

A comparative study of gravitational field modeling for Bennu asteroid

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Introduction

- Compare spatial-domain approaches (Finite Element Method - FEM) with a spectral-domain approach (Spherical Harmonics) and the polyhedral method.
- Solve three different Boundary Value Problems (BVPs) in the spatial domain.
- Apply the approaches to construct a high-resolution gravitational model of the asteroid 101955 Bennu,
- Asteroid Bennu has a complex geometry, making accurate modeling challenging.

Formulation of the Exterior Semi-Infinite Boundary Value Problem

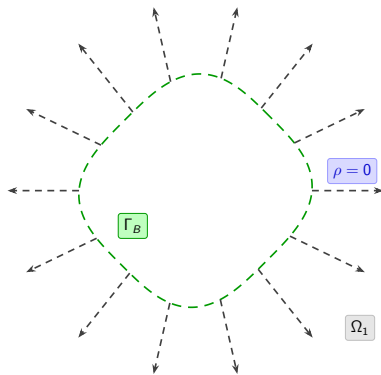


Figure: Computational domain Ω_1 for the spatial-domain approach based on solving the exterior semi-infinite BVP, where Γ_B denotes the surface of asteroid Bennu and $\rho = 0$ is the density assumed throughout the domain Ω_1 .

Formulation of the Exterior Semi-Infinite Boundary Value Problem

- In domain Ω_1 , the so-called exterior semi-infinite BVP is formulated

$$\Delta V_g(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_1, \quad (1)$$

$$\frac{\partial V_g(\mathbf{x})}{\partial \mathbf{r}} = \overline{g_g}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_B, \quad (2)$$

$$V_g(\mathbf{x}) = 0, \quad |\mathbf{x}| \rightarrow \infty \quad (3)$$

where $V_g(\mathbf{x})$ is the gravitational potential at the point $\mathbf{x} = (x, y, z)$, and $\overline{g_g}(\mathbf{x})$ is the magnitude of the gravitational vector in the radial direction.

Formulation of the Exterior Truncated Boundary Value Problem

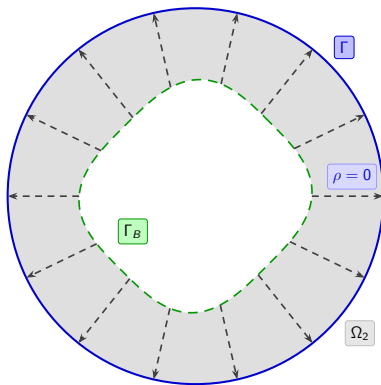


Figure: Computational domain Ω_2 used in the spatial-domain approach based on solving the exterior truncated BVP, obtained by truncating Ω_1 with an artificial spherical boundary Γ .

Formulation of the Exterior Truncated Boundary Value Problem

- The exterior truncated BVP is then formulated as

$$\Delta V_g(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_2, \quad (4)$$

$$\frac{\partial V_g(\mathbf{x})}{\partial \mathbf{r}} = \overline{g_g}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_B, \quad (5)$$

$$V_g(\mathbf{x}) = \frac{GM}{R}, \quad \mathbf{x} \in \Gamma \quad (6)$$

where G is the universal gravitational constant, M is the mass of the celestial body, and R is the radius of the spherical boundary Γ .

- The idea of introducing an artificial boundary Γ with a prescribed Dirichlet BC was first presented and solved by the FEM in [1]¹.

¹Z. Fašková, R. Čunderlík, and K. Mikula. "Finite element method for solving geodetic boundary value problems". In: *Journal of Geodesy* 84 (2010), pp. 135–144. DOI: 10.1007/s00190-009-0349-7

Formulation of the Coupled Interior–Exterior Truncated Boundary Value Problem

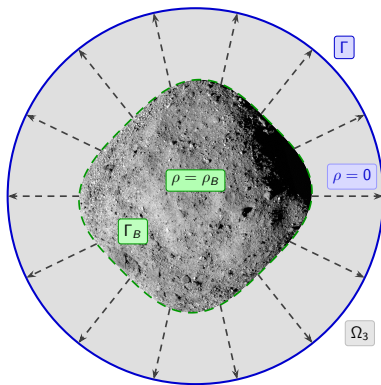


Figure: Computational domain Ω_3 for the spatial-domain approach based on solving the coupled interior–exterior BVP, where ρ_B denotes the density of the celestial body.

Formulation of the Coupled Interior–Exterior Truncated Boundary Value Problem

- Within this domain Ω_3 , we formulate the coupled interior–exterior truncated BVP

$$\Delta V_g(\mathbf{x}) = -4\pi G\rho(\mathbf{x}), \quad \mathbf{x} \in \Omega_3, \quad (7)$$

$$V_g(\mathbf{x}) = \frac{GM}{R}, \quad \mathbf{x} \in \Gamma, \quad (8)$$

where $\rho(\mathbf{x})$ denotes the density distribution, and all remaining quantities are defined as in the previous formulations. Inside the celestial body, $\rho(\mathbf{x})$ corresponds to the mass density of the body, while it is assumed to be zero in the exterior region above the body.

