



Assessing the Gravitational Force Acting on a Celestial Body from the Total Gravitational Vector field Given on an Enclosing Surface

Zhi Yin¹, Jintao Zhang¹, Nico Sneeuw², Caijun Xu³, Kefei Zhang⁴

1.School of Marine Technology and Geomatics, Jiangsu Ocean University, Lianyungang, China

2.Institute of Geodesy, University of Stuttgart, Stuttgart, Germany

3.School of Geodesy and Geomatics, Wuhan University, Wuhan, Hubei, China

4.School of Environment Science and Spatial Informatics, China University of Mining and Technology, Xuzhou, China



Outline



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- Problem Definition
- Formula Derivation
- Formula Validation
- Conclusion

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- “the gravitational force acting on an object”

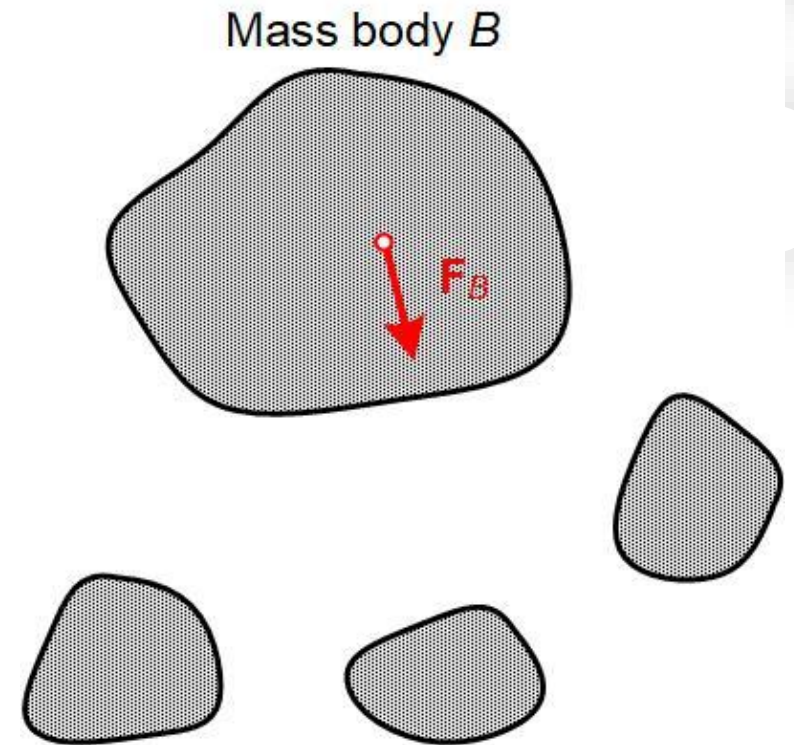
$$\mathbf{F}_B = \mathbf{F}_B^{int} + \mathbf{F}_B^{ext}$$

↑
Attraction
of itself:

↑
Attraction of
external masses:

where $\mathbf{F}_B^{int} = \mathbf{0}$

so that $\mathbf{F}_B = \mathbf{F}_B^{ext}$



The weighing problem

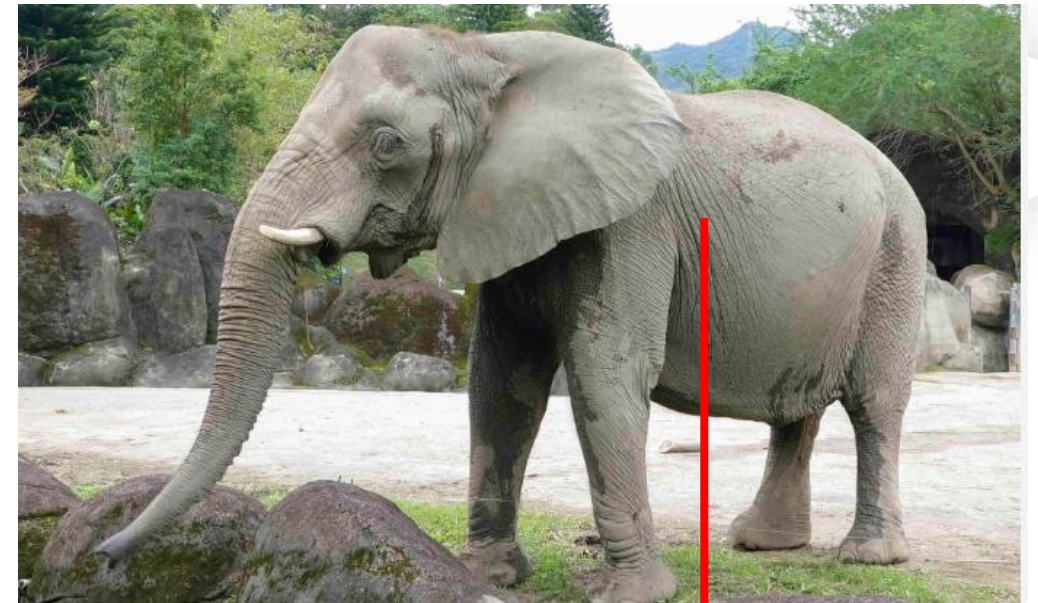
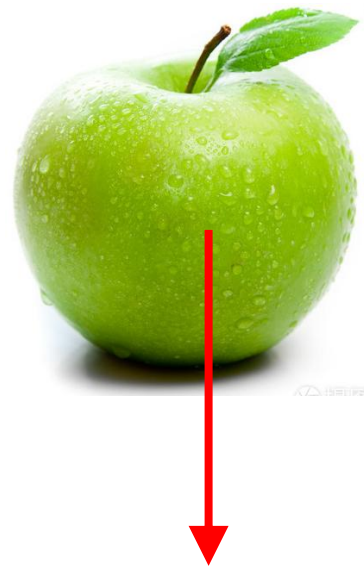
Problem Definition



□ Problem

How can we determine the gravitational force acting on a mass body?

