

FEM approach in view of cm-accurate quasigeoid modelling in high mountains

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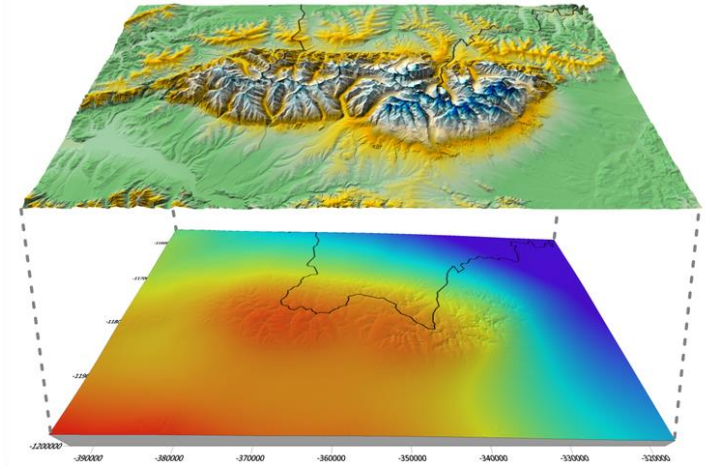
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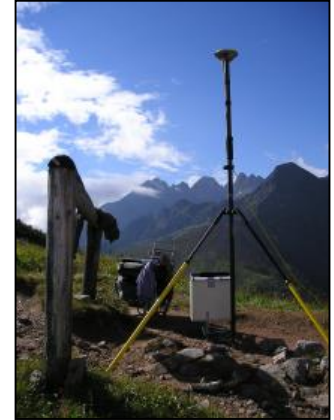
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Outline

- fixed gravimetric BVP (FGBVP)
- *FEM numerical approach to solve FGBVP*
 - *treatment of the oblique derivative problems*
- *case studies in mountainous areas:*
 - *Tatra Mountains*
 - *'EAlpG' test region in Zugspitze*
 - *Colorado*
- *comparisons with solutions of approaches solved in spectral domain*
- *Conclusions*



Fixed gravimetric BVP (FGBVP)

the exterior oblique derivative BVP for the Laplace equation

$$\begin{aligned}\Delta T(\mathbf{x}) &= 0, \quad \mathbf{x} \in R^3 - S, \\ \nabla T(\mathbf{x}) \cdot \mathbf{s}(\mathbf{x}) &= -\delta g(\mathbf{x}), \quad \mathbf{x} \in \partial S, \\ T(\mathbf{x}) &\rightarrow 0, \quad \text{as } |\mathbf{x}| \rightarrow \infty,\end{aligned}$$

Precise 3D positioning by GNSS → *the fixed boundary*



The oblique derivative problem increases in high mountains,
especially in areas of steep mountain peaks or ridges

→ *numerical solution of FGBVP in the spatial domain directly on the Earth's surface*

numerical methods vs. classical approaches

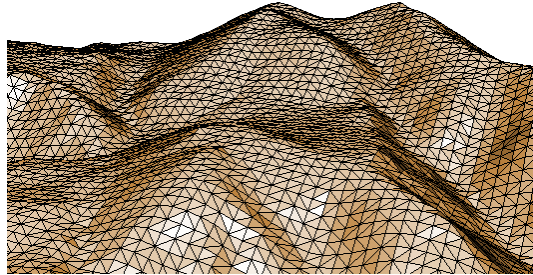
Numerical methods

(usually in spatial domain)

numerical solution directly on
the discretized Earth's surface



Oblique derivative problem



classical approaches

(usually in spectral domain)

e.g. Remove-Compute-Restore

topographic effect evaluated separately

first removed from input gravity data



Integration from residual gravity data
(approximation of the Earth's surface)



afterwards restored in terms
of disturbing potential

Numerical methods - **alternatives** to classical approaches

Numerical methods to solve FGBVP

BEM

Boundary Element Method

- based on boundary integral equations → only the Earth's surface is discretized (suitable for the exterior BVPs)

(Čunderlík et al. 2008), (Čunderlík and Mikula 2010), (Čunderlík et al. 2012)

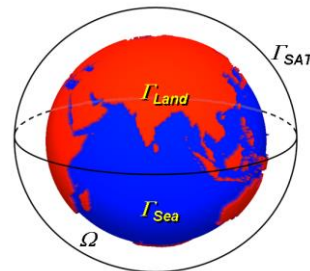


FEM

Finite Element Method

- discretization of the whole 3D computational domain → FEM solution fixed on the artificial upper boundary

(Fašková et al. 2010), (Fašková et al. 2012), (Minarechová et al. 2021)

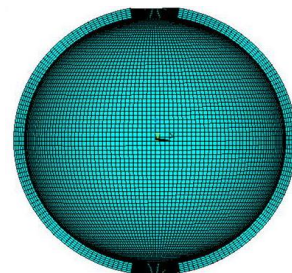


FVM

Finite Volume Method

- discretization of the whole 3D computational domain → FVM solution fixed on the artificial upper boundary

(Macák et al. 2015), (Medl'a et al. 2018), (Droniou et al. 2019), (Čunderlík et al. 2020)



FEM approach for local solutions

“satellite-fixed” FGBVP

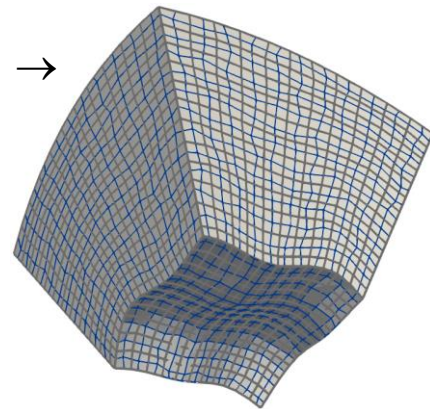
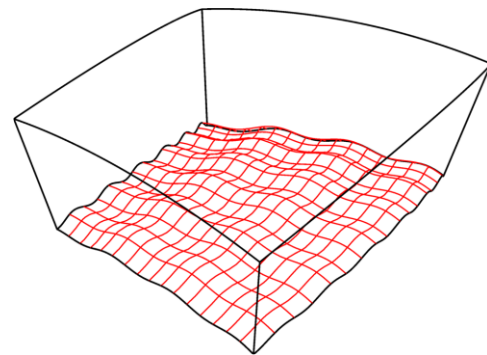
→ for bounded domains

$$\begin{aligned}\Delta T(\mathbf{x}) &= 0, \quad \mathbf{x} \in \Omega \subset \mathbf{R}^3, \\ \nabla T(\mathbf{x}) \cdot \mathbf{s}(\mathbf{x}) &= -\delta g(\mathbf{x}), \quad \mathbf{x} \in \Gamma \subset \partial\Omega, \\ T(\mathbf{x}) &= T_{SAT}(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega - \Gamma,\end{aligned}$$

→ FEM numerical solution fixed on the artificial upper boundary

- discretization of the whole 3D computational domain into finite elements $\Omega^e \rightarrow$
- weak formulation on the finite element Ω^e with weight functions w

$$\int_{\Omega^e} \nabla T \cdot \nabla w \, dx dy dz = \int_{\partial\Omega^e} \nabla T \cdot \mathbf{n} w \, d\sigma,$$

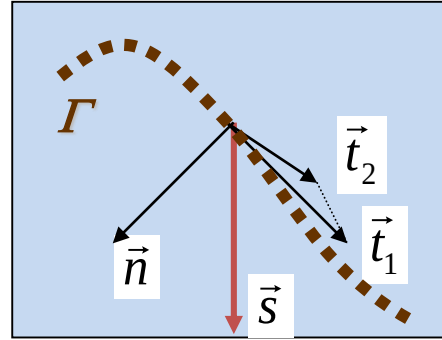


Oblique derivative problem

- treatment of the oblique derivative BC :

$$\mathbf{s} = c_1 \mathbf{n} + c_2 \mathbf{t}_1 + c_3 \mathbf{t}_2, \quad \nabla T \cdot \mathbf{s} = c_1 \nabla T \cdot \mathbf{n} + c_2 \nabla T \cdot \mathbf{t}_1 + c_3 \nabla T \cdot \mathbf{t}_2 = -\delta g.$$

$$\nabla T \cdot \mathbf{n} = \frac{-\delta g}{c_1} - \frac{c_2}{c_1} \frac{\partial T}{\partial \mathbf{t}_1} - \frac{c_3}{c_1} \frac{\partial T}{\partial \mathbf{t}_2},$$

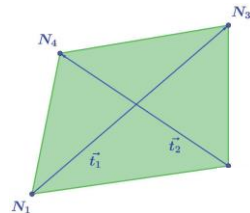


- weak formulation for finite elements on the Earth's surface

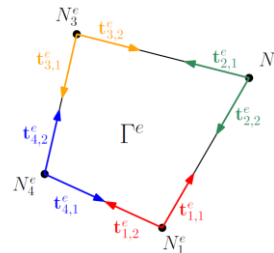
$$\int_{\Omega^e} \nabla T \cdot \nabla w \, dx dy dz + \frac{c_2}{c_1} \int_{\Gamma^e} \frac{\partial T}{\partial \mathbf{t}_1} w \, d\sigma + \frac{c_3}{c_1} \int_{\Gamma^e} \frac{\partial T}{\partial \mathbf{t}_2} w \, d\sigma = \int_{\Gamma^e} \frac{-\delta g}{c_1} w \, d\sigma + \int_{\partial \Omega^e \setminus \Gamma^e} \nabla T \cdot \mathbf{n} w \, d\sigma.$$

- discretized form for the hexahedral element

a) two tangential vector per bottom side



b) two tangential vector in every node



Macák et al. (2020) The finite element method as a tool to solve the oblique derivative boundary value problem in geodesy. Tatra Mt Math Publ 75(1): 63–80

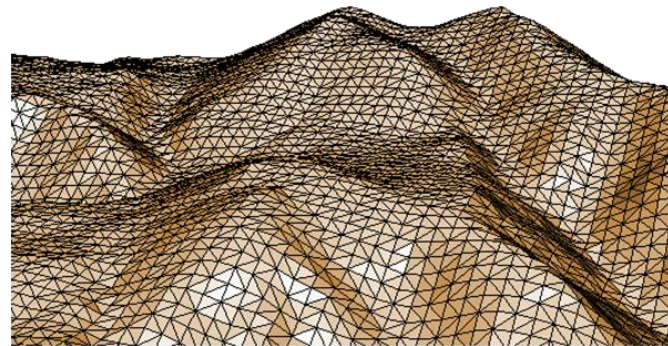
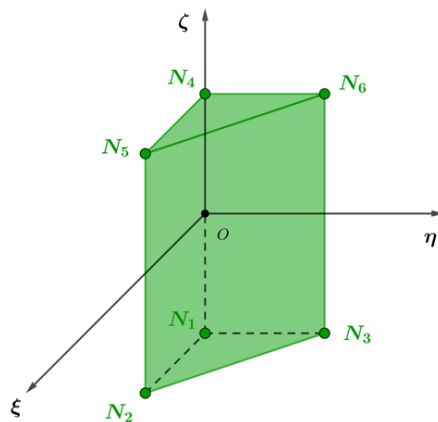
Minarechová et al. (2021) On the finite element method for solving the oblique derivative boundary value problems and its application in local gravity field modelling. J Geod 95:70

FEM implementation with pentahedral elements

- discretized form for [the pentahedral element](#) with 6 nodes (triangular prism)

$$T \approx T^e = \sum_{j=1}^6 T_j^e \psi_j$$

N_i	ξ	η	ζ	Shape functions $\psi_i(\xi, \eta, \zeta)$
N_1	0	0	-1	$(1 - \xi - \eta)(1 - \zeta)/2$
N_2	1	0	-1	$\xi(1 - \zeta)/2$
N_3	0	1	-1	$\eta(1 - \zeta)/2$
N_4	0	0	1	$(1 - \xi - \eta)(1 + \zeta)/2$
N_5	1	0	1	$\xi(1 + \zeta)/2$
N_6	0	1	1	$\eta(1 + \zeta)/2$



- local linear system of equations for each element $\Omega^e \Rightarrow$ **element stiffness matrix**
- merging all element stiffness matrices $\Rightarrow$ **global stiffness matrix**
 - continuity of the prime variables (disturbing potential)
 - “equilibrium” of the second variables (fluxes)

$$\mathbf{K}^e \mathbf{T}^e = \mathbf{Q}^e,$$

$$\mathbf{K} \mathbf{T} = \mathbf{Q},$$

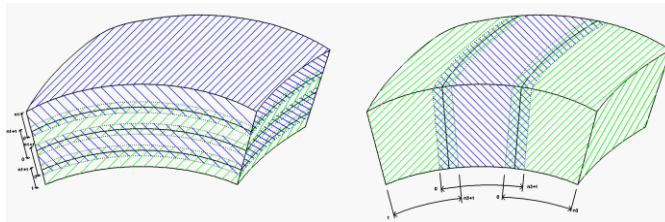
Computational aspects

Linear system of equations

⇒ *sparse system matrix (28-point stencil)*

Parallel implementation (MPI + openMP)

→ data splitting into processors/cores



Large-scale parallel computations on clusters with distributed memory

- *discretization of the 3D computational domain* ⇒ **memory requirements**

An example:

$$3750 \times 2050 \times 500 = \mathbf{3,843,750,000 \text{ elements}}$$



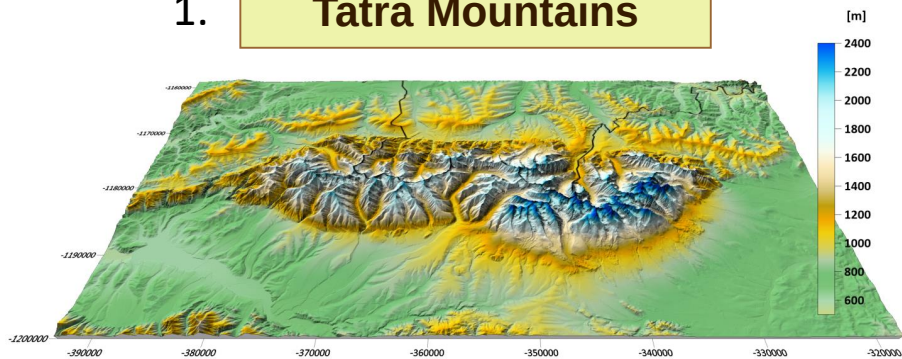
≈ 1.2 TB

- *performed on 192 cores* ⇒ 48 MPI processes (each with 4 OpenMP threads)

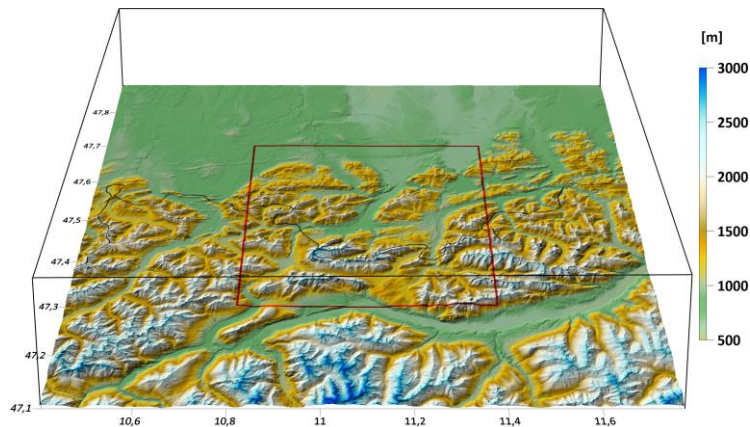
CPU time ⇒ **90 h (≈ 3.8 days)**

Numerical experiments in mountainous areas

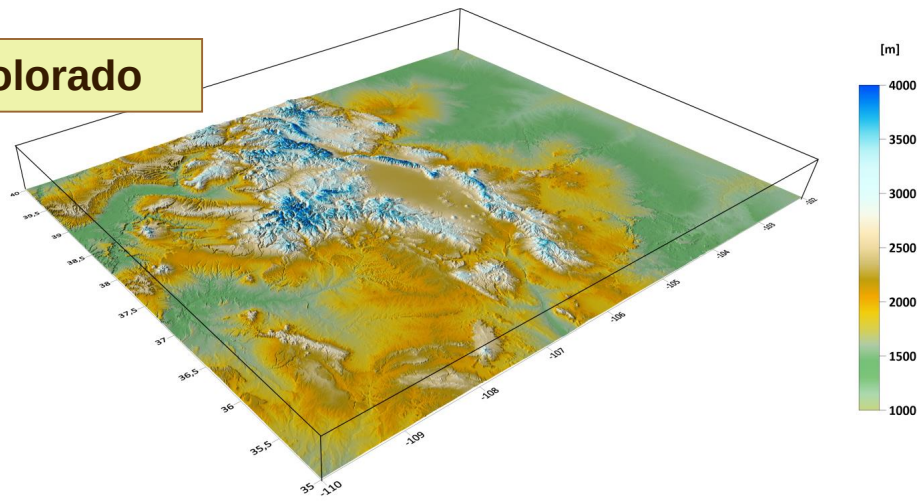
1. Tatra Mountains



2. Zugspitze area

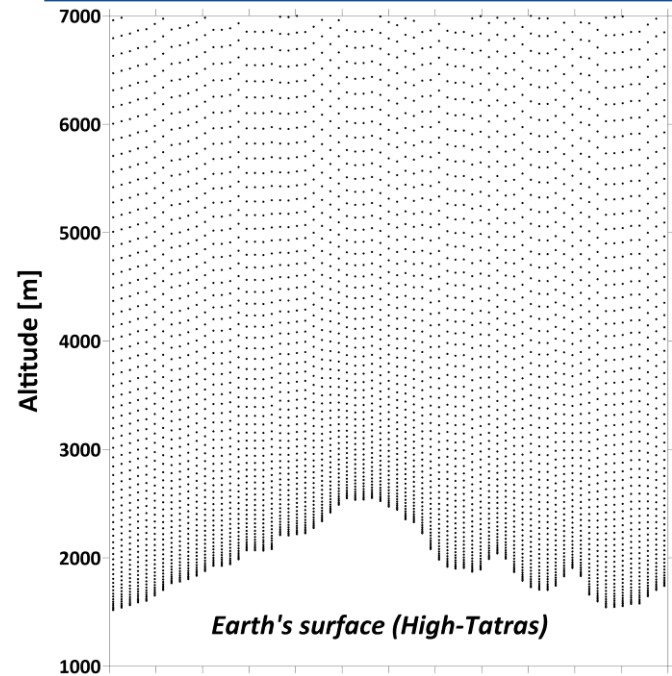


3. Colorado



Can we achieve
cm-level accuracy?