- This leads to the use of the Poisson equation inside the celestial body and the Laplace equation in the exterior part of the domain Ω_3 .

Finite Element Method

- From the standard theory of partial differential equations, it follows that there exists a unique weak solution to the BVPs (1)-(3), (4)-(6), and (7)-(8).
- Weak formulation over an arbitrary finite element Ω^e :

$$\int_{\Omega^e} \nabla V_g \cdot \nabla w \, d\mathbf{x} = \int_{\partial\Omega^e} \nabla V_g \cdot \mathbf{n} w \, d\sigma + \int_{\Omega^e} f w \, d\mathbf{x}, \quad (9)$$

where $\mathbf{n}$ denotes the outward unit normal to $\partial\Omega^e$.

- The system of equations can be written in a compact matrix form

$$\mathbf{K}^e \mathbf{V}^e = \mathbf{Q}^e + \mathbf{f}^e, \quad (10)$$

where $\mathbf{K}^e = [K_{ij}^e]$ denotes the element stiffness matrix, $\mathbf{V}^e = (V_1^e, \dots, V_6^e)^T$ is the vector of unknown nodal values.

Finite Element Method

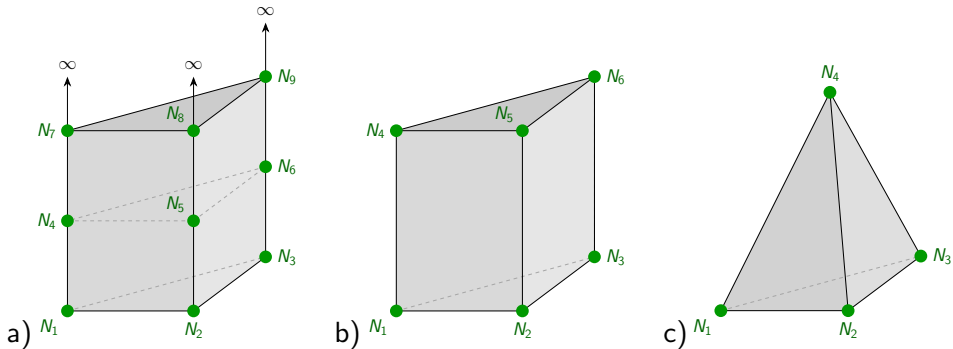


Figure: Element types used for discretization: a) linear nine-node mapped infinite element, with three nodes located at infinity; b) linear six-node prismatic element; c) linear four-node tetrahedral element.

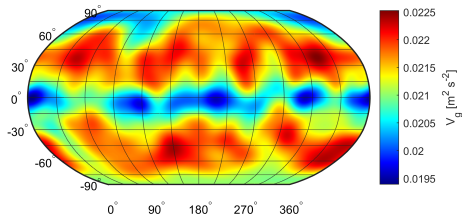
First Numerical Experiment - an analogy to the classical formulation of the geodetic BVP.

- The asteroid shape was represented by a spherical harmonic expansion of the surface radius synthesised up to degree 15 ².
- The number of nodes on the boundary Γ_B was 16 384.
- The Neumann BC (2) was computed by the approach spatial-domain gravity-forward modeling of T. Fukushima ³ while using the same degree-15 spherical harmonic expansion of Bennu's shape and the same density of $1\,260\text{ kg m}^{-3}$

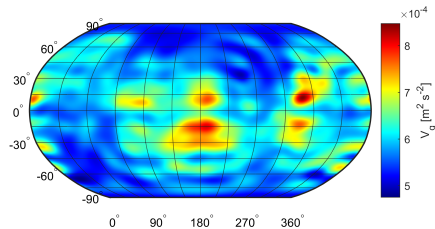
²B. Bucha and F. Sansò. "Gravitational field modelling near irregularly shaped bodies using spherical harmonics: a case study for the asteroid (101955) Bennu". In: *Journal of Geodesy* 95 (2021), p. 56. DOI: [10.1007/s00190-021-01493-w](https://doi.org/10.1007/s00190-021-01493-w)

³T. Fukushima. "Precise and fast computation of the gravitational field of a general finite body and its application to the gravitational study of asteroid Eros". In: *The Astronomical Journal* 154.145 (2017), 15pp. DOI: [10.3847/1538-3881/aa88b8](https://doi.org/10.3847/1538-3881/aa88b8)

First Numerical Experiment - an analogy to the classical formulation of the geodetic BVP.



a)



b)

Figure: a) Model of the gravitational field of asteroid Bennu. b) Differences between the gravitational field model based on spherical harmonics and the solution obtained by FEM.