$$\mathbf{F} = m\mathbf{g}$$



The weighing problem

Problem Definition



□ Solution

$$\mathbf{F}_{gravitation} = \mathbf{F}_{Hooke}$$



Hooke's law

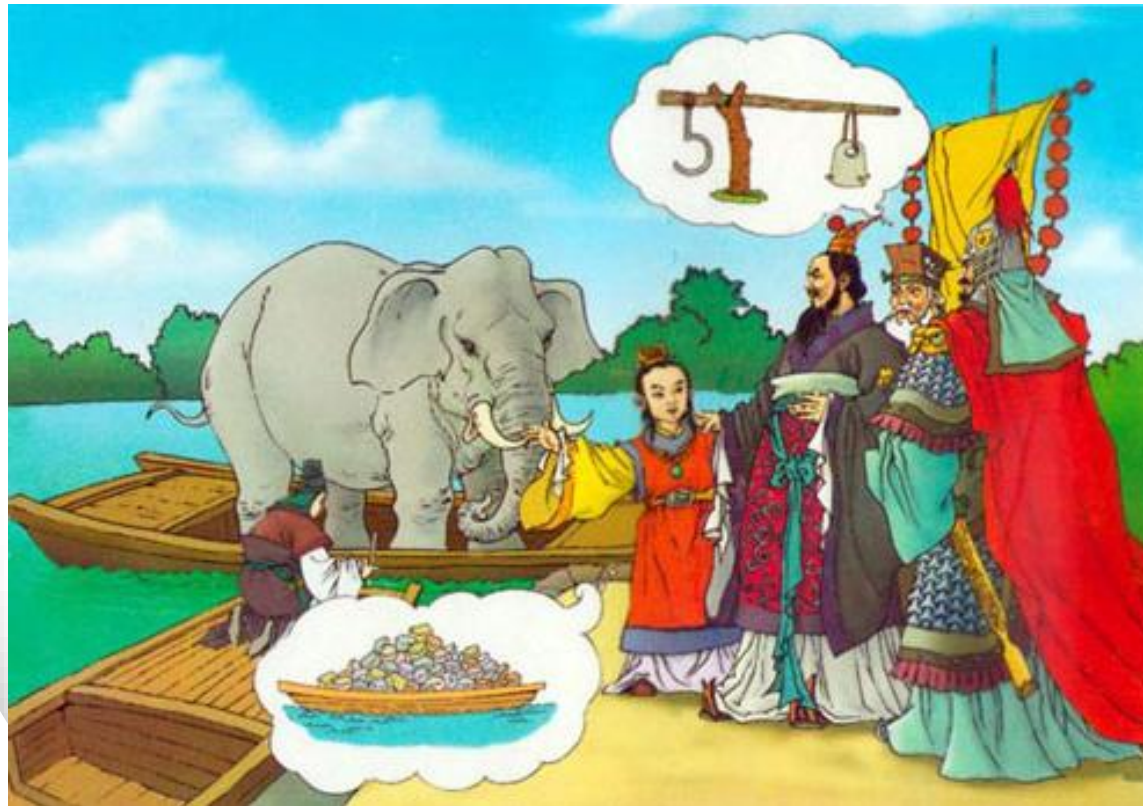
$$\mathbf{F}_{Hooke} = -kx$$

The ancient Chinese story

Problem Definition



□ Cao Chong weighs an elephant



Decompose the whole
into smaller parts

+

Archimedes' principle

$$\mathbf{F}_{elephant} = \mathbf{F}_{buoyancy}$$

Summary for the weighing problem

Problem Definition



The classical weighing problem usually involves an object that:

- has a relatively small size;
- is located within the background gravitational field generated by the Earth.



The weighing problem for larger masses

Problem Definition

□ Problem

How can we determine the total gravitational force acting on a larger mass body?



Large building



Asteroids



To determine the gravitational force acting on a massive body using gravitational information measured outside the body.

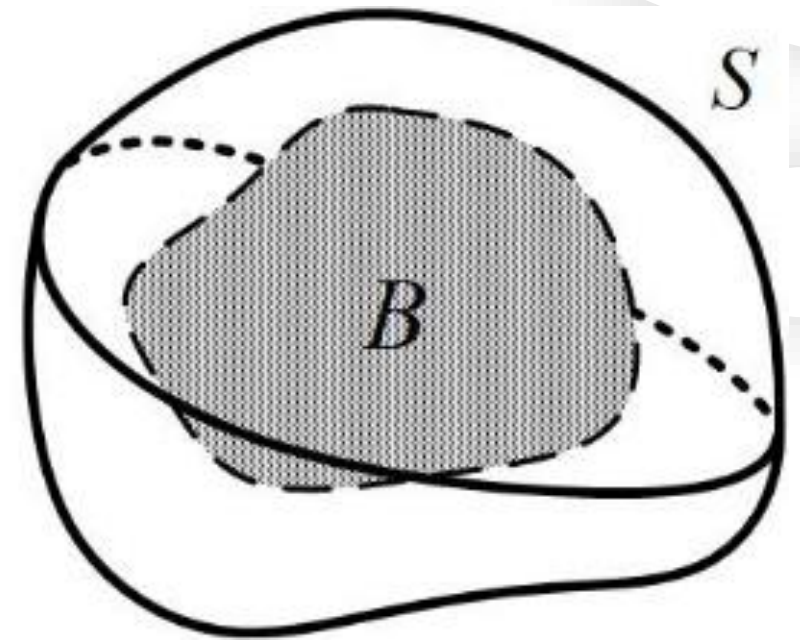
Problem statement

- Given

A closed surface S located outside a mass body B of arbitrary shape (or segmented distribution), where the gravitational vector field $\mathbf{g}$ on S is known.

- Objective

Determine the total gravitational force $\mathbf{F}_B$ acting on the entire mass body.