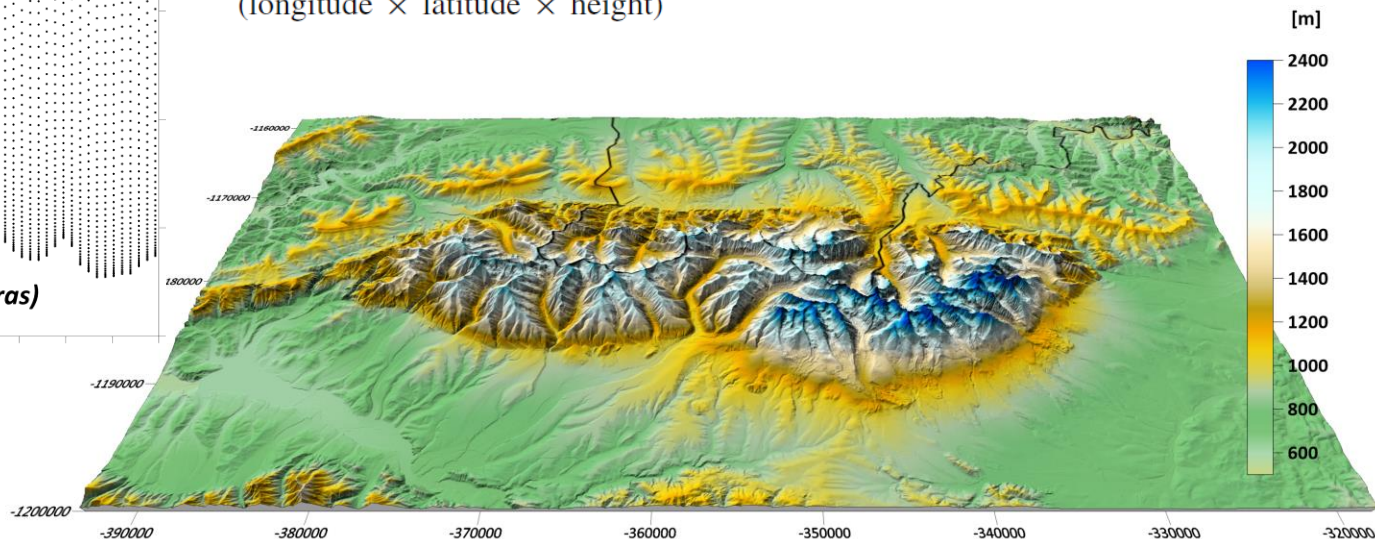
Tatra Mountains – discretization of the computational domain



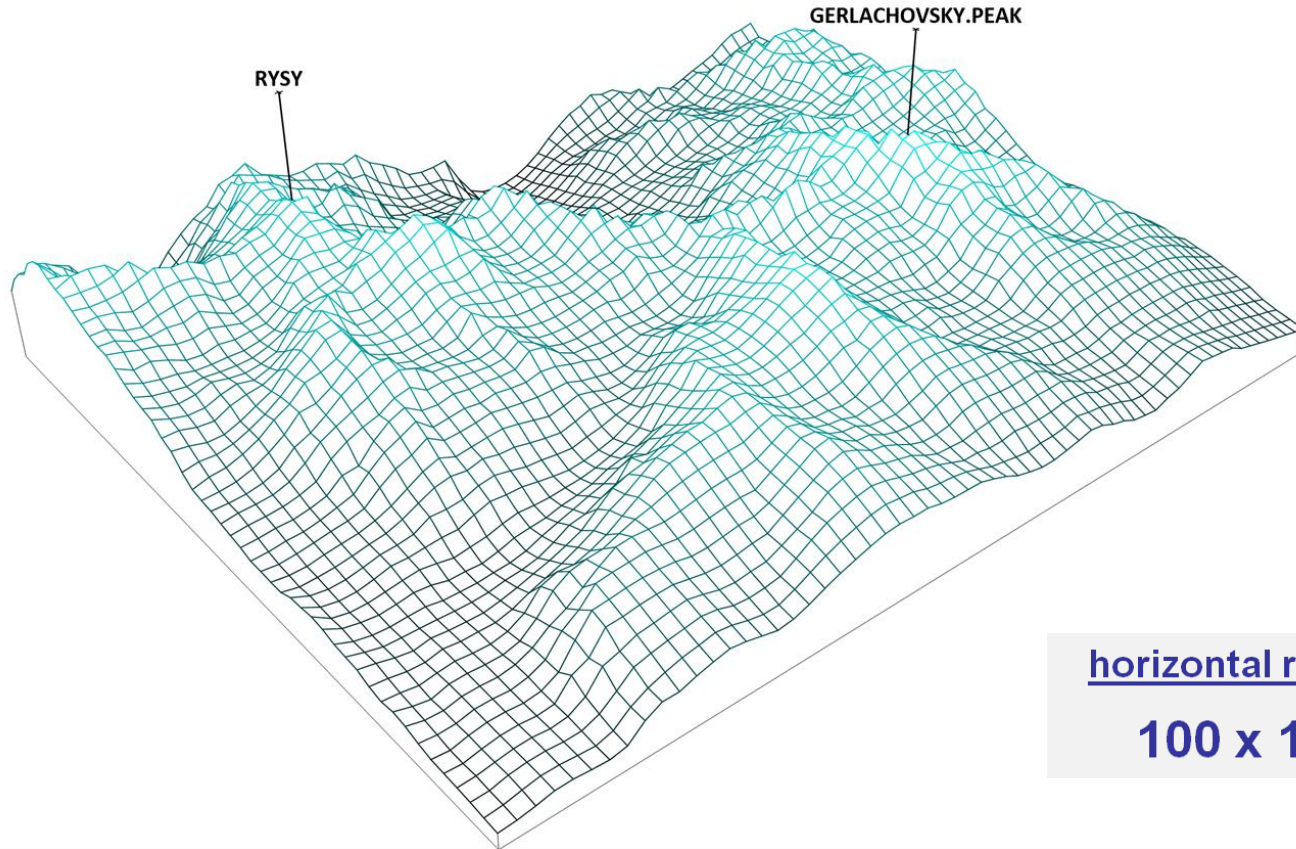
- discretized Earth's topography (horizontal resolution 20 x 20 m)
non-uniform division in the radial directions
 - 5 m on the Earth's surface (the bottom boundary)
 - 1 km at 230 km (the upper boundary)

$$3750 \times 2050 \times 500 = 3,843,750,000 \text{ elements}$$

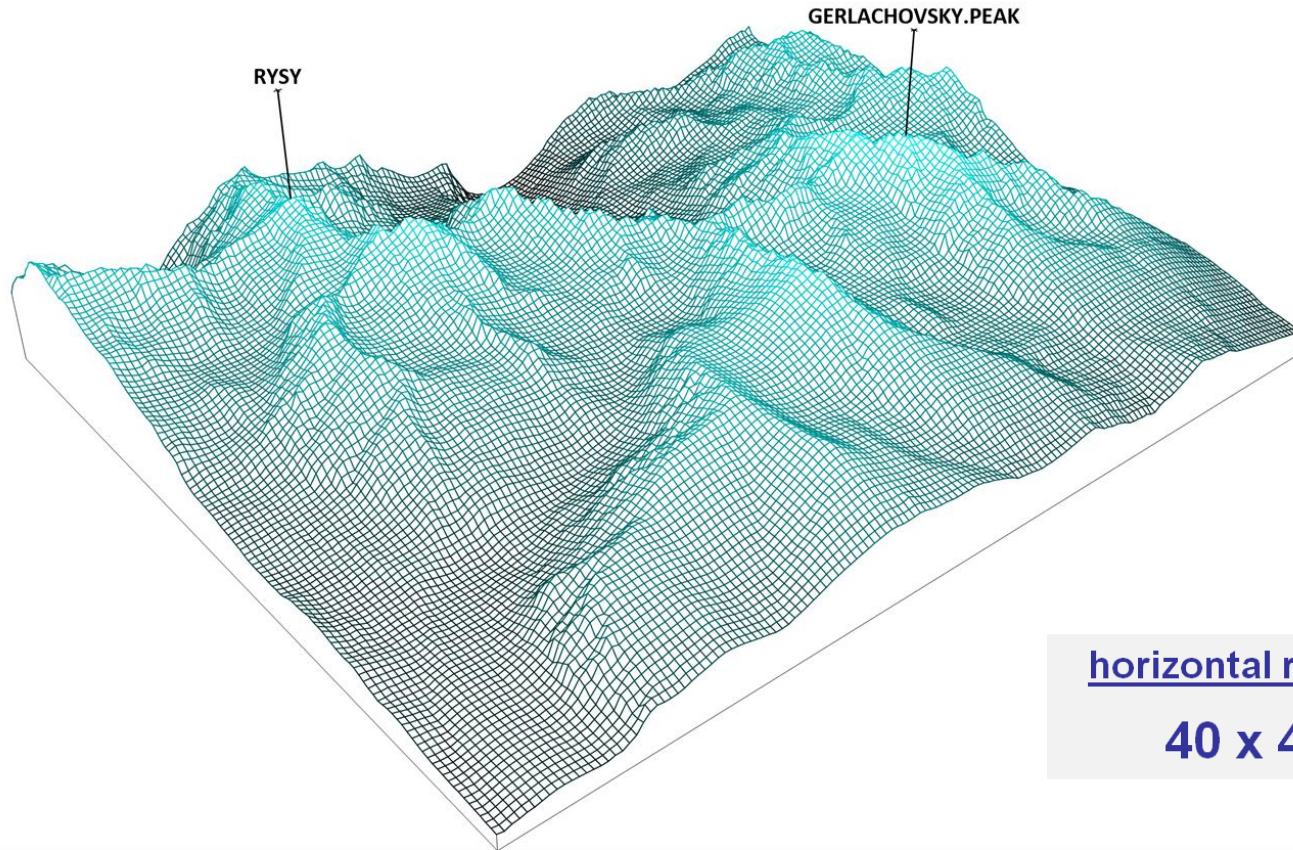
(longitude $\times$ latitude $\times$ height)