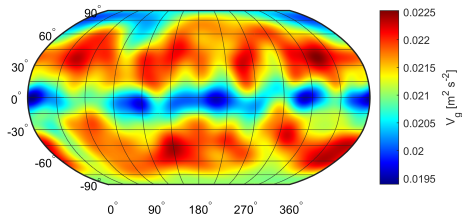
Second Numerical Experiment - studying the effect of truncating an unbounded computational domain.

- The asteroid shape was represented by a spherical harmonic expansion of the surface radius synthesised up to degree 15 ⁴.
- The number of nodes on the boundary Γ_B was 16 384.
- The Neumann BC (5) was computed by the approach spatial-domain gravity-forward modeling of T. Fukushima ⁵ while using the same degree-15 spherical harmonic expansion of Bennu's shape and the same density of $1\,260\text{ kg m}^{-3}$
- The constant Dirichlet BC at a constant distance of 5 km was applied.

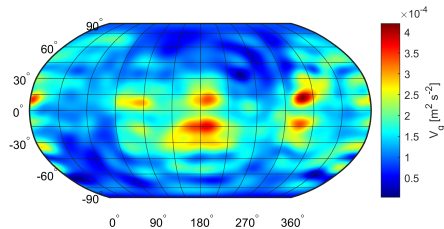
⁴B. Bucha and F. Sansò. "Gravitational field modelling near irregularly shaped bodies using spherical harmonics: a case study for the asteroid (101955) Bennu". In: *Journal of Geodesy* 95 (2021), p. 56. DOI: [10.1007/s00190-021-01493-w](https://doi.org/10.1007/s00190-021-01493-w)

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Figure: a) Model of the gravitational field of asteroid Bennu. b) Differences between the gravitational field model based on spherical harmonics and the solution obtained by FEM.

Summary of the First and Second Numerical Experiments

Statistical Parameter	First Experiment	Second Experiment
Minimum	4.755×10^{-4}	3.607×10^{-6}
Maximum	8.475×10^{-4}	4.208×10^{-4}
Mean	6.066×10^{-4}	1.526×10^{-4}
Std. Deviation	5.779×10^{-5}	6.283×10^{-5}

Table: Comparison of statistical characteristics for the First and Second Experiments.

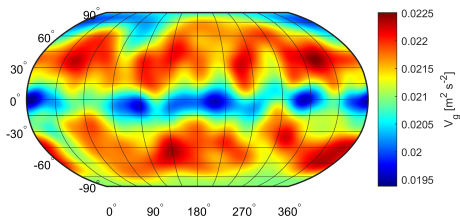
Third Numerical Experiment - solving a BVP with a discontinuous right-hand side

- The 3D model used in the computation was OLA v20 PTM ⁶.
- The number of nodes on the boundary Γ_B was 74 863.
- The reference gravitational field is now obtained by evaluating the gravitational field of a polyhedron having a constant density using the exact method of D. Tsoulis ⁷.

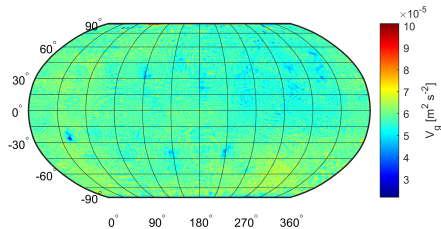
⁶K. Elkins et al. *Asteroid Bennu 3D Models, OLA v20 PTM*. <https://svs.gsfc.nasa.gov/5069>. 2023

⁷D. Tsoulis. "Analytical computation of the full gravity tensor of a homogeneous arbitrarily shaped polyhedral source using line integrals". In: *Geophysics* 77 (2012), F1–F11. DOI: [10.1190/GE02010-0334.1](https://doi.org/10.1190/GE02010-0334.1)

Third Numerical Experiment - solving a BVP with a discontinuous right-hand side



a)



b)

Figure: a) Detailed model of the gravitational field of asteroid Bennu. b) Differences between the solution obtained using the polyhedral method and the FEM.

Summary of the Third Numerical Experiment

Statistical Parameter	Third Experiment
Minimum	$2.18289430 \times 10^{-5}$
Maximum	$1.01143565 \times 10^{-4}$
Mean	$5.79905376 \times 10^{-5}$
Std. Deviation	$3.98726818 \times 10^{-6}$

Table: Comparison of statistical characteristics for the Third Experiment.

Conclusion and summary

- We have presented three spatial-domain approaches for modeling the gravitational field of the asteroid 101955 Bennu.
- Within the spatial-domain framework, three boundary value problems (BVPs) are formulated:
 - an exterior semi-infinite BVP,
 - an exterior truncated BVP,
 - a coupled interior–exterior truncated BVP.
- The proposed framework was validated on a challenging numerical example involving a highly complex geometry.
- The experiment confirmed ability of FEM to accurately represent irregularly shaped celestial bodies .

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Thanks for your attention.