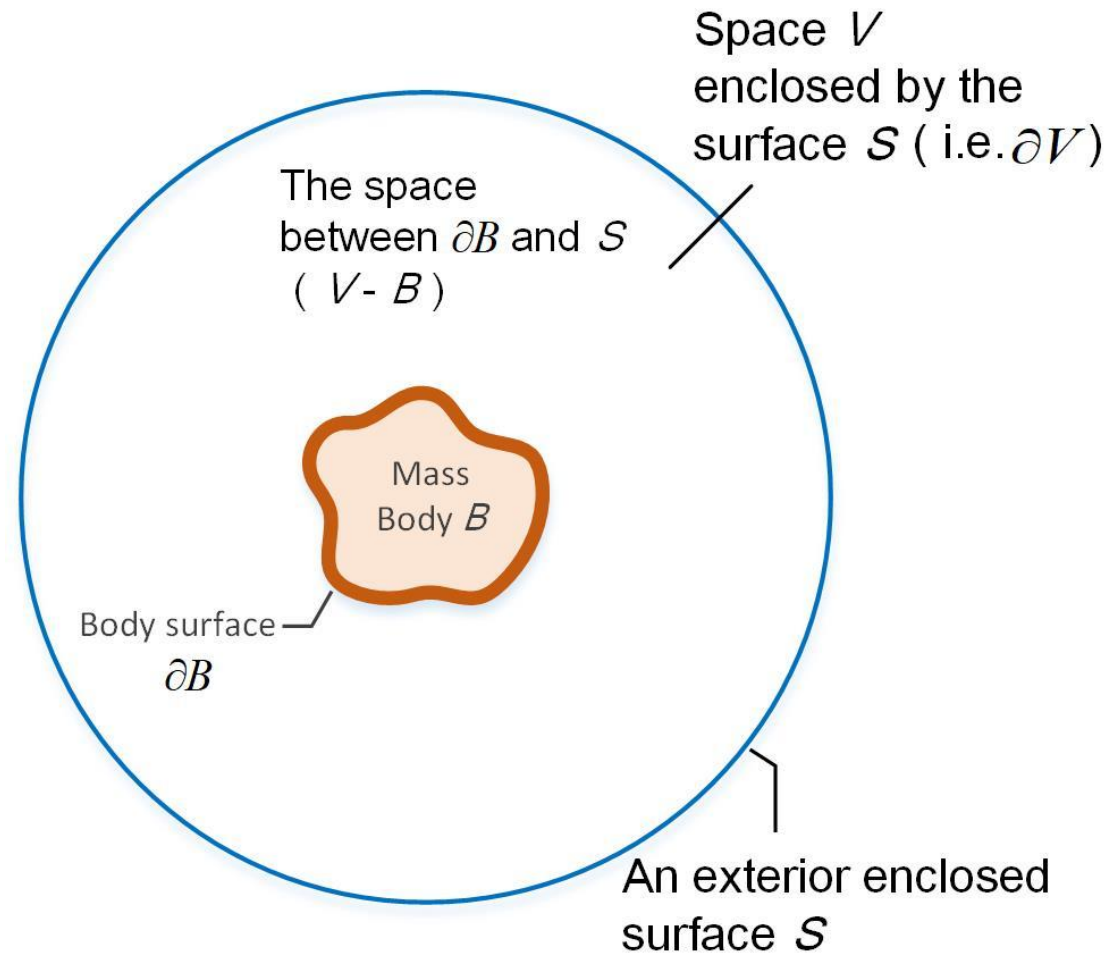


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- Conclusion





Let the gravitational vector be

$$\mathbf{g} = [g_1 \quad g_2 \quad g_3]^T$$

Define the outer product of $\mathbf{g}$ with itself as

$$\mathbf{g} \otimes \mathbf{g} = \begin{bmatrix} g_1 g_1 & g_1 g_2 & g_1 g_3 \\ g_2 g_1 & g_2 g_2 & g_2 g_3 \\ g_3 g_1 & g_3 g_2 & g_3 g_3 \end{bmatrix}$$



Identity: $\nabla \cdot (\mathbf{g} \otimes \mathbf{g}) = (\nabla \cdot \mathbf{g})\mathbf{g} + \mathbf{g} \cdot \nabla \mathbf{g}$



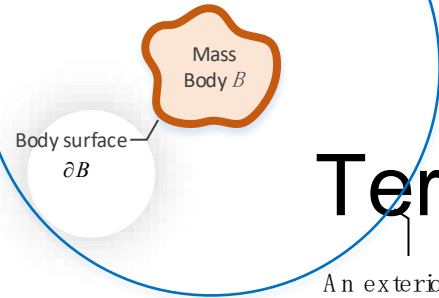
$$\iiint_V \nabla \cdot (\mathbf{g} \otimes \mathbf{g}) dV = \iiint_V (\nabla \cdot \mathbf{g})\mathbf{g} dV + \iiint_V \mathbf{g} \cdot \nabla \mathbf{g} dV$$



$$\underbrace{\iiint_V \nabla \cdot (\mathbf{g} \otimes \mathbf{g}) dV}_{\textcircled{1}} = \left[\underbrace{\iiint_{V-B} (\nabla \cdot \mathbf{g})\mathbf{g} dV}_{\textcircled{2}} + \underbrace{\iiint_B (\nabla \cdot \mathbf{g})\mathbf{g} dV}_{\textcircled{3}} \right] + \underbrace{\iiint_V \mathbf{g} \cdot \nabla \mathbf{g} dV}_{\textcircled{4}}$$

Formula derivation

Space V
enclosed by the
surface S (i.e. ∂V)