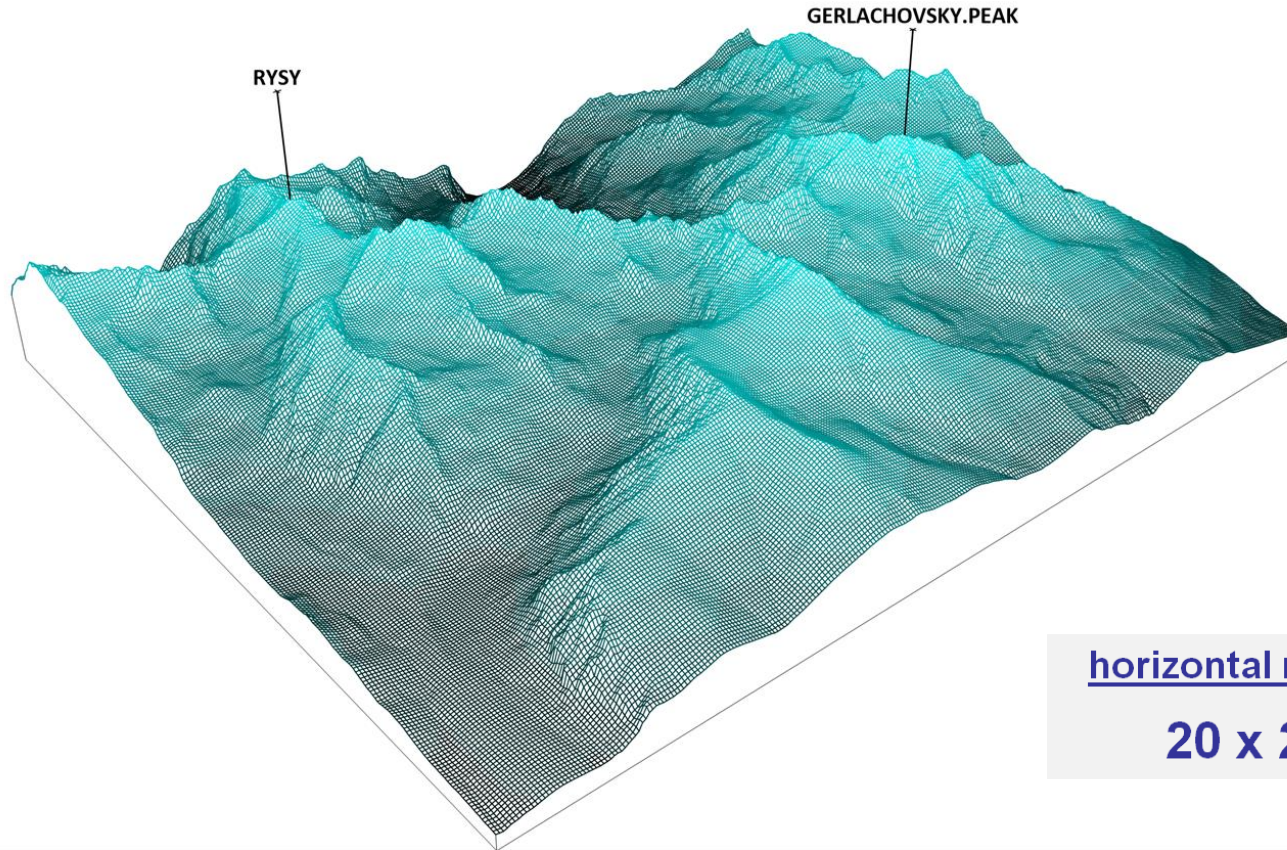
Level of discretization



Level of discretization

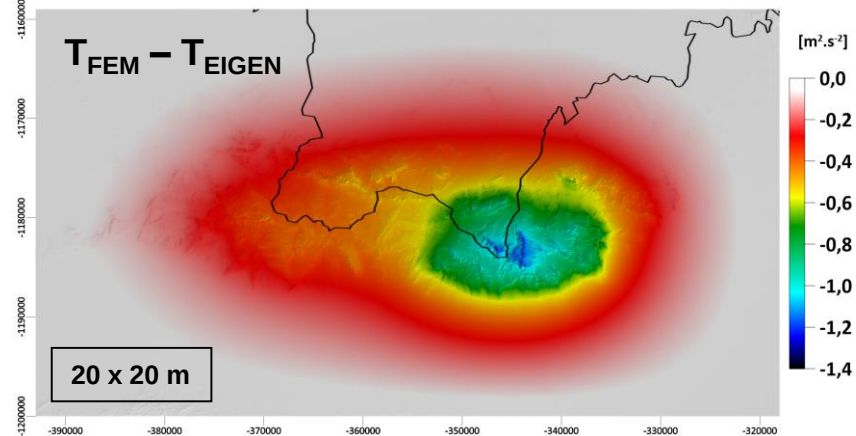
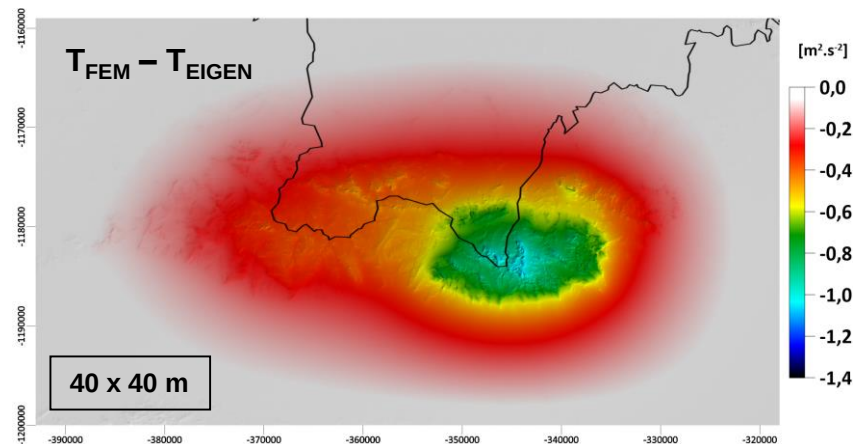
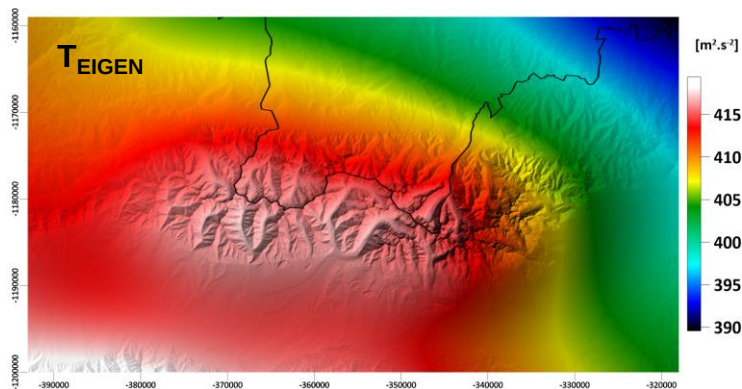
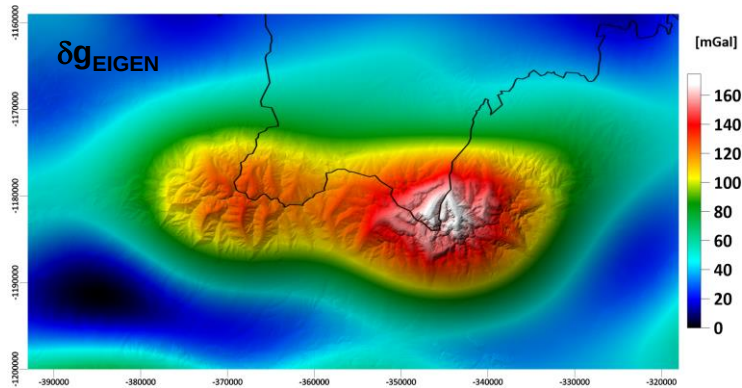


Level of discretization

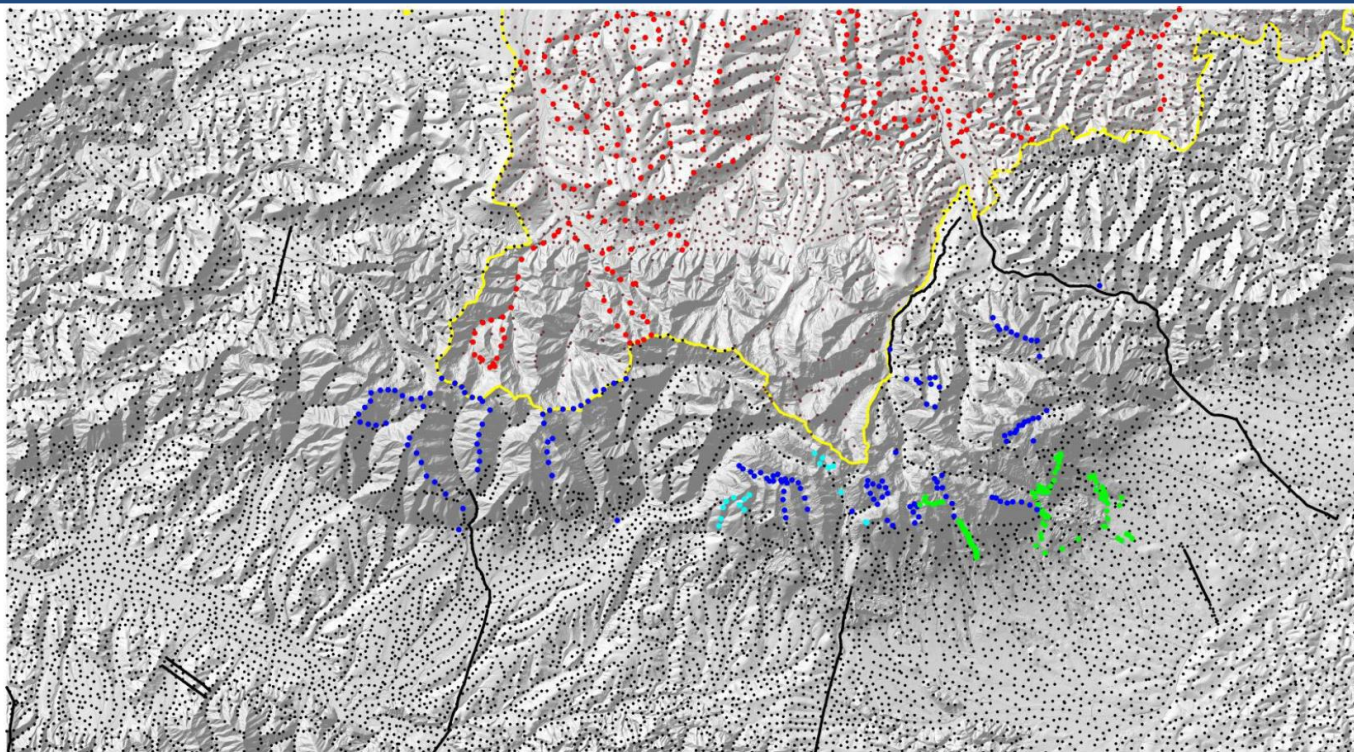


Estimates of the distretization error

Reconstruction of EIGEN6C4



Gravimetric mapping in Tatra Mountains

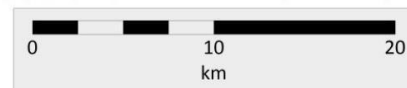


old measurements:

- - in Slovakia
- - in Poland

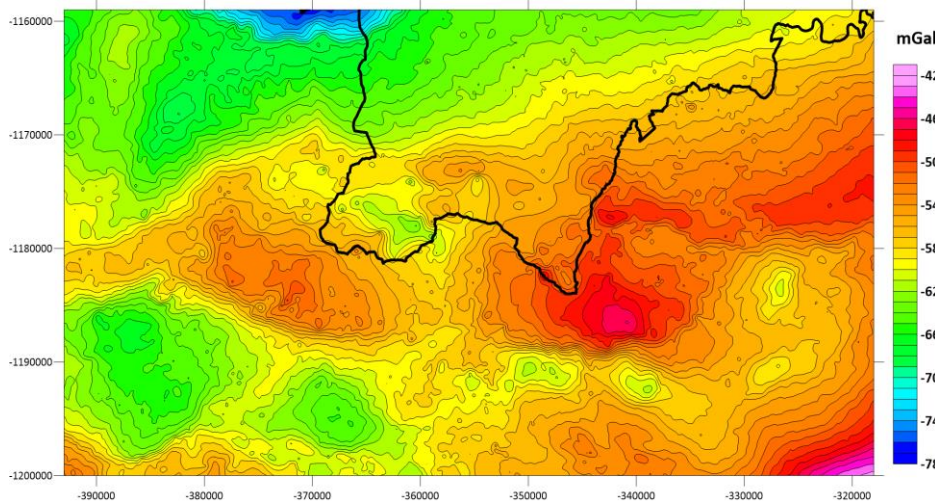
new measurements:

- - campaign 2004 (Slovakia)
- - campaign 2004 – 2005 (Poland)
- - campaign 2010 (Slovakia)
- - campaign 2024 – 2025 (Slovakia)



Generating input data on the Earth's surface

Complete Bouguer Anomalies (CBA)

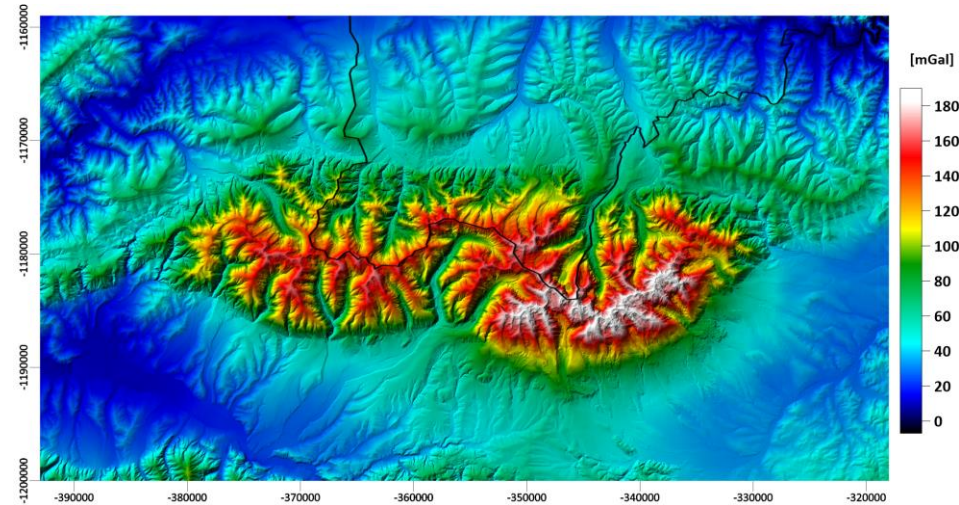


Topo_SK software

(Zahorec et al. 2017)

Surface gravity disturbances

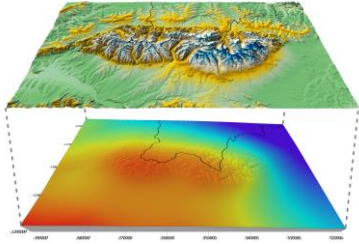
(resolution: 20 x 20 m)



CBA2G_SK software

(Marušiak et al. 2015)

