01      | 1.7596e-01      | 5.5502e-04      | 1.7499e-01      | 4.1456e-04      |
| Min  | 5.8352e-11                             | 6.5788e-10                                      | 5.0944e-08     | 1.8295e-07     | 5.2453e-08                                      | 2.7385e-06      | 3.8264e-06      | 1.3473e-07      | 1.1901e-06      | 3.5562e-08      |
| Max  | 6.8180e-07                             | 2.0829e-05                                      | 2.3584e-02     | 3.9604e-01     | 2.3422e-03                                      | 1.2731e01       | 5.0463e00       | 3.5281e-03      | 4.4791e00       | 2.6430e-03      |

Outer and inner expansion

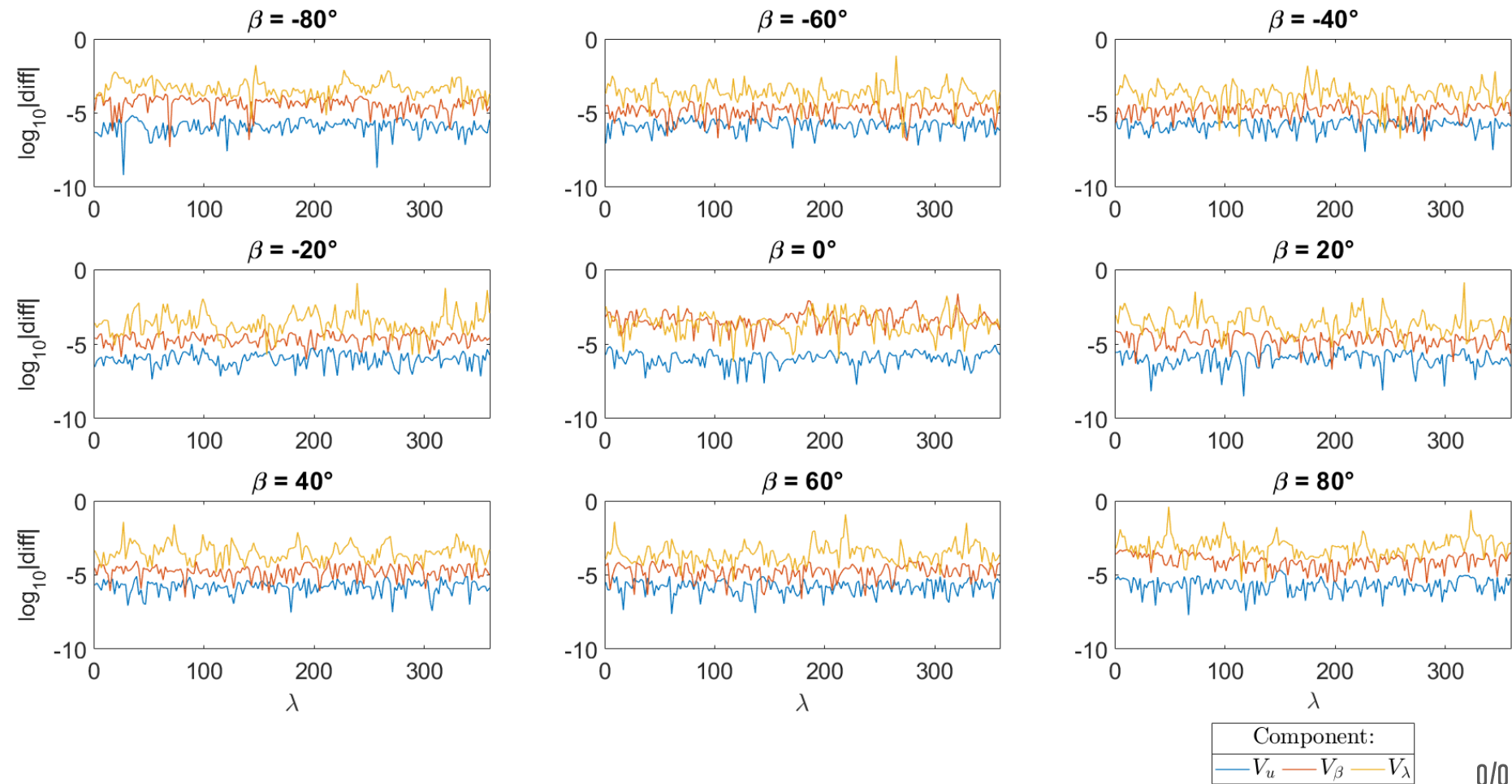
|      | Pot. [m <sup>2</sup> /s <sup>2</sup> ] | 1 <sup>st</sup> derivatives [m/s <sup>2</sup> ] |                |                | 2 <sup>nd</sup> derivatives [1/s <sup>2</sup> ] |                 |                 |                 |                 |                 |
|------|----------------------------------------|-------------------------------------------------|----------------|----------------|-------------------------------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
|      | V                                      | V <sub>u</sub>                                  | V <sub>β</sub> | V <sub>λ</sub> | V <sub>uu</sub>                                 | V <sub>uβ</sub> | V <sub>uλ</sub> | V <sub>ββ</sub> | V <sub>βλ</sub> | V <sub>λλ</sub> |
| Mean | 1.5093e-07                             | 4.8352e-07                                      | 5.0521e-05     | 7.7329e-04     | 1.5927e-06                                      | 4.2156e-04      | 7.9588e-04      | 2.8710e-06      | 7.3163e-04      | 2.7799e-06      |
| Std  | 9.1237e-08                             | 3.7729e-07                                      | 5.6803e-04     | 8.8877e-03     | 1.5423e-06                                      | 3.1907e-03      | 6.8299e-03      | 2.6204e-06      | 4.9184e-03      | 2.3572e-06      |
| Min  | 1.1104e-10                             | 2.5164e-10                                      | 6.3482e-09     | 3.2982e-07     | 2.9013e-10                                      | 2.0235e-07      | 9.8135e-08      | 5.5562e-10      | 1.3833e-07      | 1.5032e-09      |
| Max  | 4.1168e-07                             | 2.4594e-06                                      | 2.1543e-02     | 2.9938e-01     | 1.5766e-05                                      | 8.9516e-02      | 2.6295e-01      | 2.0718e-05      | 1.5255e-01      | 1.7584e-05      |