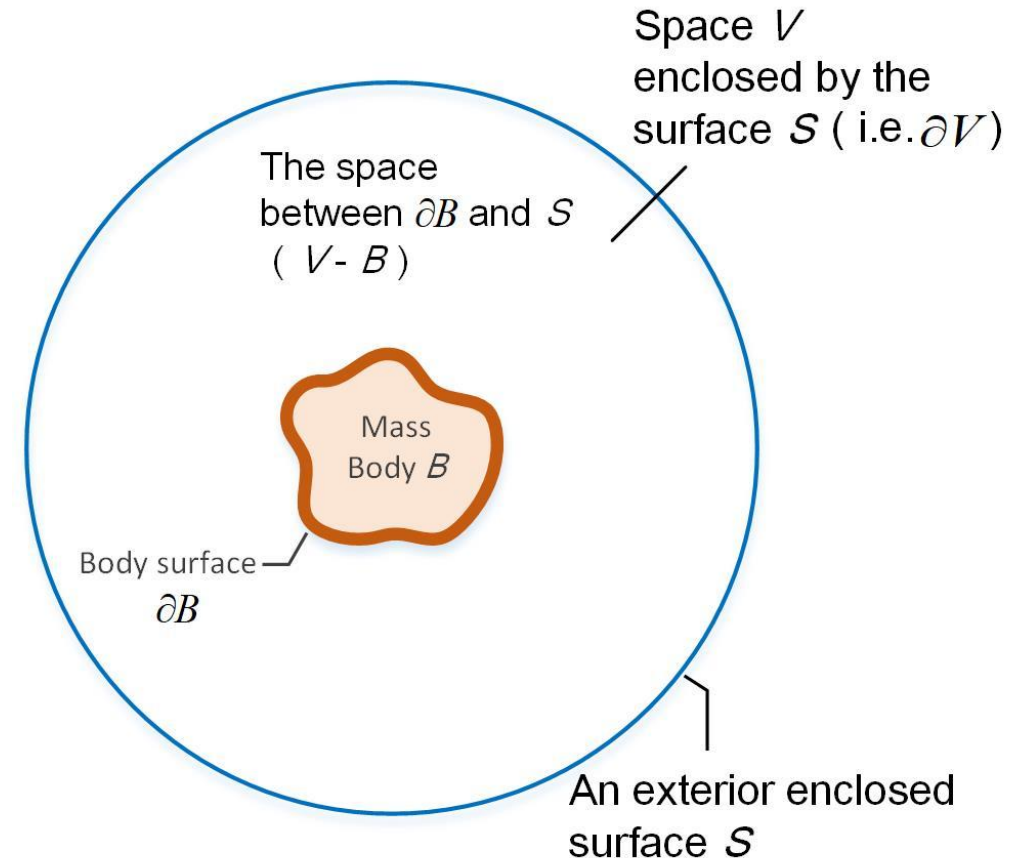


Term ①:

An exterior enclosed surface S

$$\iiint_V \nabla \cdot (\mathbf{g} \otimes \mathbf{g}) dV = \oint_S (\mathbf{g} \otimes \mathbf{g}) \cdot \mathbf{n} dS$$

Here, applied with Gauss's divergence theorem for a tensor function

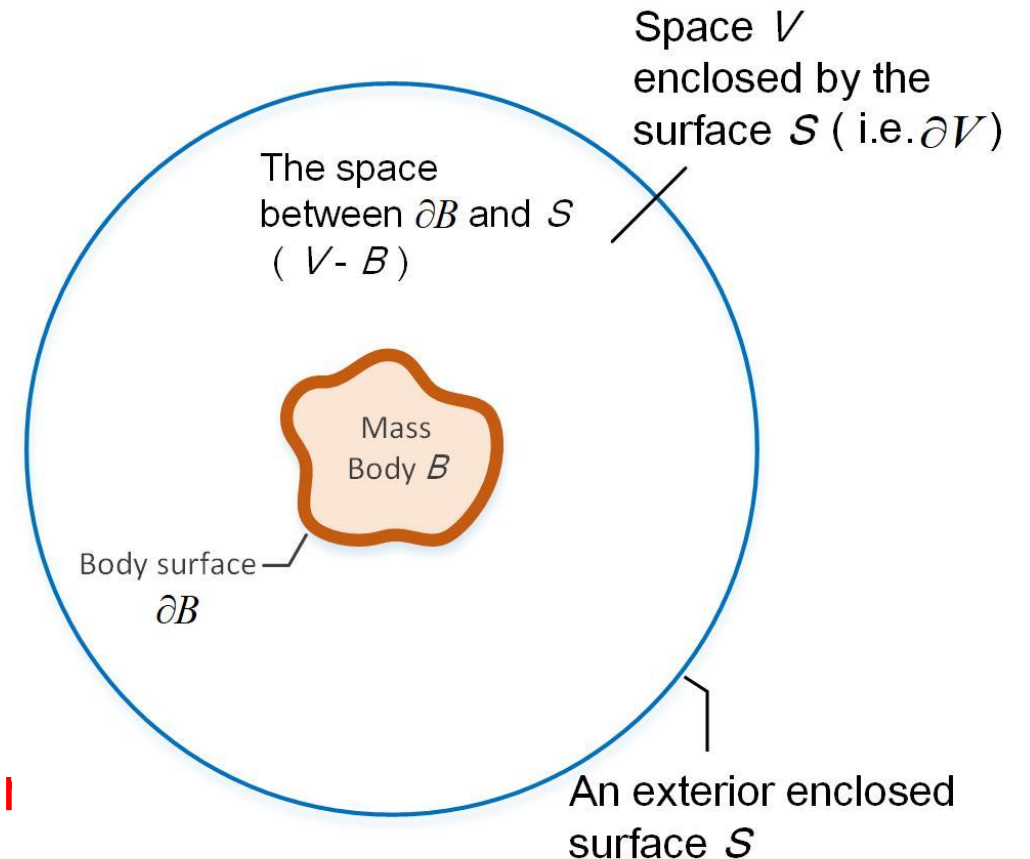


Term ②:

$$\iiint_{V-B} (\nabla \cdot \mathbf{g}) \mathbf{g} dV = 0$$

$$\nabla \cdot \mathbf{g} = 0$$

Here, applied with Laplace Equation

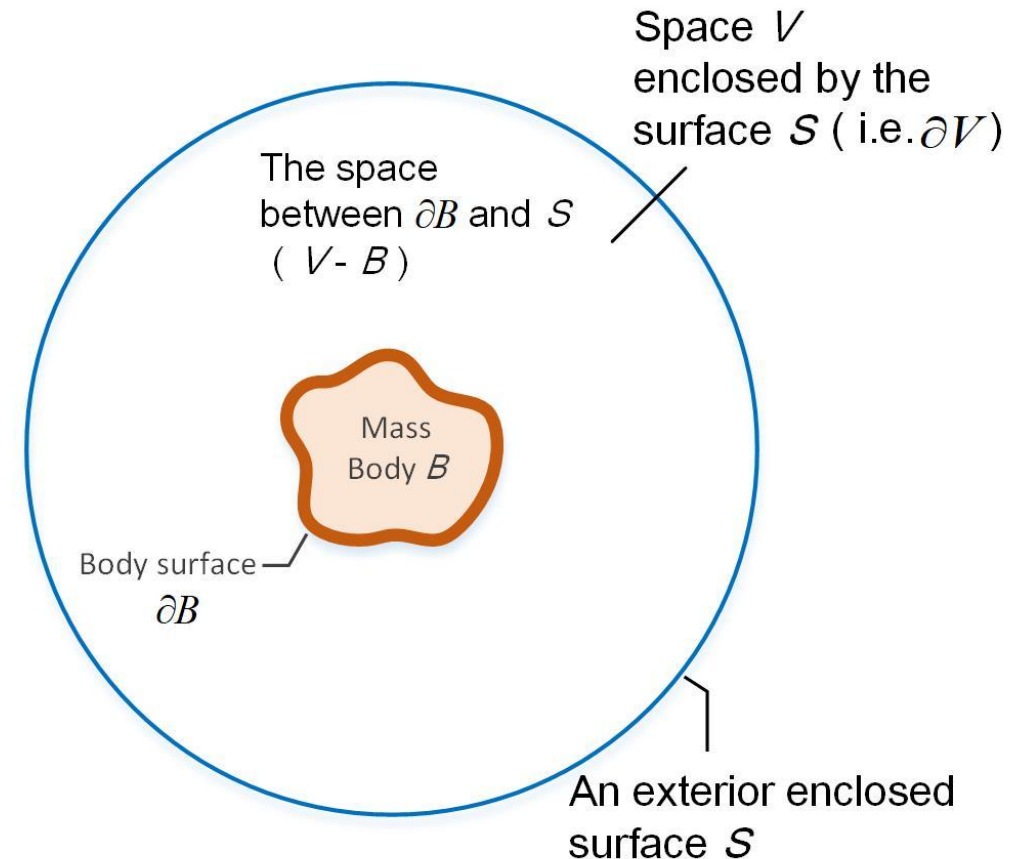


Term ③:

$$\iiint_B \underbrace{(\nabla \cdot \mathbf{g}) \mathbf{g}}_{\substack{\uparrow \\ \nabla \cdot \mathbf{g} = -4\pi G \rho}} dV = -4\pi G \underbrace{\iiint_B \rho \mathbf{g}}_{\substack{\uparrow \\ \mathbf{F}_B}} dV$$

Here, applied with
Poisson Equation

$\mathbf{F}_B$



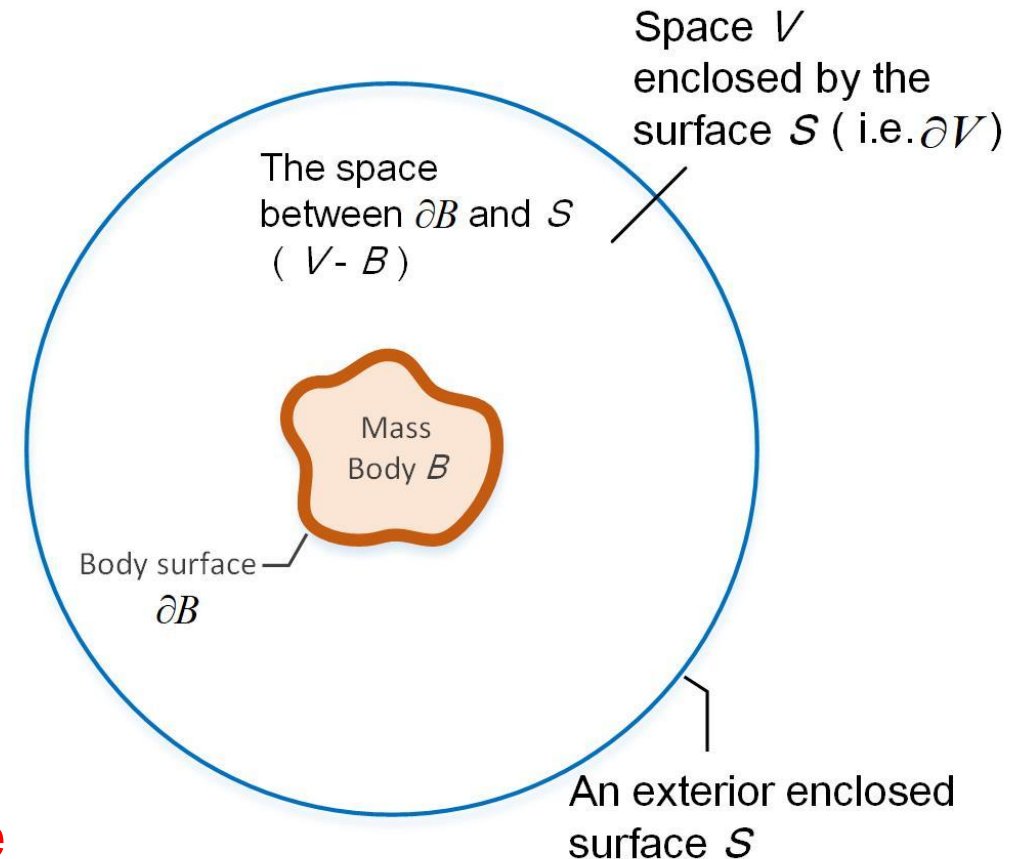
Term ④:

$$\iiint_V \mathbf{g} \cdot \nabla \mathbf{g} \, dV = \oiint_S \left(\frac{1}{2} g^2 \right) \mathbf{n} \, dS$$

$$\nabla \left(\frac{1}{2} g^2 \right)$$

$$\mathbf{g} \cdot \nabla \mathbf{g} = \nabla \left(\frac{1}{2} \mathbf{g} \cdot \mathbf{g} \right) = \nabla \left(\frac{1}{2} g^2 \right)$$

Here, applied with Gauss's divergence theorem for a scalar function





Combining the four terms gives the total gravitational attraction from all external masses:

$$\mathbf{F}_B = -\frac{1}{4\pi G} \oiint_S \mathbf{K} \cdot \mathbf{n} \, dS$$

where

$$\mathbf{K} = \mathbf{g} \otimes \mathbf{g} - \frac{1}{2} g^2 \mathbf{I}$$

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A uniform background gravitational field is established with a gravitational acceleration of

$$g = 10 \text{ m/s}^2$$

Different mass distributions with the same total mass,

$$m = 4.0 \times 10^{10} \text{ kg}$$

are placed in the gravitational field.