Local quasigeoid model in Tatra Mountains

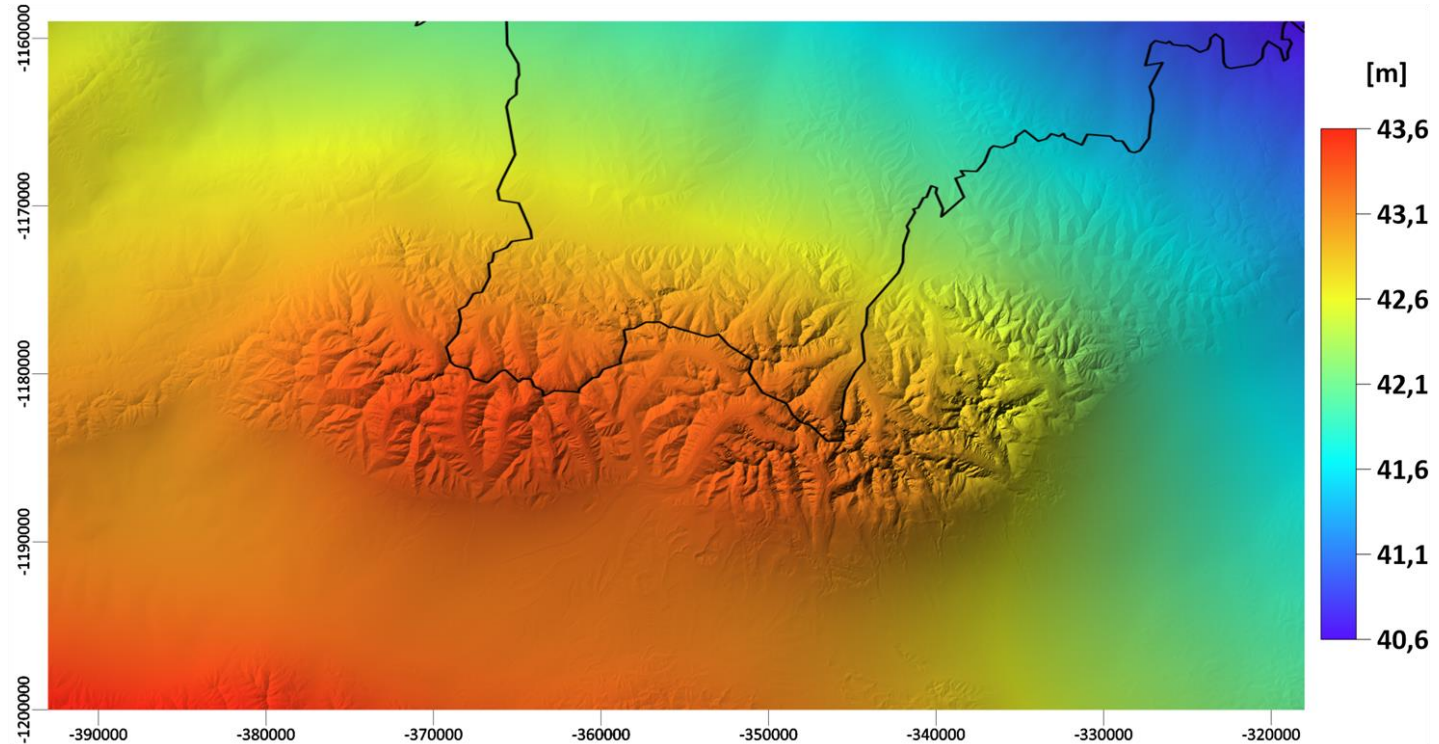


$$\zeta_P = \frac{T_P}{\gamma_P}$$

horizontal
resolution

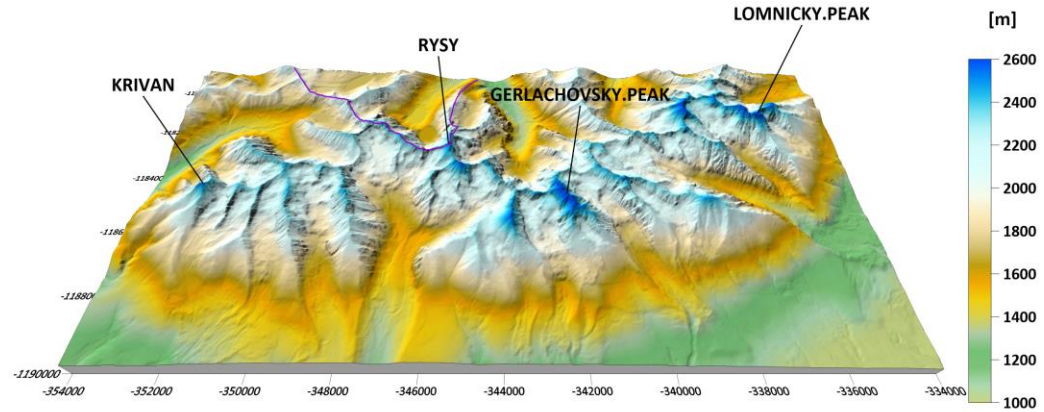
20 x 20 m

FEM numerical solution to FGBVP

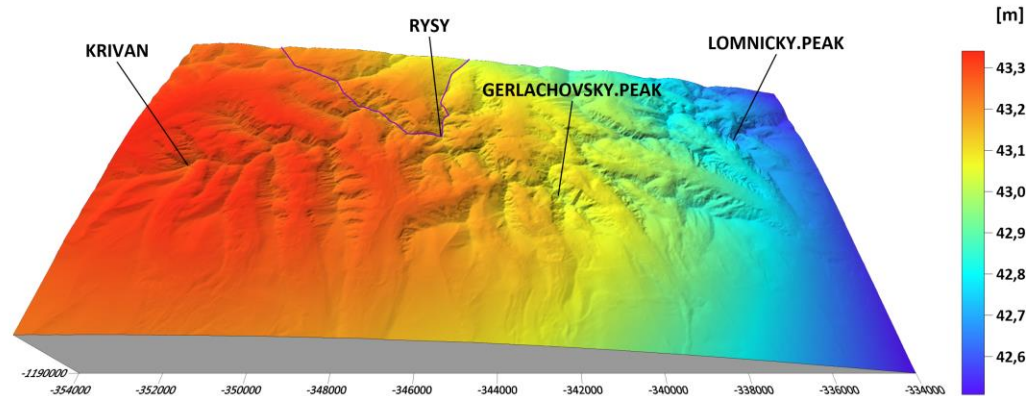


Detail in High Tatras

Topography

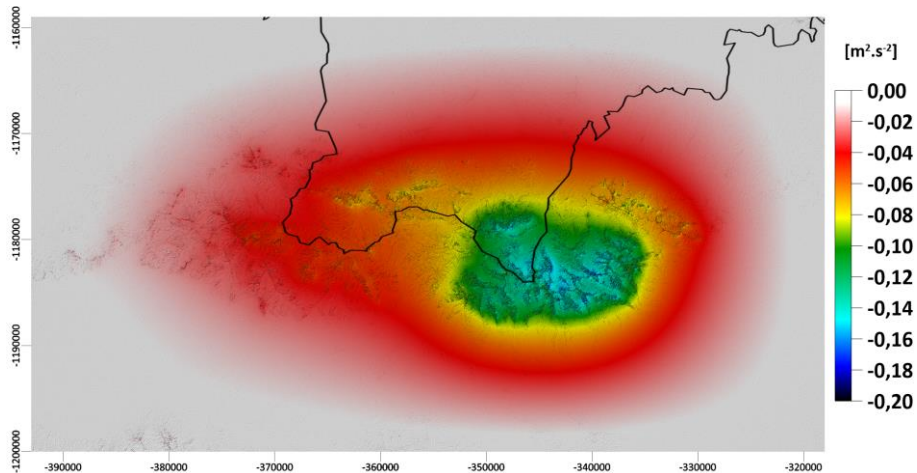


Quasigeoid

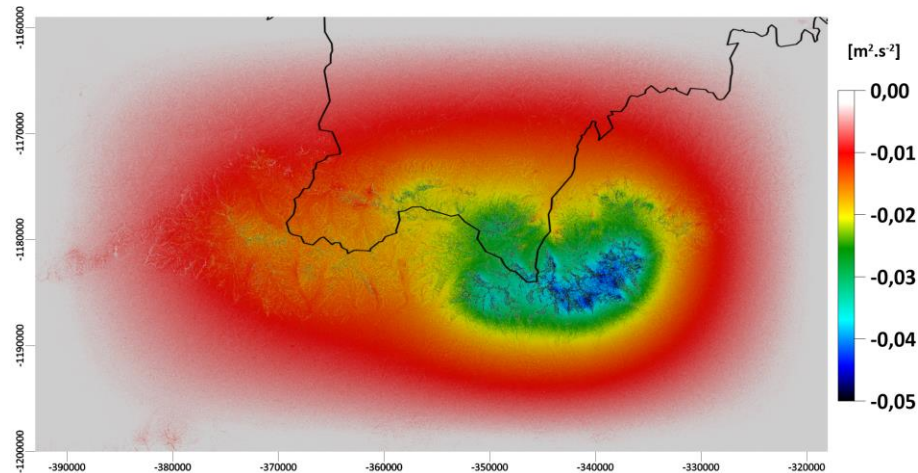


Contribution of grid refinements

without the distretization error



with the distretization error

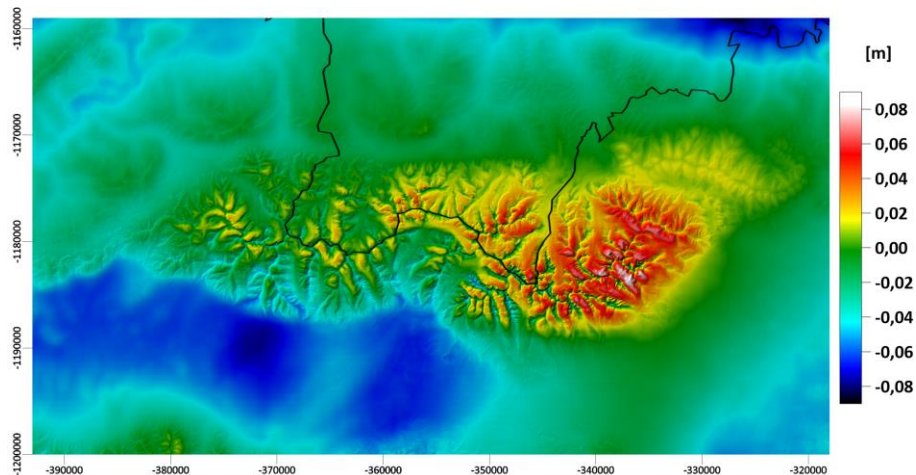


40 x 40 m $\Rightarrow$ 20 x 20 m

Comparison with quasigeoid models

DVRM05

(Klobušíak et al. 2005)

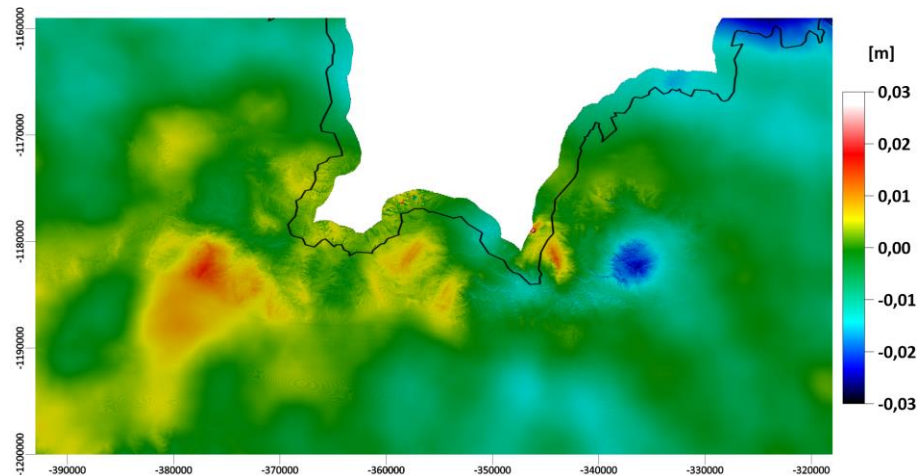


Method: R-C-R strategy + FFT integration
+ polynomial fitting to GNSS/levelling data

Resolution: 20 x 30 arc sec

GMSQ-2019

(Bucha et al. 2016)



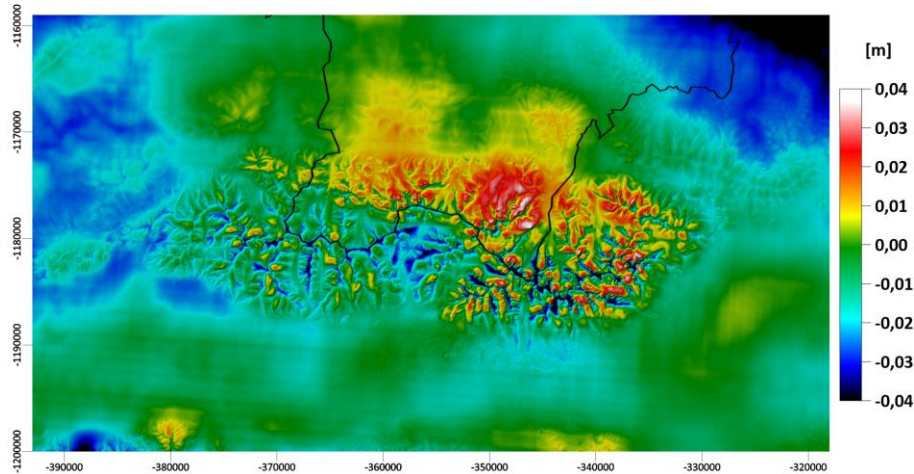
Method: SH + band limited SRBF + residual RTM
- gravimetric-only

Resolution: 2 x 2 arc sec

Comparison with quasigeoid models

PLgeoid21

(Trojanowicz and Owczarek-Wesołowska 2023)

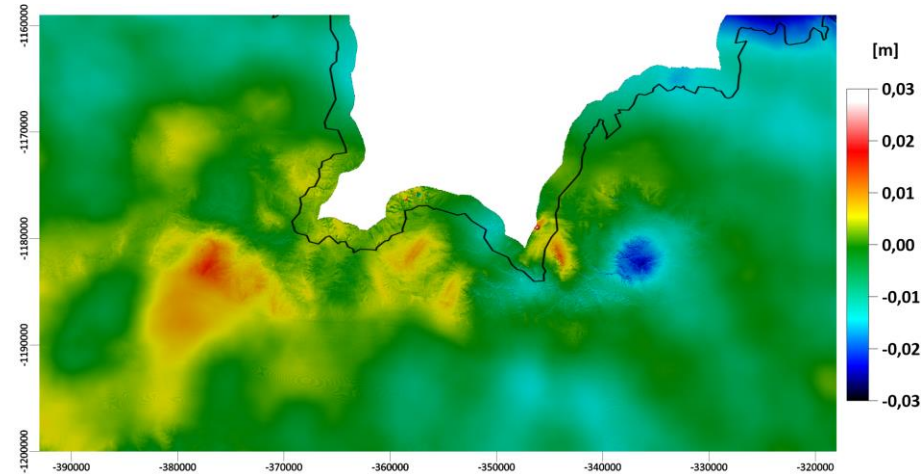


Method: **GGI (Geophysical Gravity data Inversion)**
+ fitted to GNSS/levelling data in Poland

Resolution: **0.01 x 0.01 deg**

GMSQ-2019

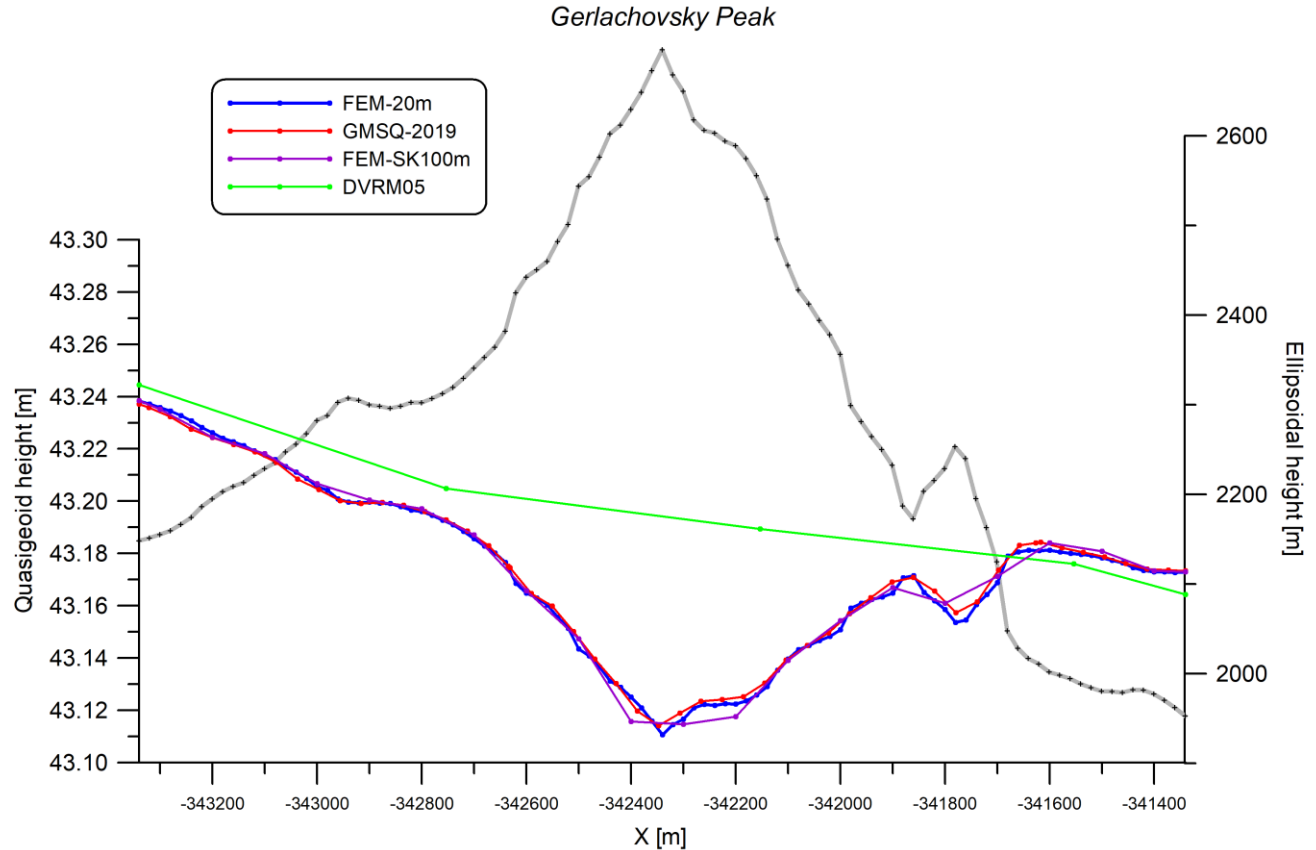
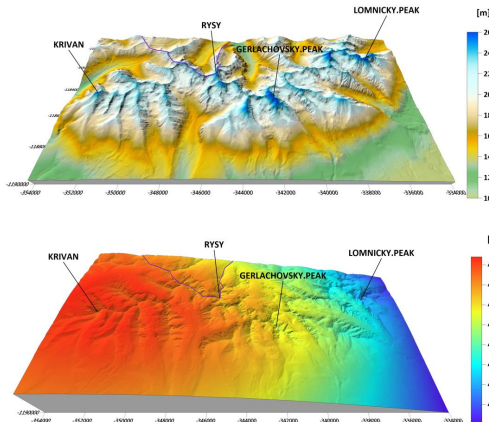
(Bucha et al. 2016)



Method: **SH + band limited SRBF + residual RTM**
- gravimetric-only

Resolution: **2 x 2 arc sec**

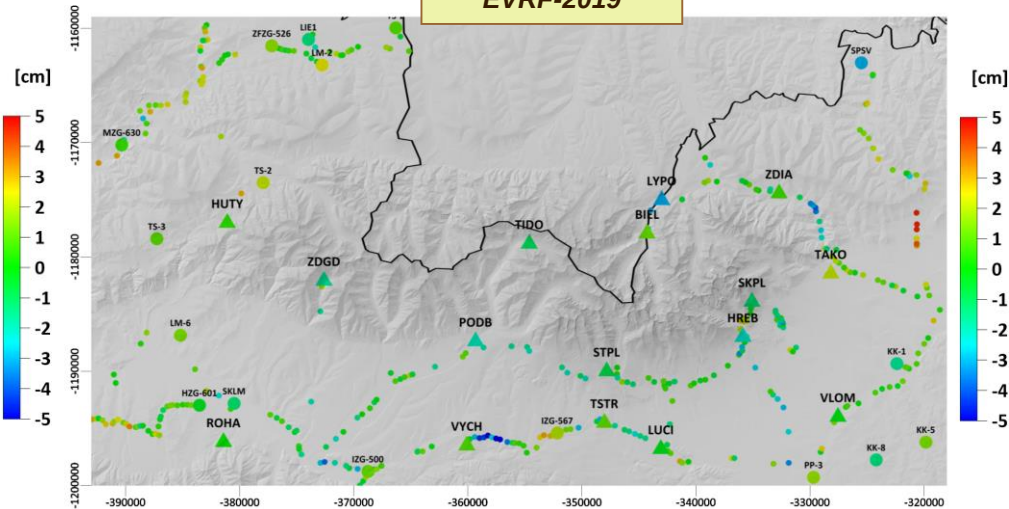
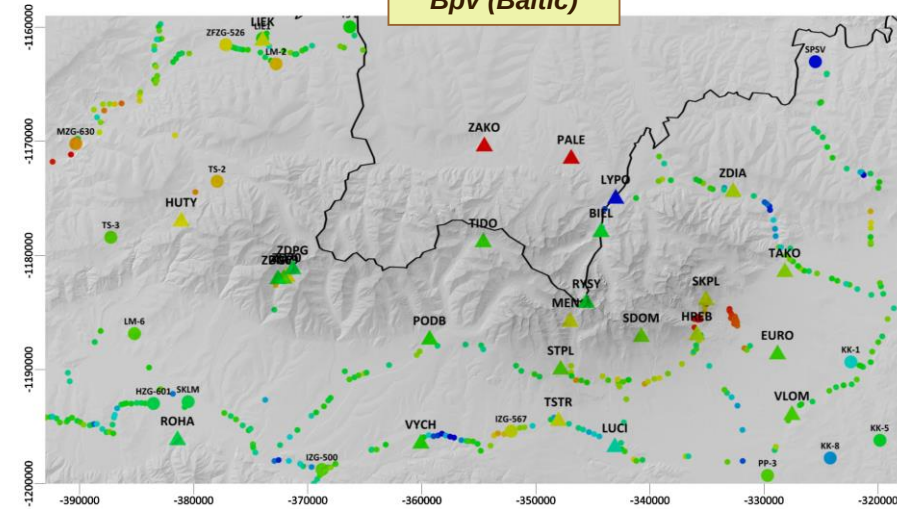
Profile across the Gerlachovsky Peak



GNSS-levelling test

Bpv (Baltic)