# Relative differences along parallels - potential

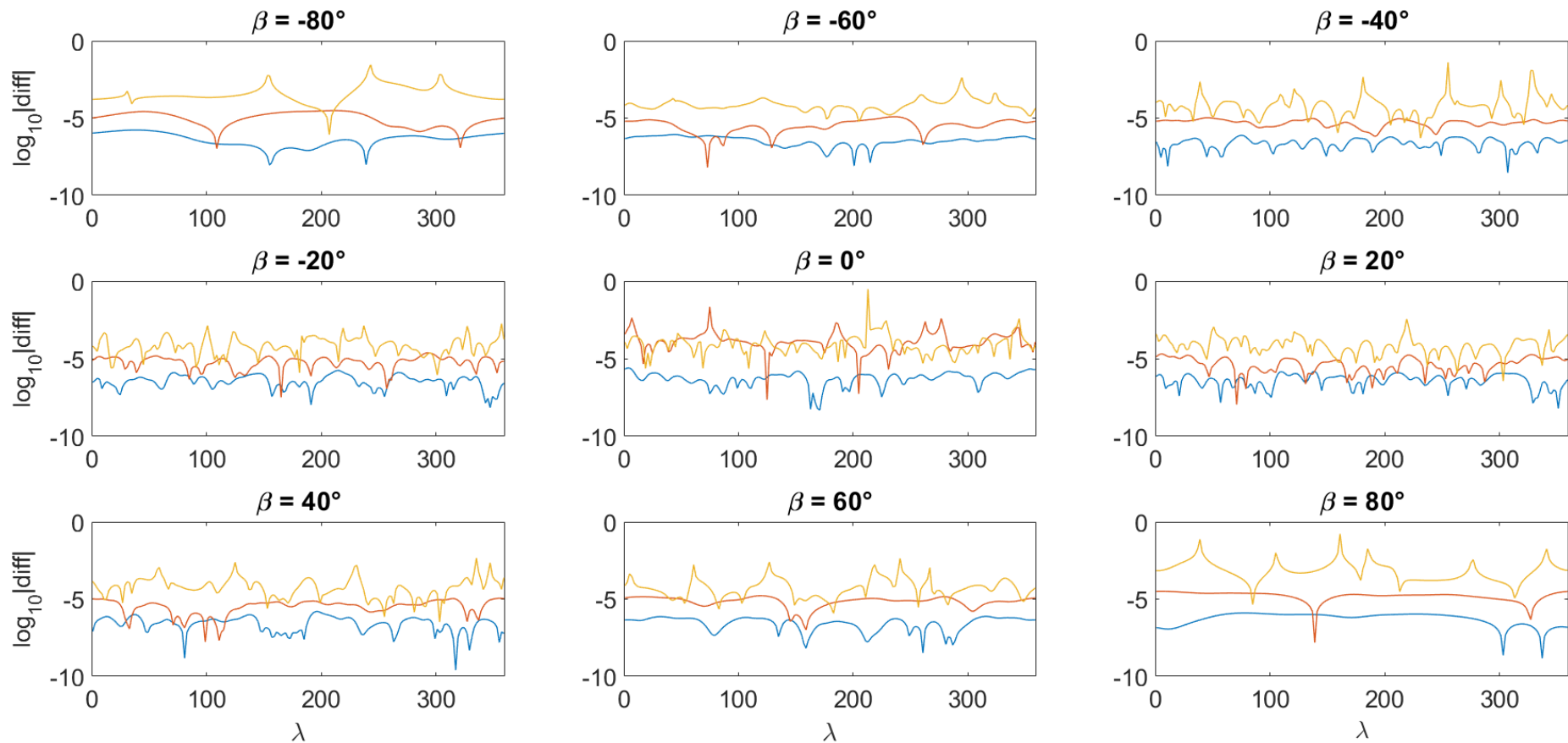


Component:  
 $V_{out}$   $V_{in}$

## Relative differences along parallels – 1<sup>st</sup> derivatives (outer expansion only)

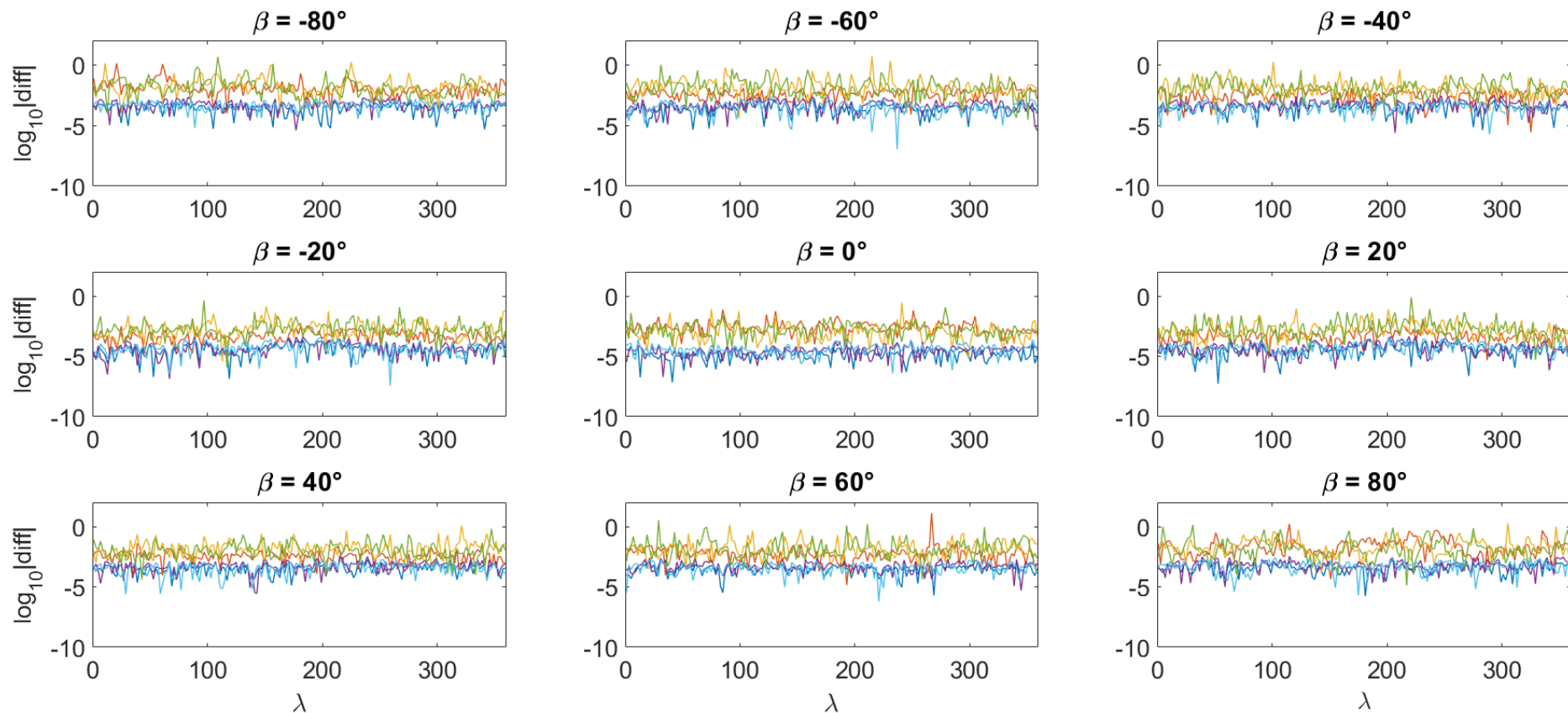


## Relative differences along parallels - 1<sup>st</sup> derivatives (combination of outer and inner expansion)



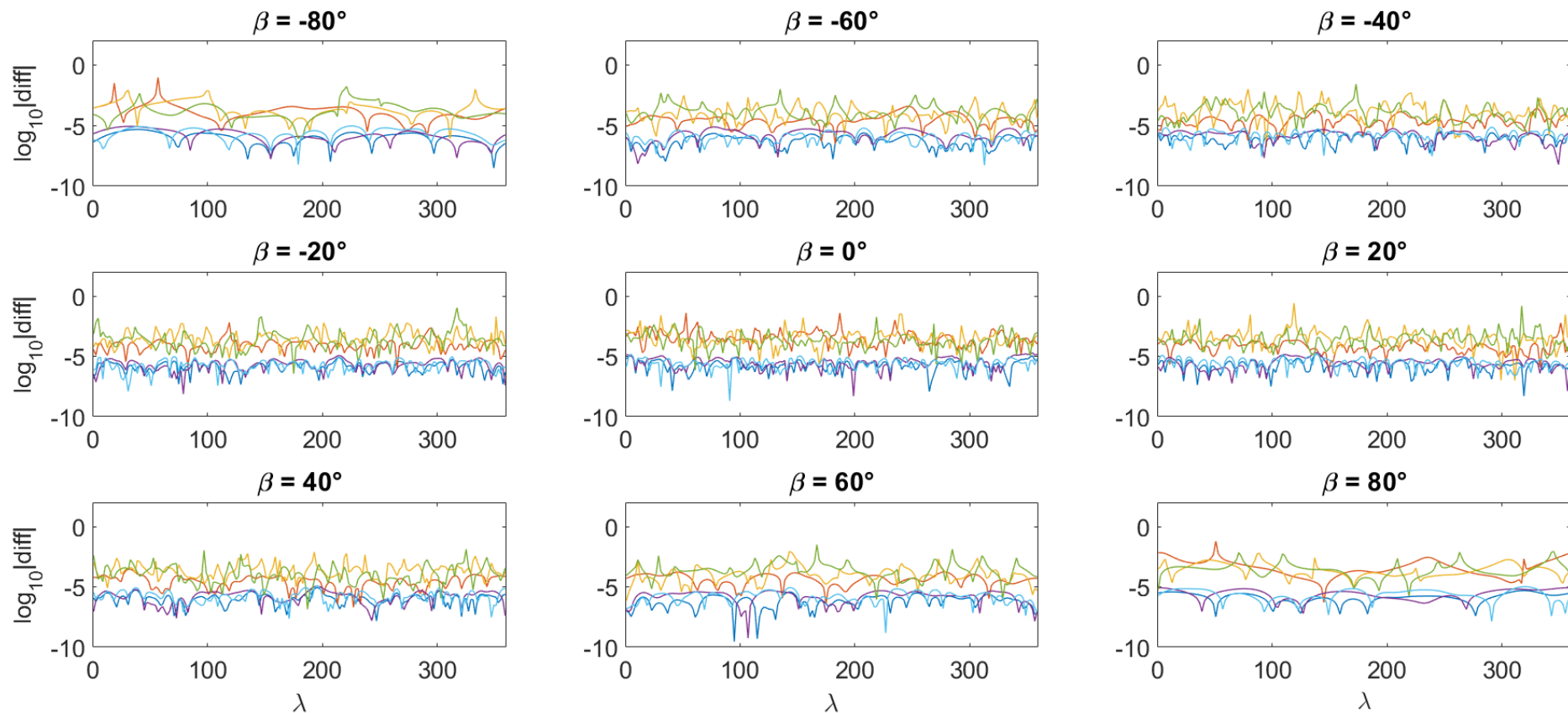
Component:  
— V<sub>u</sub> — V<sub>β</sub> — V<sub>λ</sub>

## Relative differences along parallels – 2<sup>nd</sup> derivatives (outer expansion only)



| Component:   |                  |                      |
|--------------|------------------|----------------------|
| $V_{uu}$     | $V_{u\lambda}$   | $V_{\beta\lambda}$   |
| $V_{u\beta}$ | $V_{\beta\beta}$ | $V_{\lambda\lambda}$ |

## Relative differences along parallels - 2<sup>nd</sup> derivatives (combination of outer and inner expansion)



| Component:   |                  |                      |
|--------------|------------------|----------------------|
| $V_{uu}$     | $V_{u\lambda}$   | $V_{\beta\lambda}$   |
| $V_{u\beta}$ | $V_{\beta\beta}$ | $V_{\lambda\lambda}$ |