The total gravitational force is calculated independently using:

1. Newton's second law

$$\mathbf{F}_N = m\mathbf{g}$$

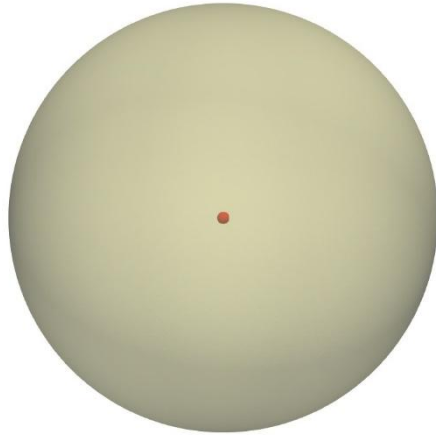
2. Formula derived in this study

$$\mathbf{F}_B = -\frac{1}{4\pi G} \oint \oint_S \mathbf{K} \cdot \mathbf{n} \, dS$$

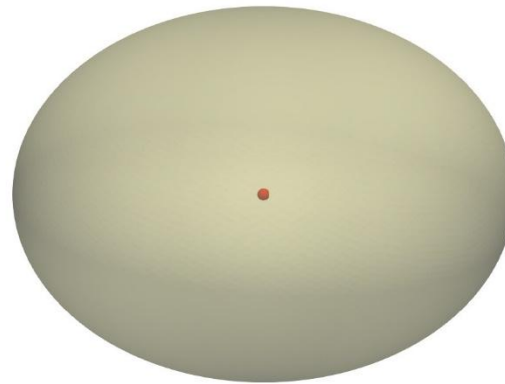
The two results, $\mathbf{F}_N$ and $\mathbf{F}_B$, are then compared.

Case group 1

Formula Validation



One point mass
enclosed by a
sphere



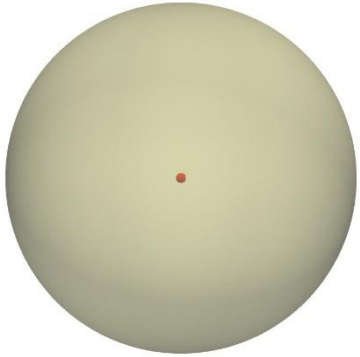
One point mass
enclosed by an
ellipsoid



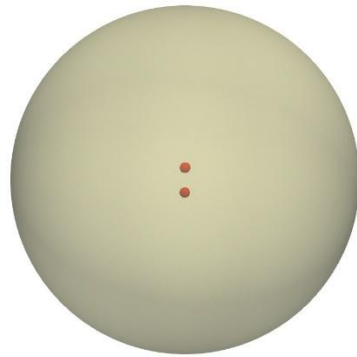
One point mass
enclosed by a **heart-**
shaped surface