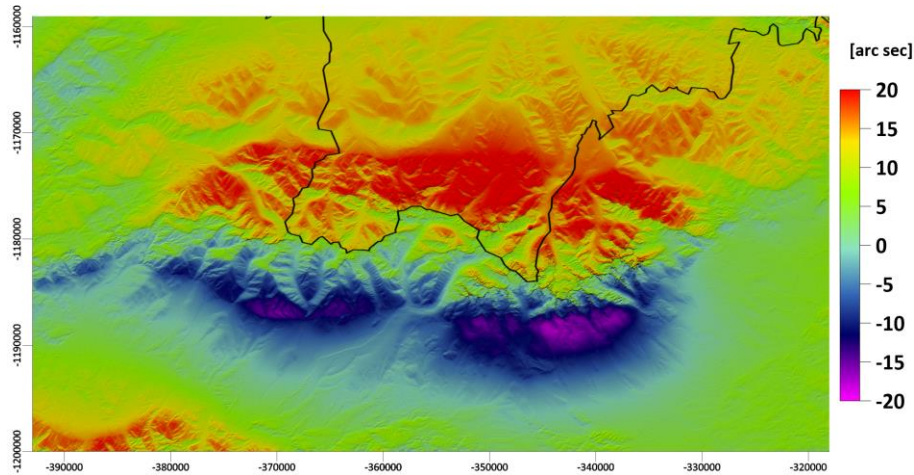
EVRF-2019



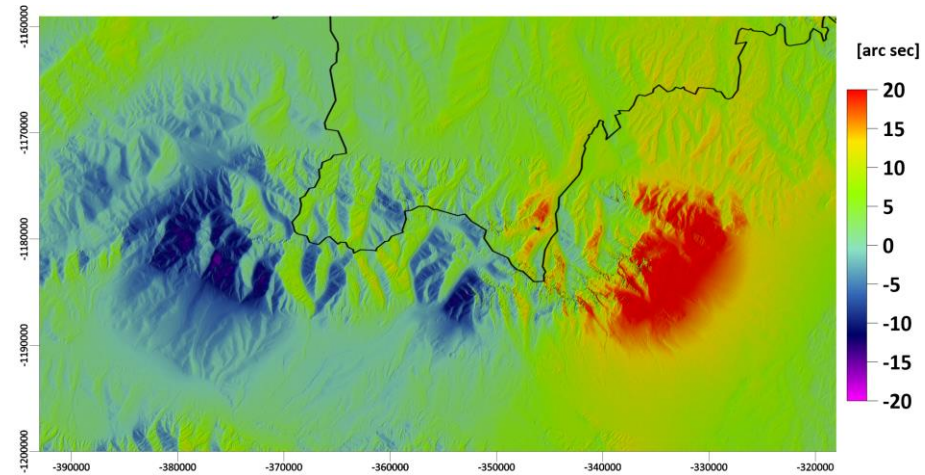
Vertical datum		<i>Bpv (Baltic)</i>				<i>EVRF-2019</i>			
Dataset	Points	<u>Quasigeoid</u>	FEM-20m	GMSQ2019	DVRM05	FEM-20m	GMSQ2019	DVRM05	
		Offset	-41.0	+55.0	0.0	-56.0	+40.0	-12.0	
SET 1	24 / 16 (Bpv / EVRF)	Mean	+0.65	-0.44	-4.35	-0.39	-0.57	-0.12	
		Range	7.49	8.64	12.80	5.32	5.61	16.47	
		<u>StD</u>	1.72	1.93	3.25	1.44	1.63	4.58	
SET 2	17	Mean	+0.32	+0.21	-1.36	+0.43	+0.32	+0.75	
		Range	8.19	7.60	7.86	5.64	5.10	5.86	
		<u>StD</u>	2.28	2.31	2.32	1.63	1.64	1.74	
SET 3	373	Mean	+0.10	+0.03	-3.06	+0.20	+0.13	+0.04	
		Range	14.73	15.54	15.54	11.03	10.96	16.40	
		<u>StD</u>	2.26	2.40	2.80	1.80	1.85	3.24	

Derived deflections of vertical (DoV)

north-south component

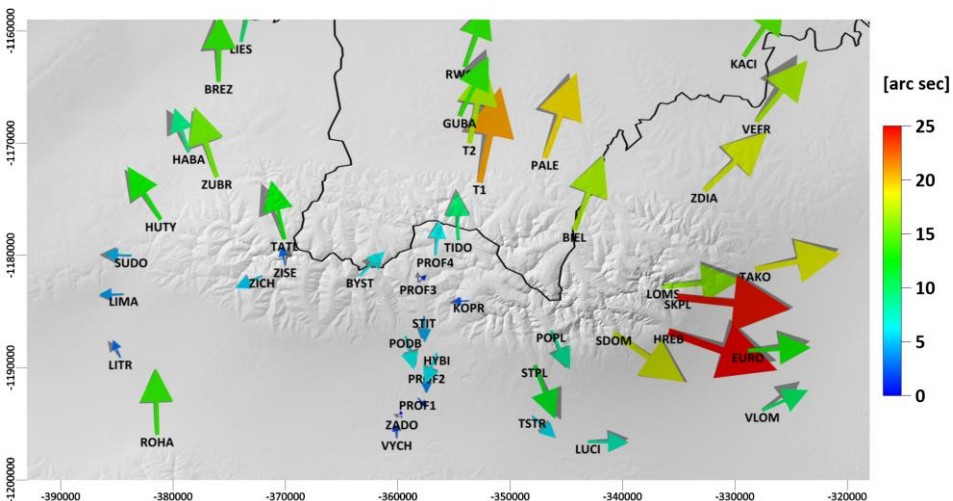


east-west component

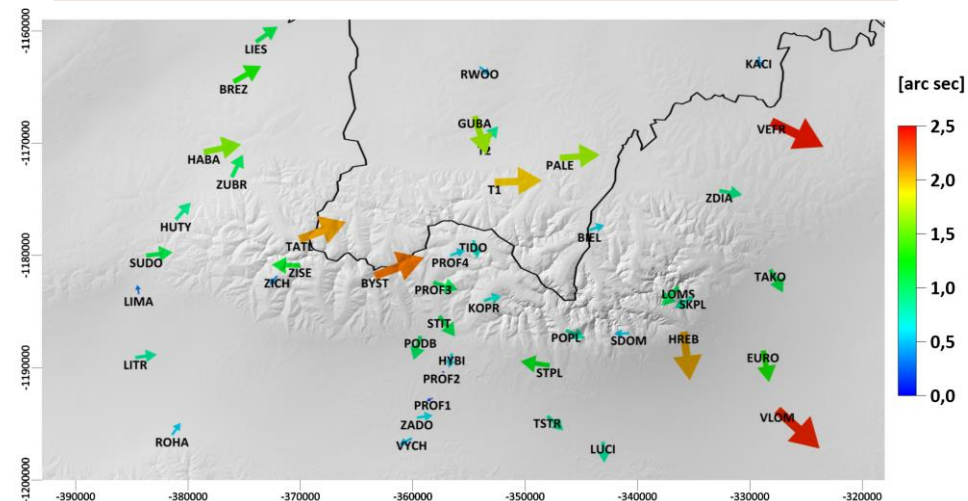


Validation by astronomical DoV

Derived & Astronomical DoV



Residuals: Derived DoV – Astronomical DoV



DoV	Astro-nomical		Derived						Residuals from Astronomical					
			FEM-20m		GMSQ2019*		CG05**		FEM-20m		GMSQ2019*		CG05**	
	ξ	η	ξ	η	ξ	η	ξ	η	$d\xi$	$d\eta$	$d\xi$	$d\eta$	$d\xi$	$d\eta$
Min	-10.9	-7.3	-10.7	-6.7	-11.7	-6.9	-11.5	-7.8	-2.08	-1.17	-2.78	-2.14	-1.78	-0.94
Max	19.9	24.5	20.0	23.8	16.5	24.9	19.7	25.4	0.90	2.20	1.26	2.07	0.79	1.67
Mean	4.4	3.5	4.2	3.9	2.4	3.5	4.0	3.4	-0.20	0.49	-0.33	0.24	-0.38	-0.02
StD	8.0	7.4	8.2	7.2	7.4	7.8	7.4	7.4	0.68	0.82	0.86	0.82	0.60	0.60

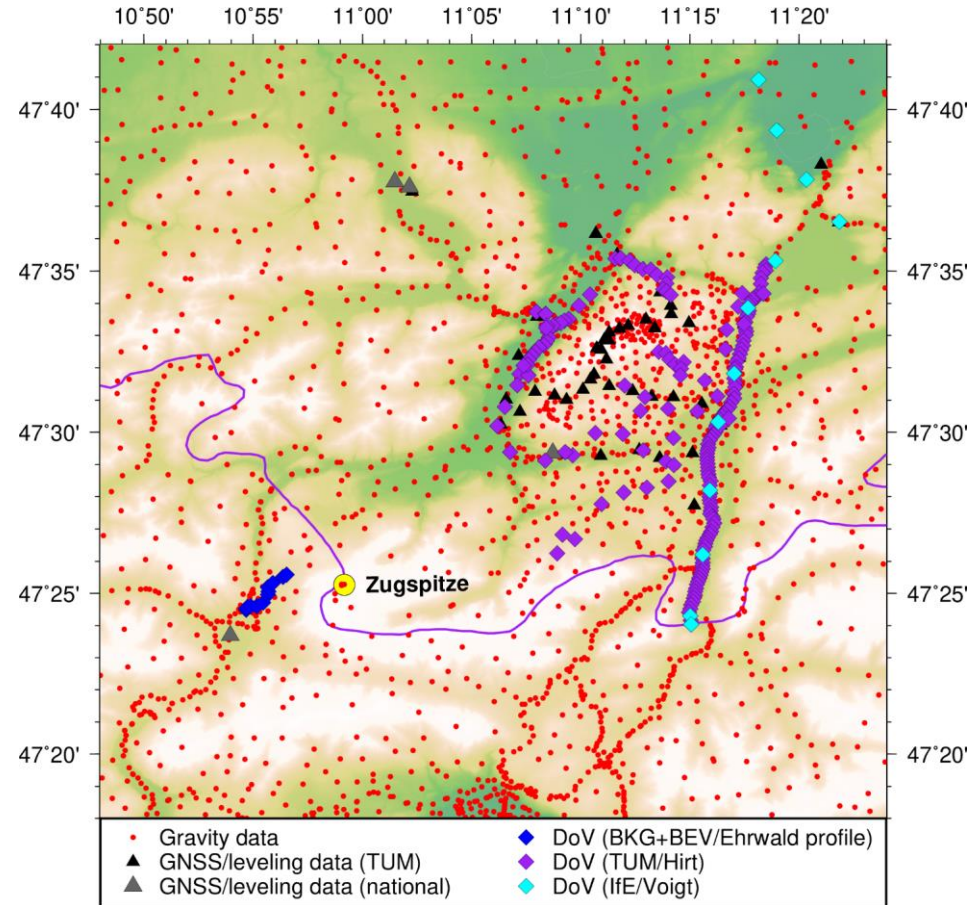
Zugspitze area – geoid test computations

EAlpG

The European Alps Geoid

(a joint initiative)

- improved cross-border regional geoid model of the Alpine area
- improved height transformation between the Alpine countries

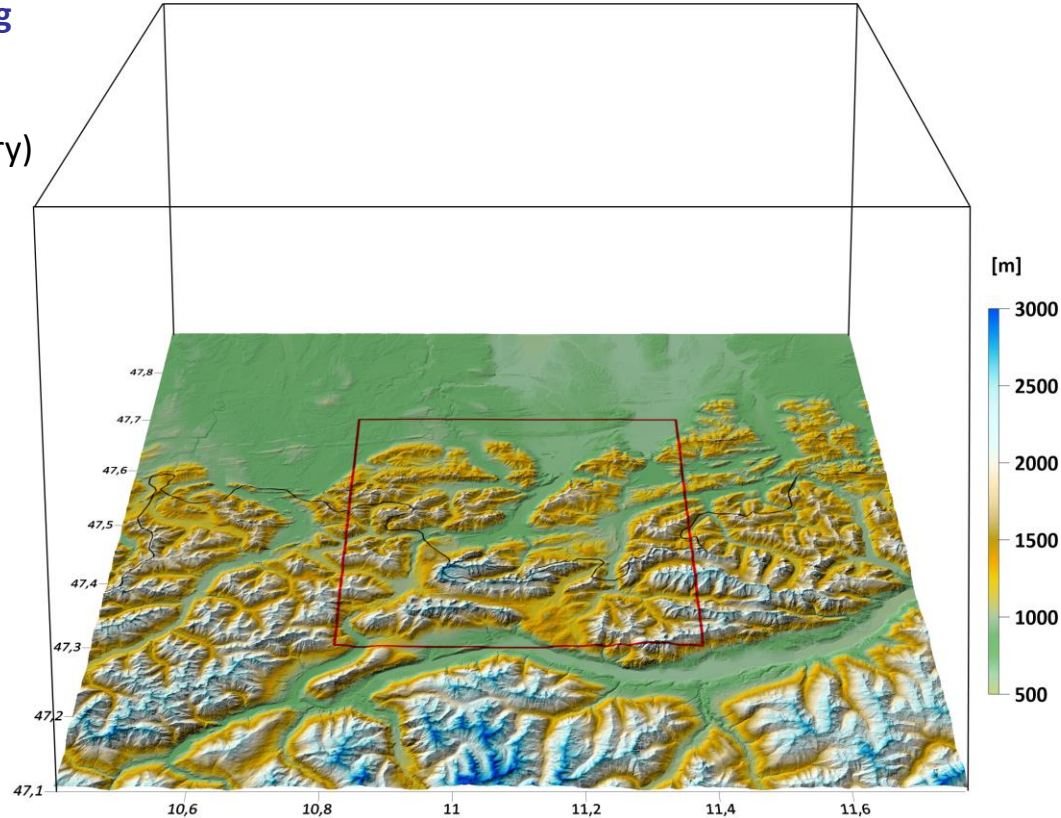
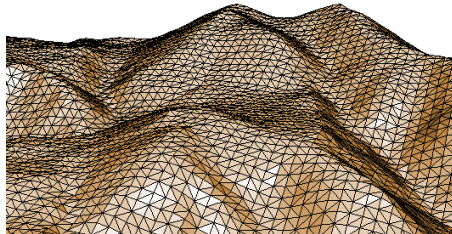


Discretization of the computational domain

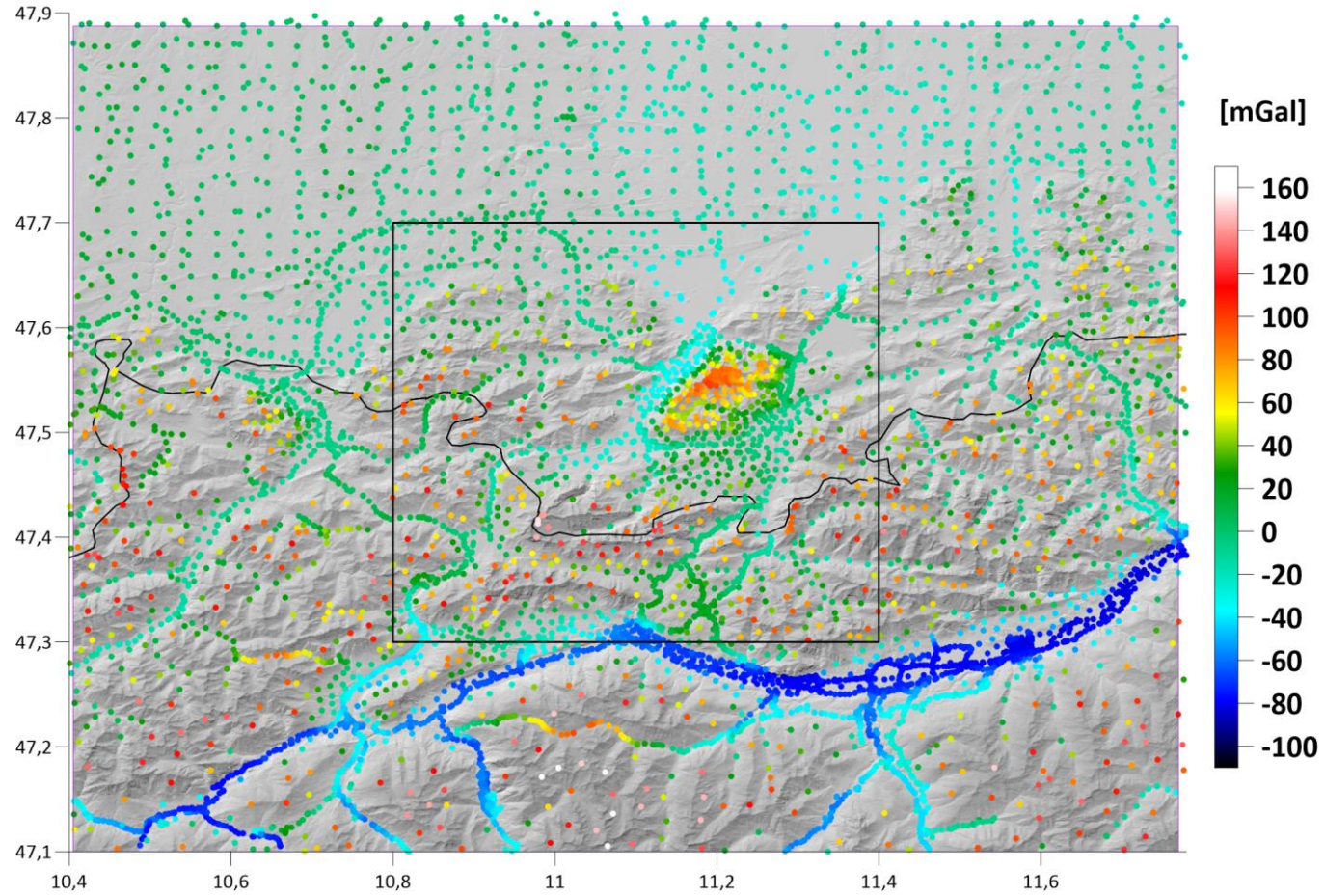
Discretized Earth's topography:

- horizontal resolution: 0.000225×0.000325 deg
($\approx 25 \times 25$ m)
- non-uniform division in the radial directions
5 m on the Earth's surface (the bottom boundary)
480 m at 120 km (the upper boundary)

$$\begin{aligned} &4200 \times 3497 \times 300 = \\ &\text{(longitude} \times \text{latitude} \times \text{height)} \\ &= \mathbf{4,406,220,000 \text{ elements}} \end{aligned}$$

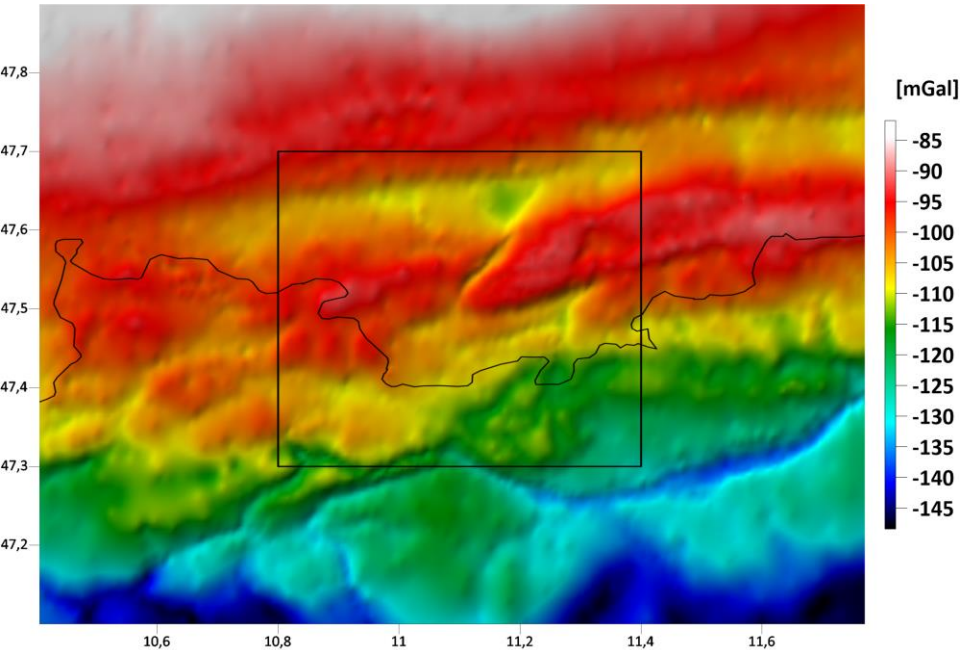


Discrete gravimetric measurements in the test area

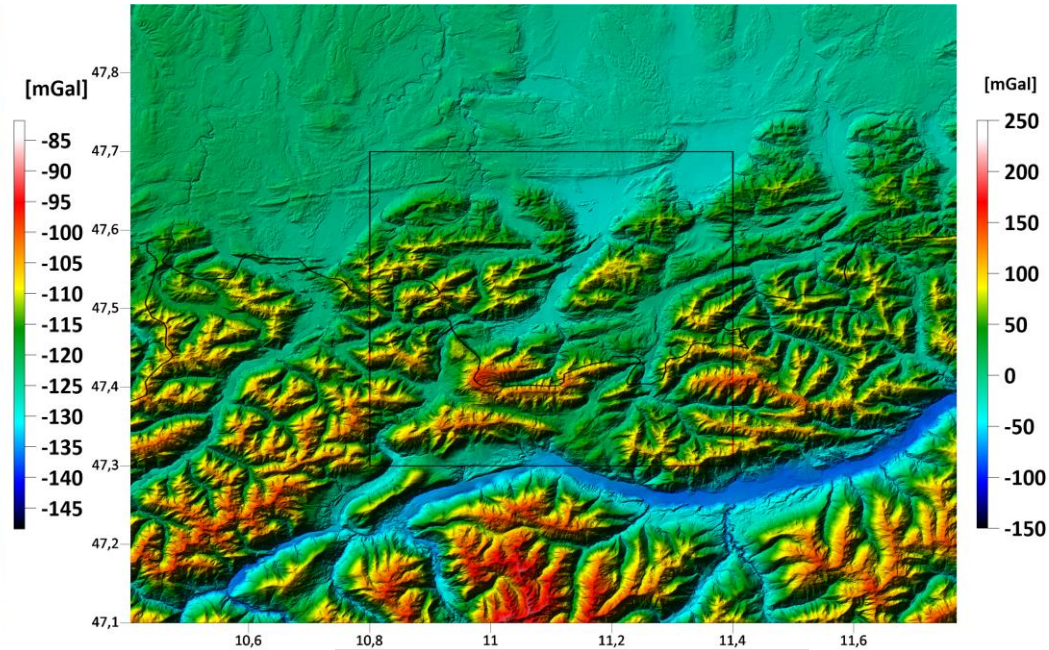


Generating gravity disturbances on the Earth's surface

Complete Bouguer Anomalies (CBA)



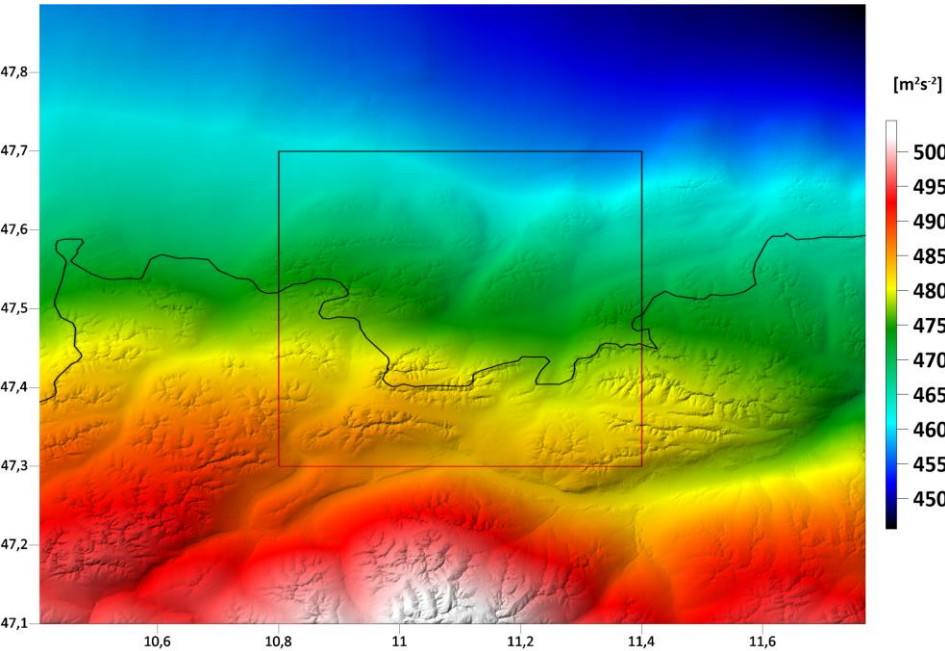
Surface gravity disturbances



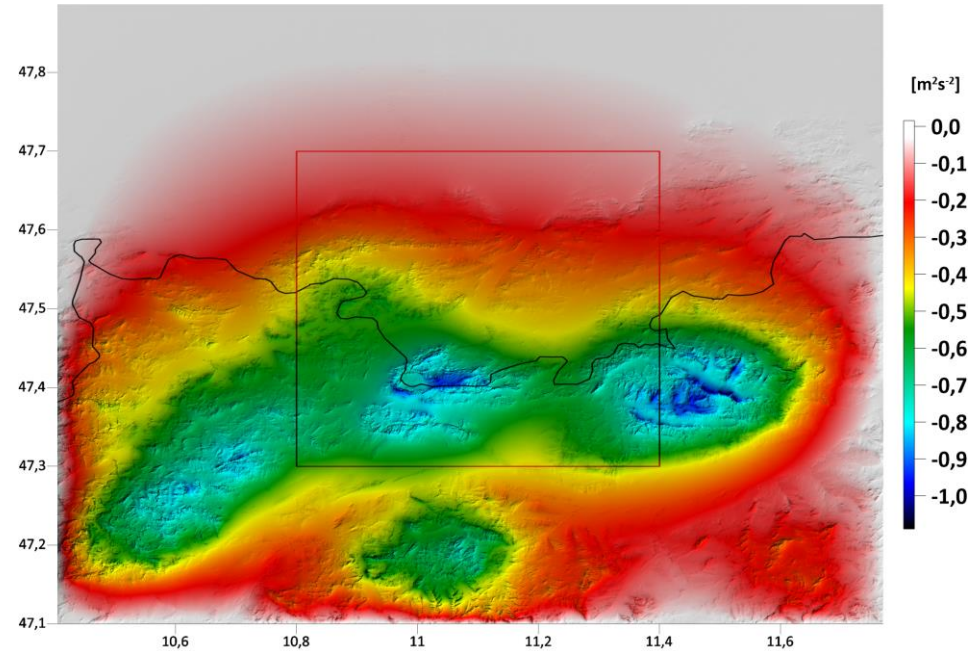
grid: 0.000225 x 0.000325 deg
($\approx 25 \times 25$ m)

FEM numerical solution in the Zugspitze test area

Disturbing potential

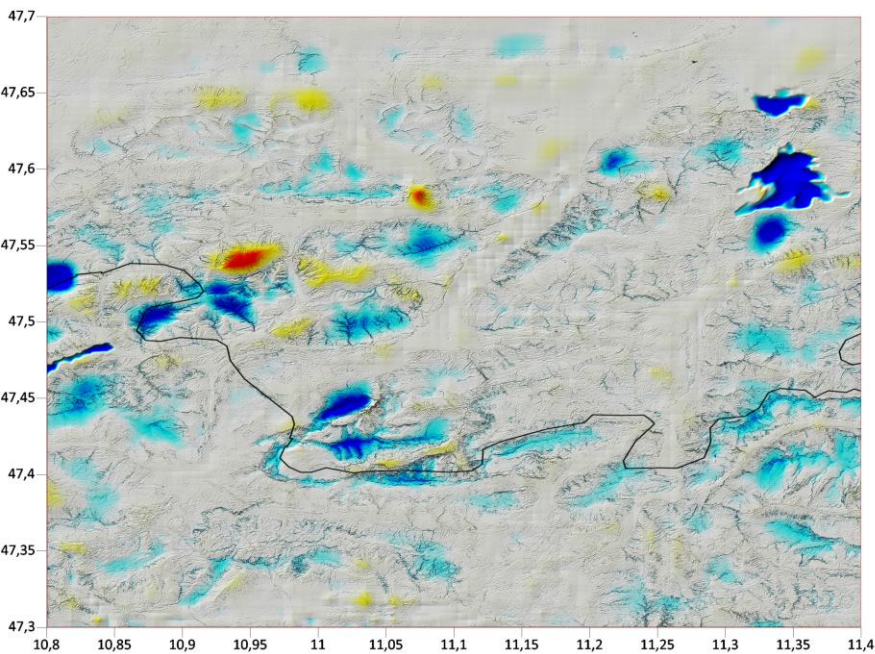


Discretization error

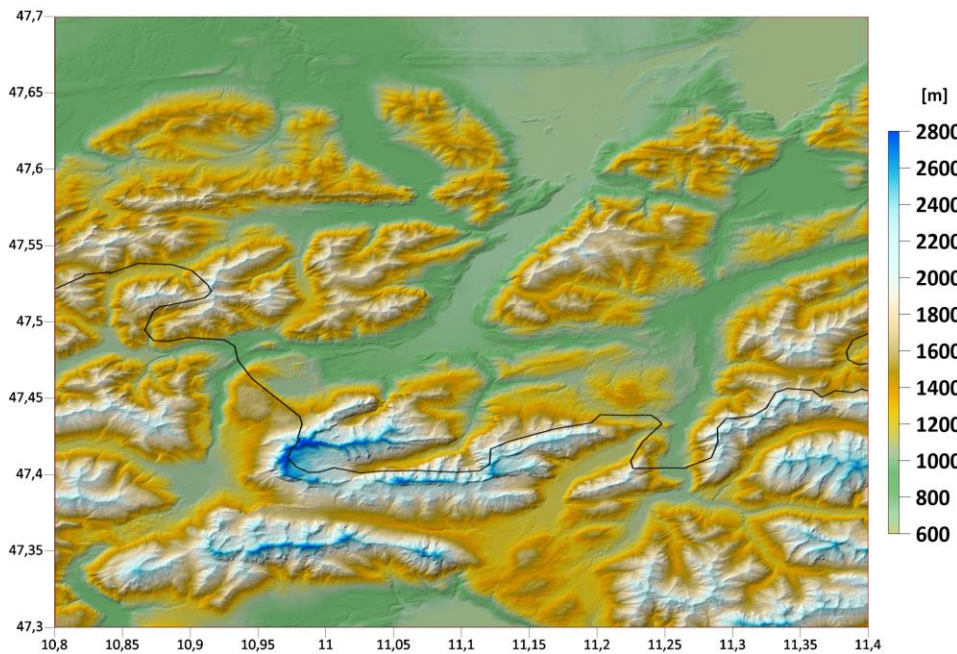


Comparison with the RSHTP solution (J.Schwabe - BKG)

Differences in input gravity disturbances

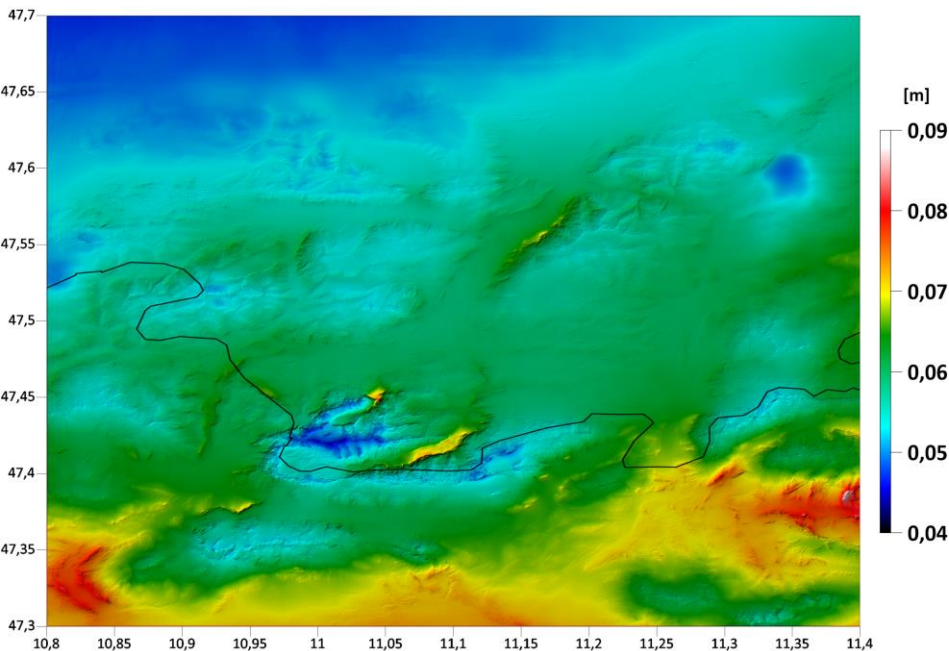


Topography

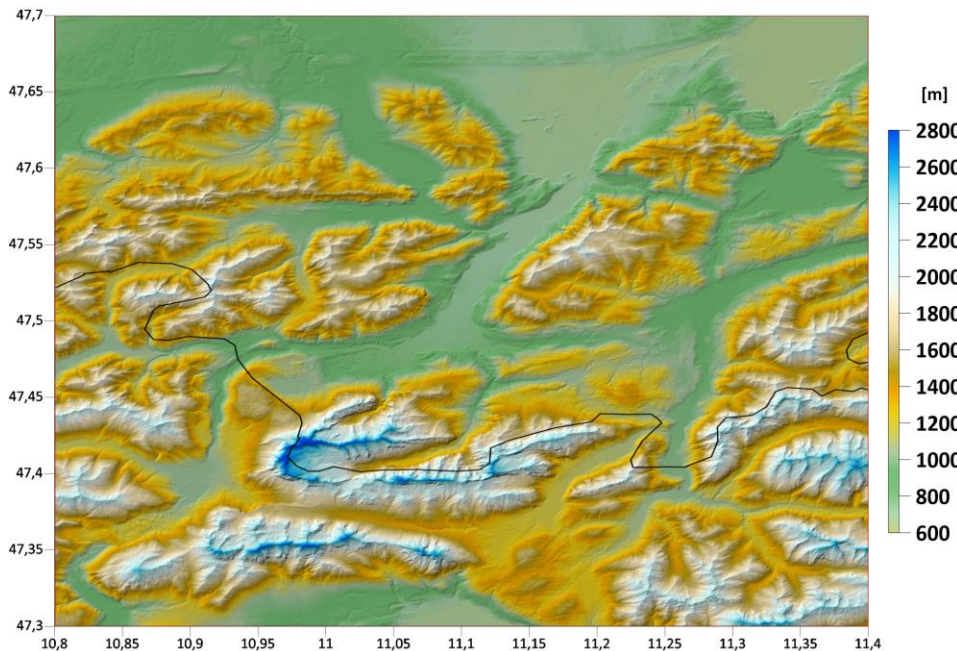


Comparison with the RSHTP* solution (J.Schwabe - BKG)