Case group 2

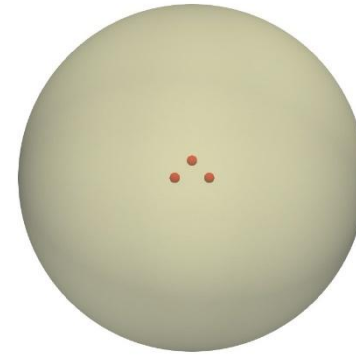
Formula Validation



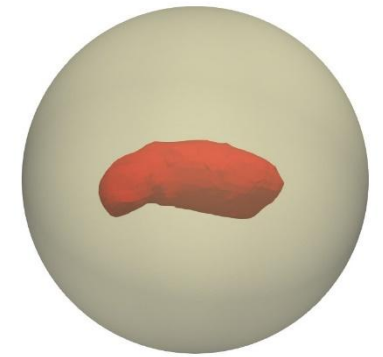
One point mass
enclosed by a
sphere



Two point mass
enclosed by a
sphere



Three point mass
enclosed by a
sphere

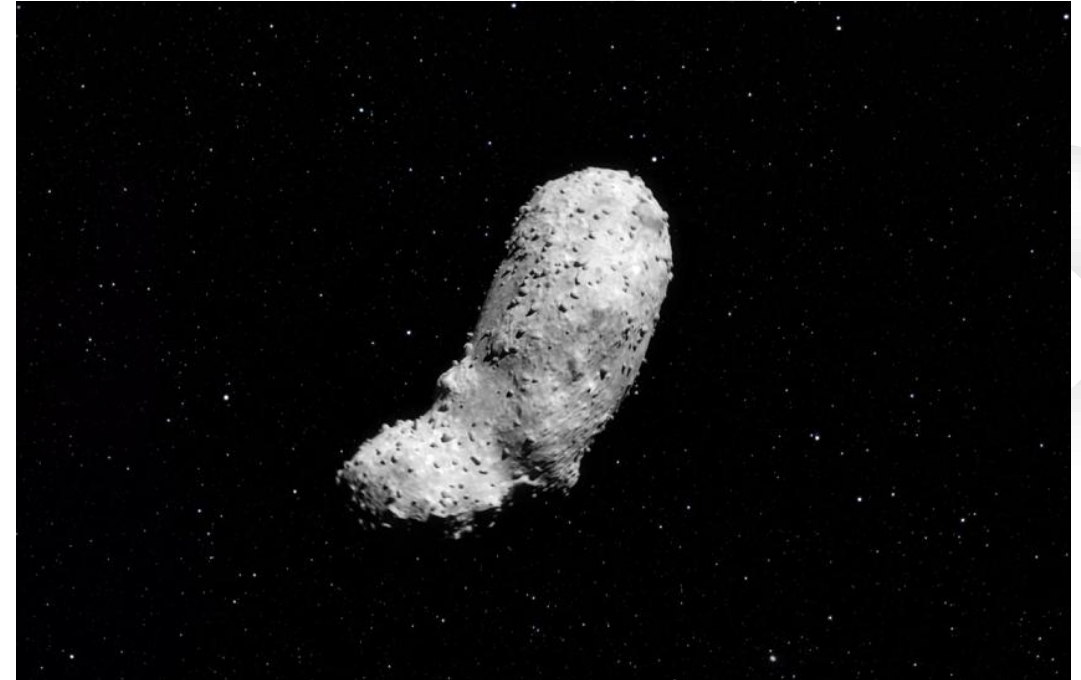


Asteroid Itokawa
enclosed by a
sphere

Asteroid Itokawa

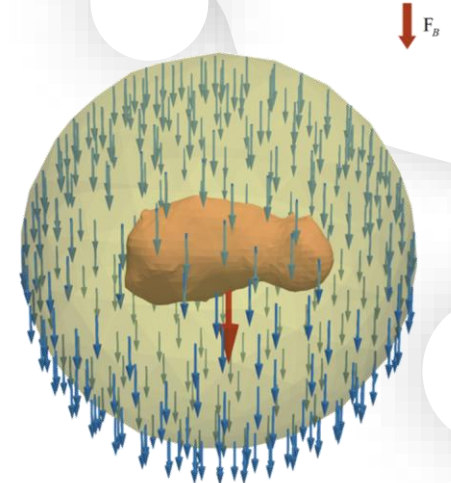
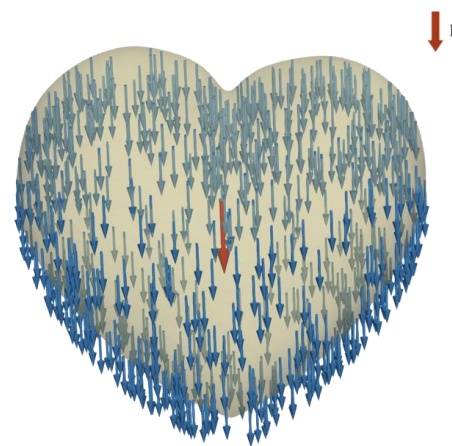
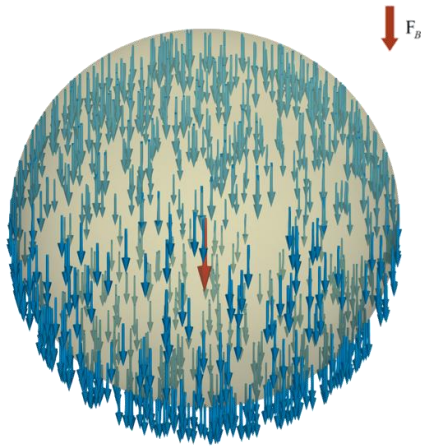
Basic information

Dimensions	$535 \times 294 \times 209 \text{ m}$
Mass (kg)	$(3.51 \pm 0.105) \times 10^{10} \text{ kg}$
Density	$1.9 \pm 0.13 \text{ g/cm}^3$



Result (examples)

Formula Validation



$$\mathbf{F}_N = m\mathbf{g} = 4.0 \times 10^{11} \text{ N}$$

$$\mathbf{F}_B = -\frac{1}{4\pi G} \oint_S \mathbf{K} \cdot \mathbf{n} \, dS \approx 4.0 \times 10^{11} \text{ N}$$

$$\Delta \mathbf{F} = |\mathbf{F}_N - \mathbf{F}_B| < 0.05\%$$

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- Formula derivation

A formula was derived for assessing the total gravitational force acting on a massive body from the external gravitational vector field measured on an enclosing surface.

- Formula validation

The numerical experiments show that the proposed formula agrees with the theoretical gravitational force, with relative differences $< 0.05\%$ in the tested cases.

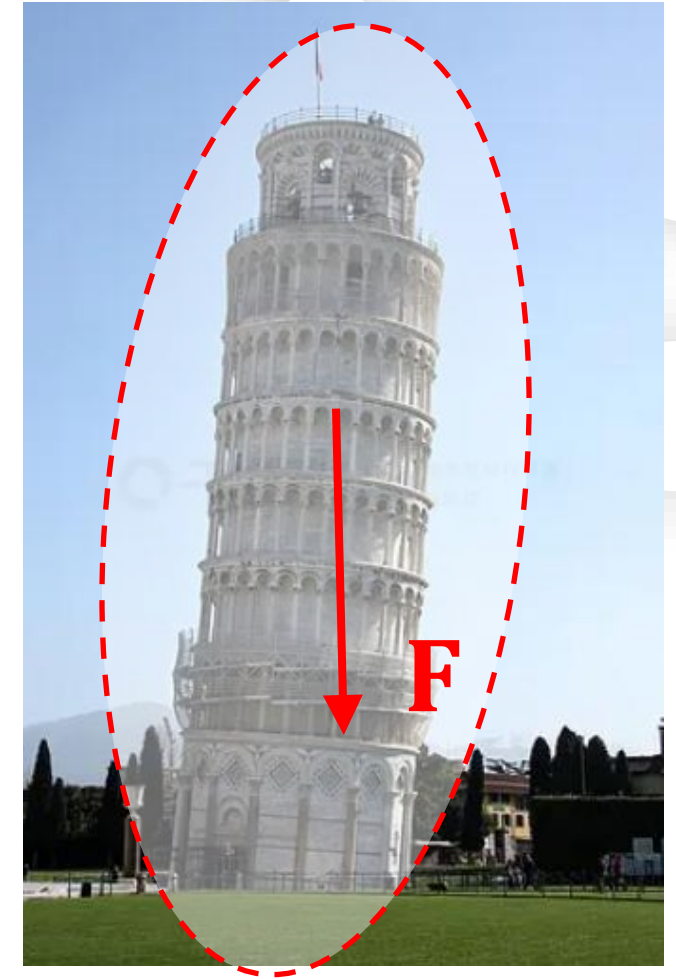
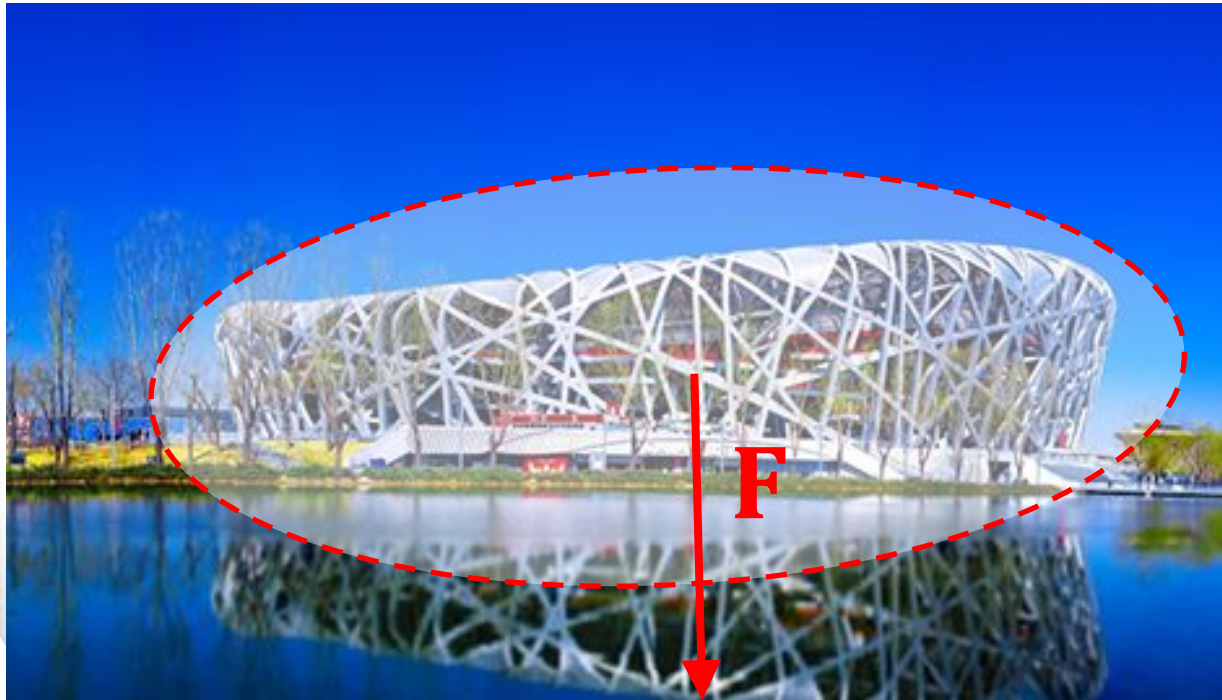
Thanks for your attention.

Comments are questions are welcome.

Email: zyin@jou.edu.cn

Appendix I: discussion

□ The static weighing of large objects



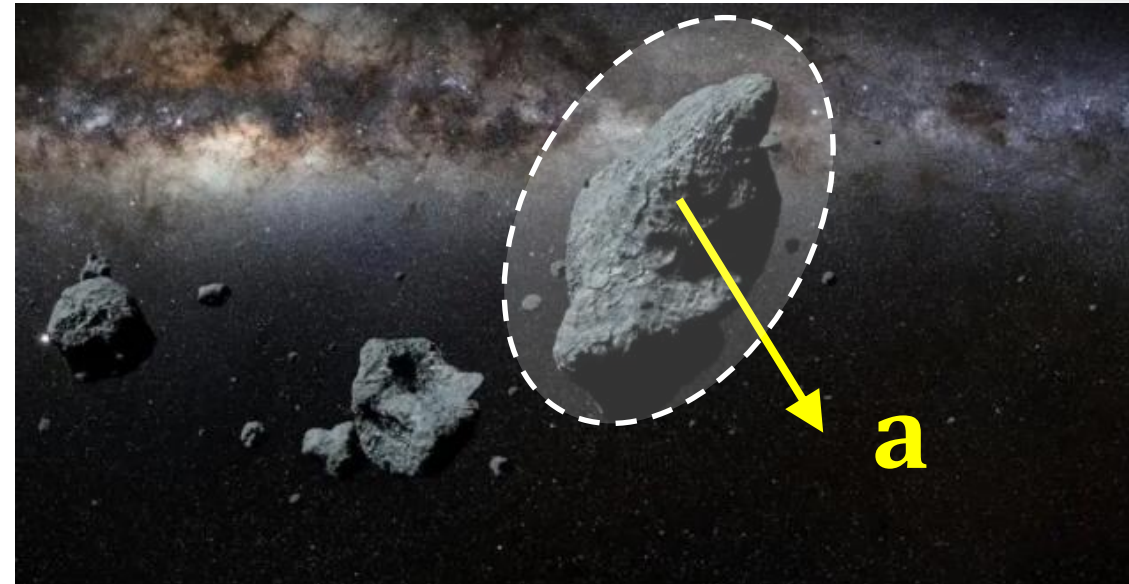
Appendix I: discussion

□ The orbital dynamics of a multi-body system

Net force: $\mathbf{F}_B = -\frac{1}{4\pi G} \oiint_S \mathbf{K} \cdot \mathbf{n} dS$

Mass: $m = -\frac{1}{4\pi G} \oiint_S \mathbf{g} \cdot \mathbf{n} dS$

Acceleration: $\mathbf{a} = \frac{\mathbf{F}_B}{m}$



Appendix II: outlook for future studies

- High-precision **gravitational vector** measurement
- Fast-speed **exterior** gravitational field **inversion**

These developments will provide the observational and computational basis for the application of the proposed formula.

Appendix III: mathematical tool

□ Gauss's Divergence Theorem

for a tensor function $\mathbf{T}$:

$$\iiint_V \nabla \cdot \mathbf{T} dV = \oiint_S \mathbf{T} \cdot \mathbf{n} dS$$

for a vector function $\mathbf{F}$:

$$\iiint_V \nabla \cdot \mathbf{F} dV = \oiint_S \mathbf{F} \cdot \mathbf{n} dS$$

for a scalar function f :

$$\iiint_V \nabla f dV = \oiint_S f \mathbf{n} dS$$