Differences between quasigeoid models: FEM - RSHTP



Topography



* RSHTP (Residual Spherical Harmonic Topographic Potential) → [*harmonic correction*](#)

Colorado test experiment

Discretized Earth's topography:

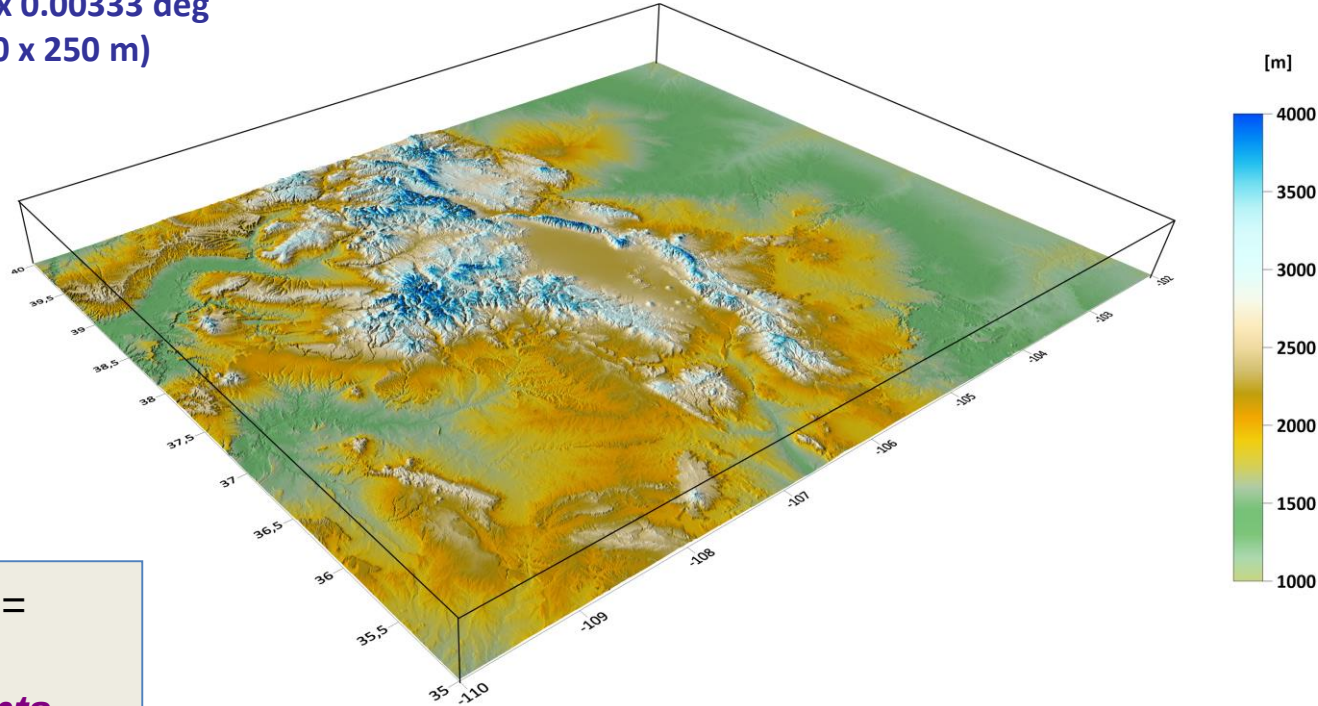
- horizontal resolution: 0.0025×0.00333 deg
($\approx 250 \times 250$ m)

Airborne gravity data



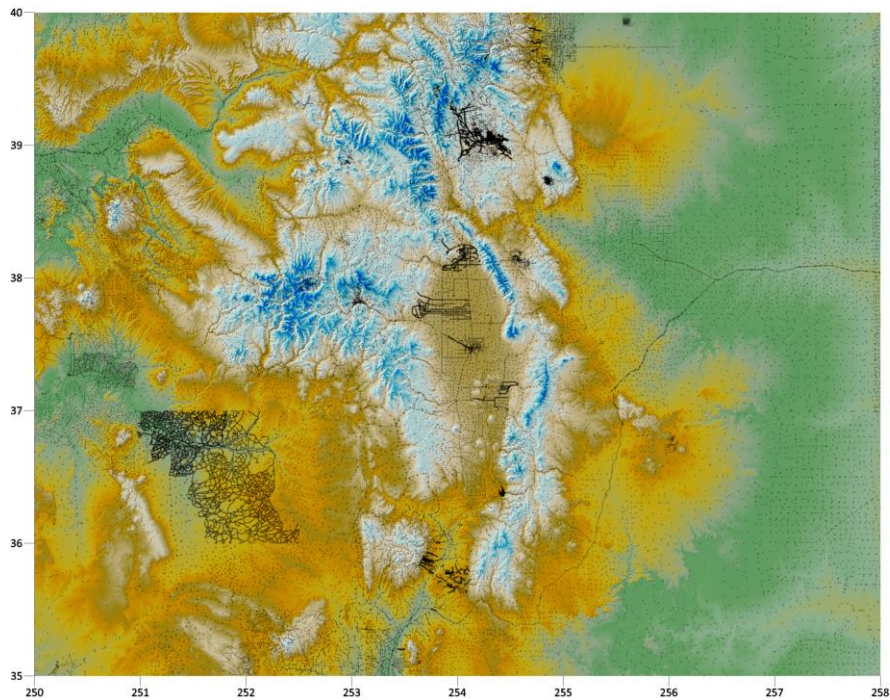
- the upper boundary at **6 km**

$$\begin{aligned} 3200 \times 1500 \times 90 &= \\ (\text{longitude} \times \text{latitude} \times \text{height}) \\ &= \mathbf{432,000,000 \text{ elements}} \end{aligned}$$

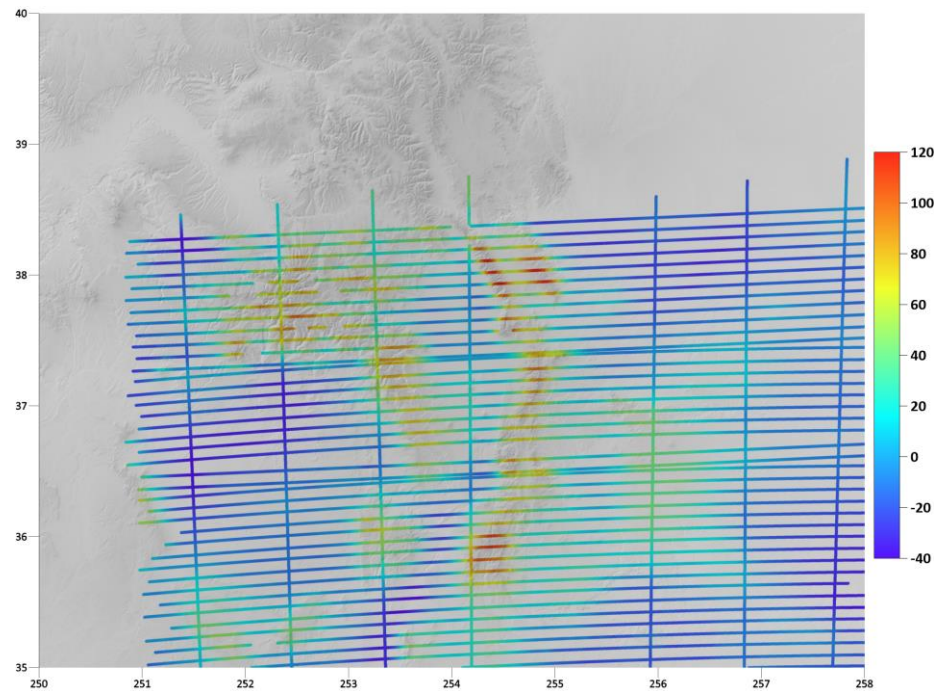


Discrete gravity data in the test area

Terrestrial

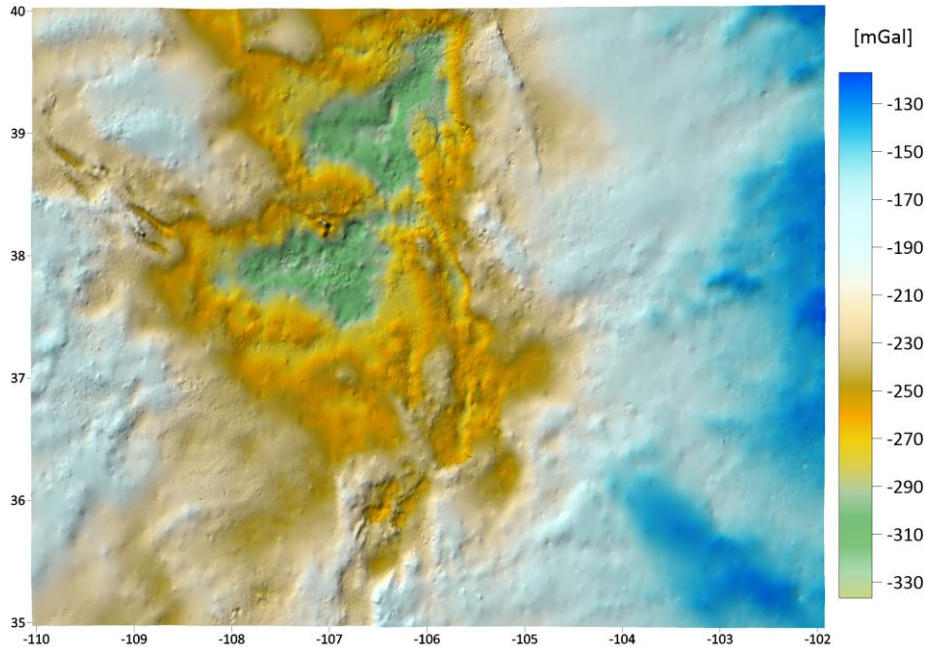


Airborne

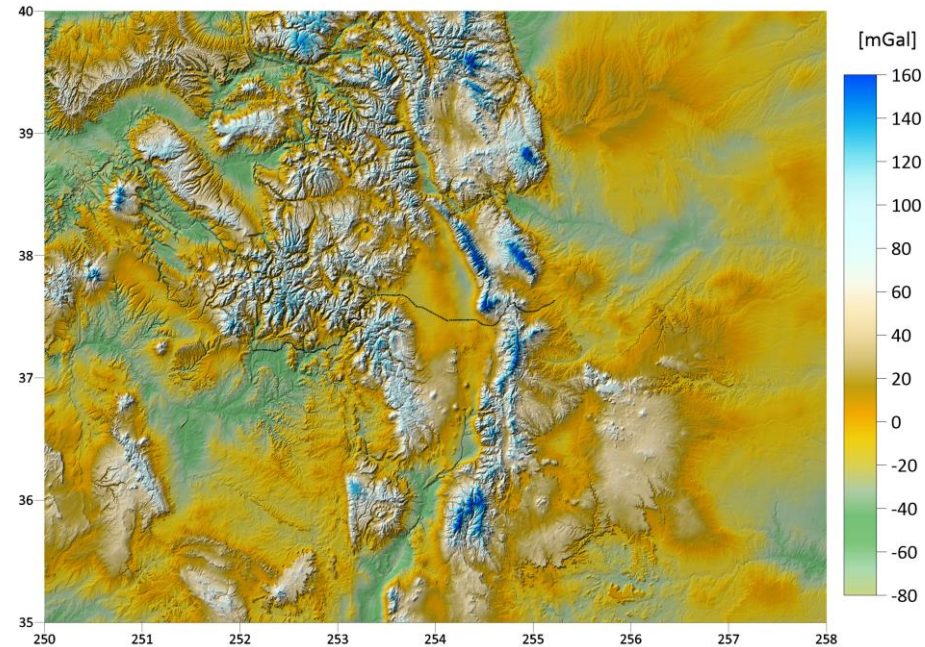


Generating gravity disturbances on the Earth's surface

Complete Bouguer Anomalies (CBA)



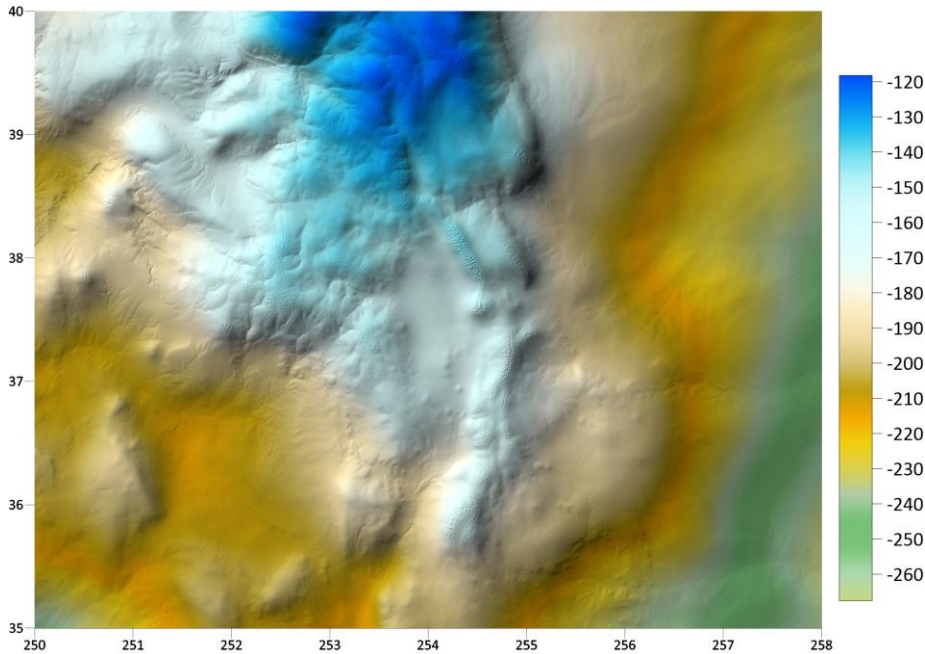
Surface gravity disturbances



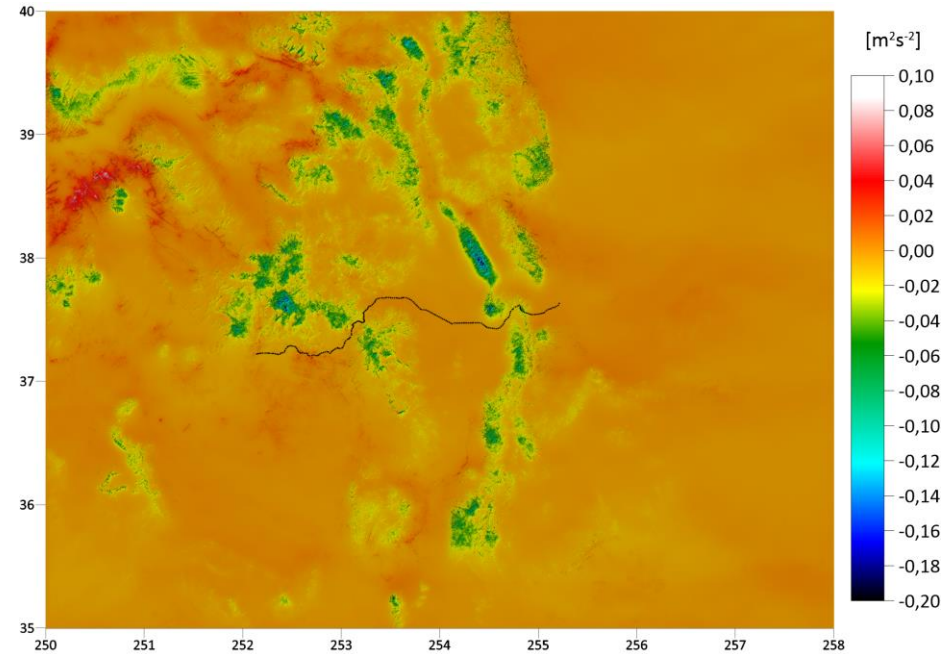
grid: 0.0025 x 0.00333 deg

FEM numerical solution in the Colorado test area

Disturbing potential

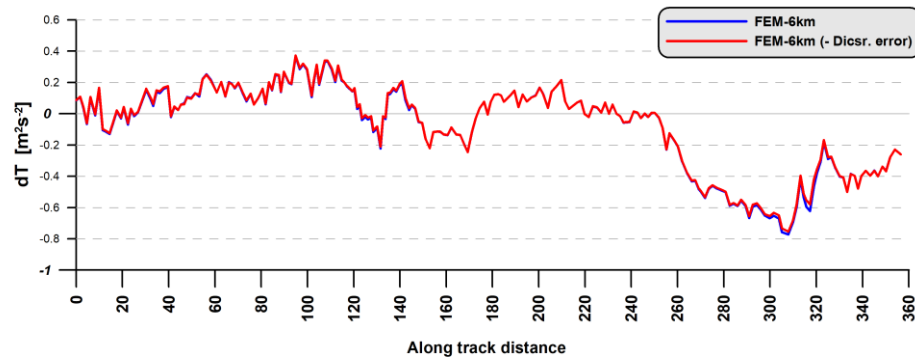
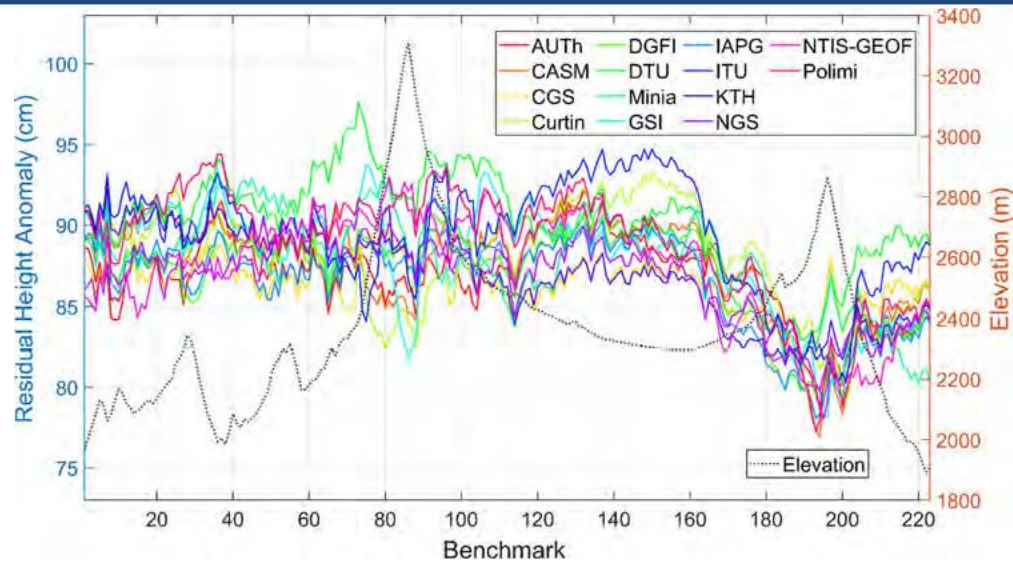
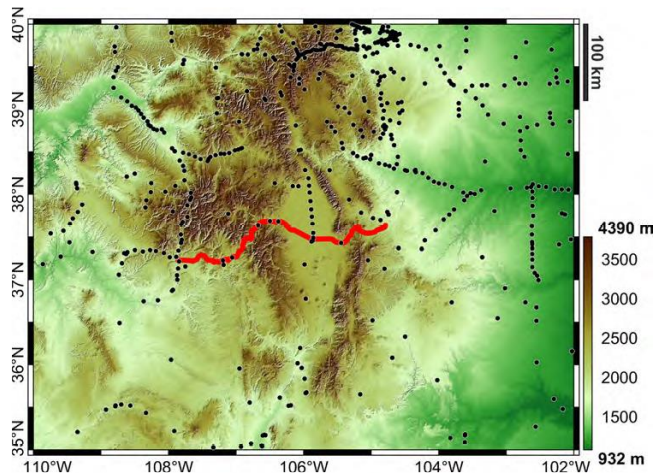


Discretization error



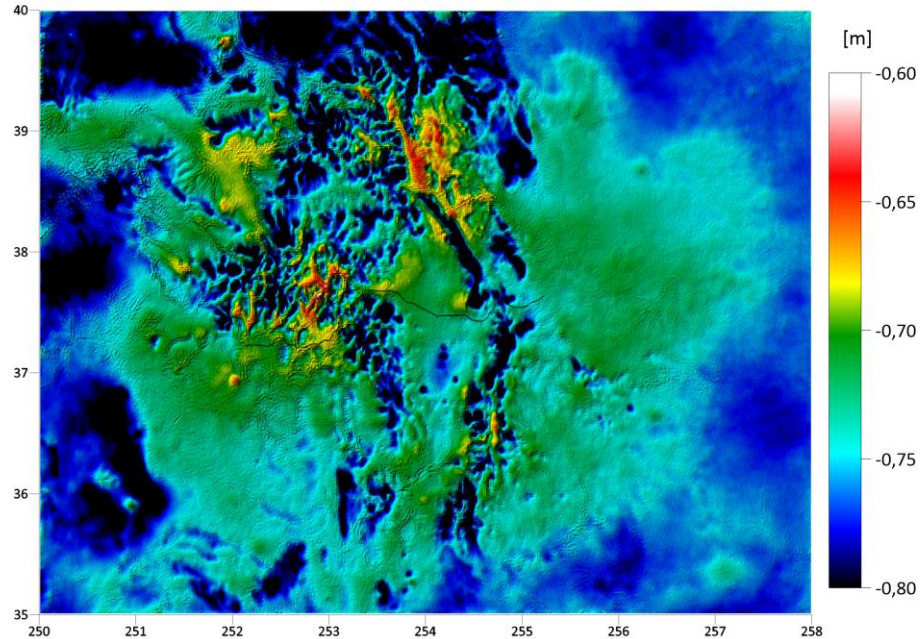
Testing along the GSVS17 profile

Wang et al. 2021

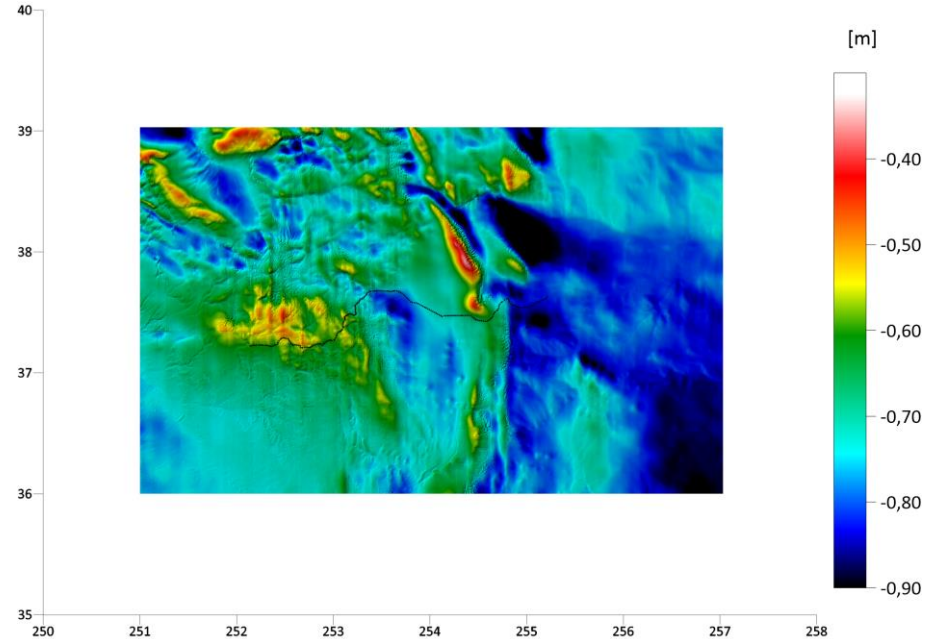


Comparison with other approaches in the whole area

CGS

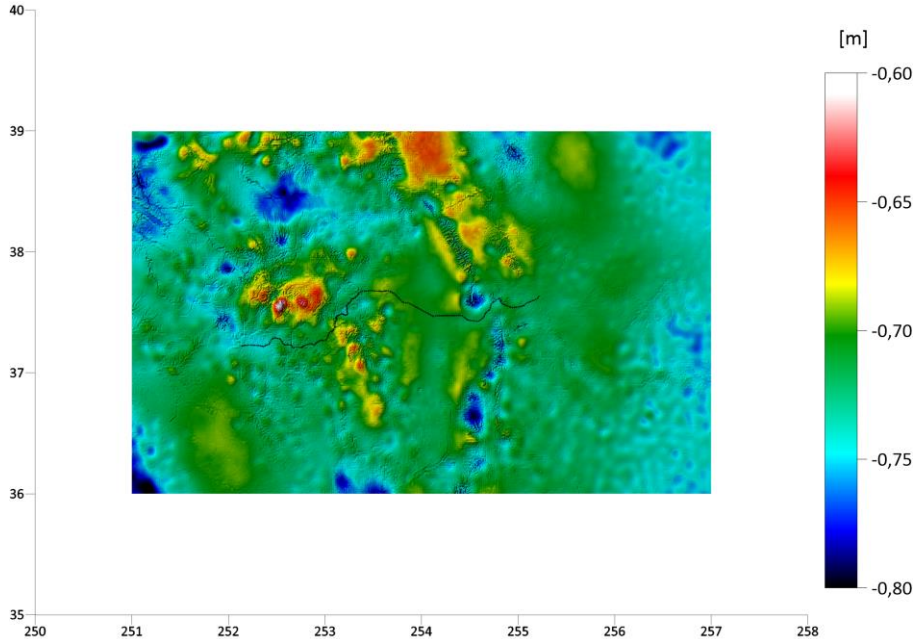


NGS

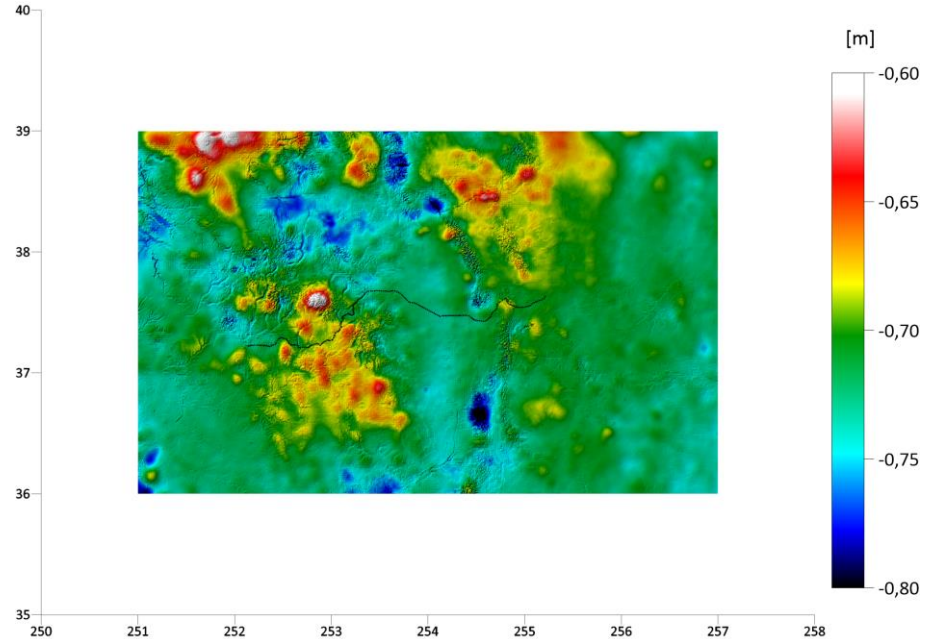


Comparison with other approaches in the whole area

KTH

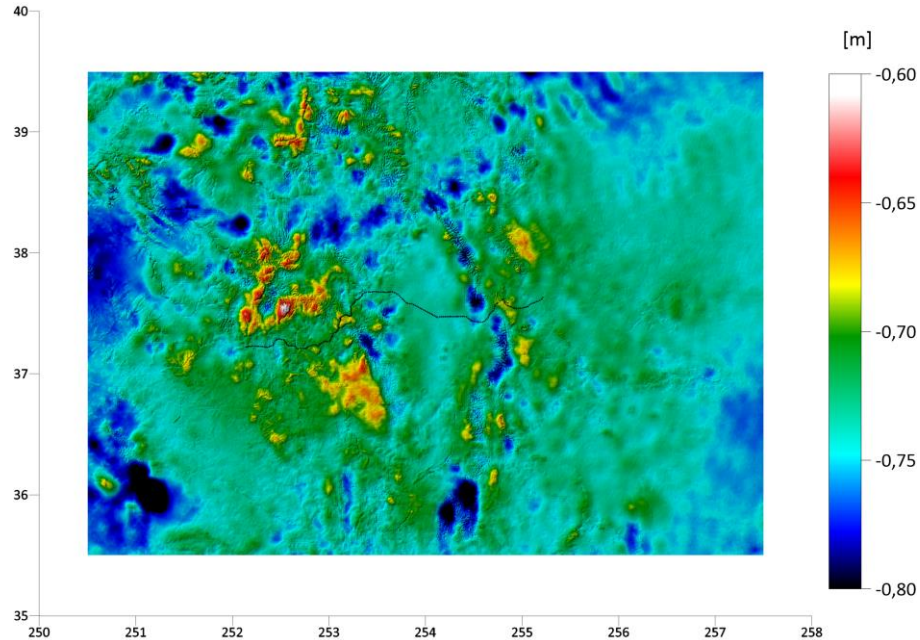


IAPG

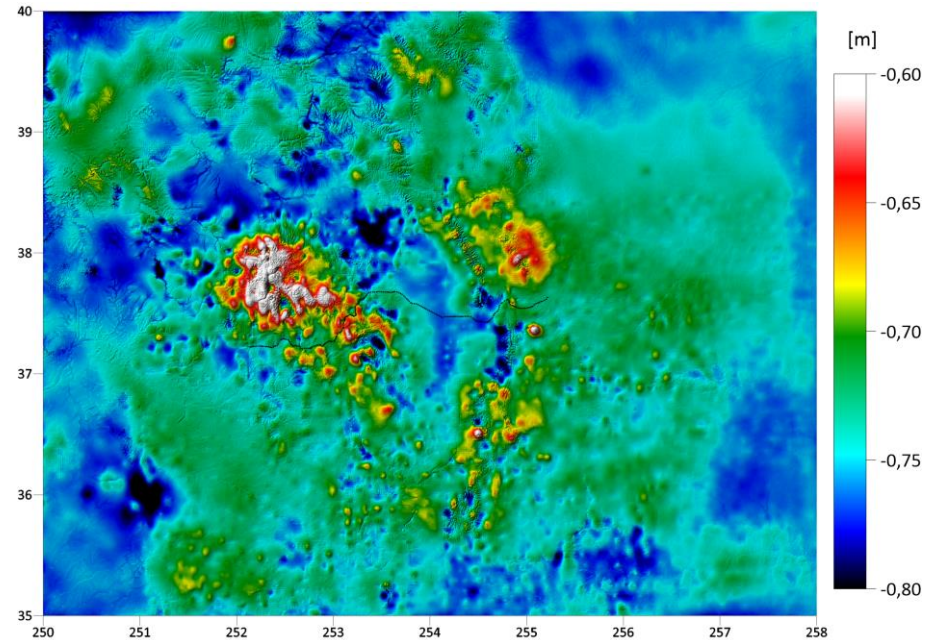


Comparison with other approaches in the whole area

CASM



Curtin



Conclusions

- **FEM approach** is efficient for local gravity field modelling in mountainous areas
 - is able to solve the fixed gravimetric BVP in **the spatial domain** (directly on the Earth's surface)
- to achieve cm-accuracy → **discretization error** has to be estimated and removed from FEM solutions
- **better convergence** of the oblique derivative on detailed grids (rugged Earth's topography) in case of the FEM implementation with **pentahedral finite elements**
- case studies show:
 - very good agreement with results of other approaches treated in the spectral domain, if dense terrestrial gravity data are available (Tatra Mountains, Zugspitze region)
 - ⇒ GNSS/leveling test indicates **accuracy about 2 cm**
 - in case of poor coverage of terrestrial gravity data, accuracy in high mountains can hardly achieve cm-level

Thank you For Your attention !

